# On the boundary behavior of space mappings

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**Abstract**

It is investigated the boundary behavior of the so-called Q-homeomorphisms. We give a series of conditions on Q(x) and boundaries under which every Q-homeomorphism admits a continuous or homeomorphic extension to the boundary. Note that the Q-homeomorphisms contain conformal, quasiconformal mappings as well as wide modern classes of mappings with finite distortion.

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## 1 Introduction

The work is the continuation of the preprint [IR] as well as [$\mathrm{MRSY}_1$]. In [IR] we investigated the behavior of Q-homeomorphisms at isolated boundary points and proved an analogue of the well-known Painleve theorem for analytic functions in the case when Q(x) has finite mean oscillation in a singular set of the length zero. The reader can also find the history of the question and further references there. Recall the main conception proposed by Professor Olli Martio.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is called a **Q–homeomorphism** if

$$M(f\Gamma) \leq \int\limits_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{1.1}$$

for every family $\Gamma$ of paths in $D$ and every admissible function $\rho$ for $\Gamma$.

Given a family of paths $\Gamma$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is said to be **admissible** for $\Gamma$, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_\gamma \rho(x)\, |dx| \geq 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The **modulus** of $\Gamma$ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\, dm(x). \tag{1.3}$$

In section 2, we extend our lemma in [IR] as well as [$\mathrm{MRSY}_1$] on functions $f : D \to \mathbb{R}$ with finite mean oscillation to the case of boundary points where the domain D satisfies the condition of doubling measure in $\mathbb{R}^n$.

In section 3, we prove the main lemma and its consequence on the continuous and homeomorphic extension of Q-homeomorphisms to the boundary.

In section 4, we formulate the main consequences on the continuous and homeomorphic extension for quasiextremal distance (QED) domains by Gehring-Martio [GM].

Finally, in Section 5, we give conditions under which the null-sets for extremal distances (NED sets) by Vaisala [Va] are unessential.

## 2 Doubling in the theory of finite mean oscillation

We say that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, satisfies the **condition of doubling (Lebesgue) measure** at a point $x_0 \in \partial D$ if

$$|B(x_0, 2\varepsilon) \cap D| \leq c \cdot |B(x_0, \varepsilon) \cap D| \tag{2.1}$$

for some $c > 0$ and for all small enough $\varepsilon > 0$, cf. [HKM] and [He].

We also say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $x_0 \in \overline{D}$, abbr. $\varphi$ is of **FMO** at $x_0$, if

$$\overline{\lim_{\varepsilon \to 0}} \; \frac{1}{|D(x_0, \varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi_\varepsilon}|\, dm(x) < \infty \tag{2.2}$$

where

$$\overline{\varphi_\varepsilon} = \frac{1}{|D(x_0, \varepsilon)|} \int\limits_{D(x_0,\varepsilon)} \varphi(x)\, dm(x) \tag{2.3}$$

is the mean value of the function $\varphi(x)$ over $D(x_0, \varepsilon) = D \cap B(x_0, \varepsilon)$. Here

$$B(x_0, \varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\} \tag{2.4}$$

and the condition (2.2) includes the assumption that $\varphi$ is integrable in a neighborhood of the point $x_0$.

Note, under (2.2) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$, say in the example $\varphi(x) = log\frac{1}{|x|}$ of the class BMO by John and Nirenberg [JN] and [RR], p. 5. Recall that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of **bounded mean oscillation** in $D$, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in$ **BMO**, if

$$\|\varphi\|_* = \sup_{B \subset D} \frac{1}{|B|} \int\limits_B |\varphi(x) - \varphi_B|\, dm(x) \;\; < \;\; \infty \tag{2.5}$$

where the supremum is taken over all balls $B$ in $D$ and

$$\varphi_B = \frac{1}{|B|} \int\limits_B \varphi(x)\, dm(x) \tag{2.6}$$

is the mean value of the function $\varphi$ over $B$. It is well–known that $\mathrm{L}^\infty(\mathrm{D}) \subset \mathrm{BMO}(\mathrm{D}) \subset \mathrm{L}^p_{loc}(\mathrm{D})$ for all $1 \le p < \infty$.

**2.7. Lemma.** *Let a domain $D \subset \mathbb{R}^n, n \ge 3$, satisfy the condition of doubling measure at $0 \in \partial D$. If a nonnegative function $\varphi : D \to \mathbb{R}$ has finite mean oscillation at 0, then*

$$\int\limits_{D(\varepsilon_0)} \frac{\varphi(x)\, dm(x)}{\left(|x| log \frac{1}{|x|}\right)^n} < \infty \tag{2.8}$$

*where $D(\varepsilon_0) = \{x \in D : |x| < \varepsilon_0\}$ for some $\varepsilon_0 \in (0,1)$.*

**2.9. Remark.** Under the conditions of Lemma 2.7, for $n \ge 2$,

$$\int\limits_{D\cap A(\varepsilon,\varepsilon_0)} \frac{\varphi(x)\, dm(x)}{\left(|x| log \frac{1}{|x|}\right)^n} = O\left(loglog \frac{1}{\varepsilon}\right) \tag{2.10}$$

as $\varepsilon \to 0$ where $A(\varepsilon, \varepsilon_0) = \{x \in \mathbb{R}^n : \varepsilon < |x| < \varepsilon_0\}$ for some $\varepsilon_0 \in (0, 1/2)$.

As the example $\varphi(x) = log \frac{1}{|x|} \in BMO$ shows the convergence of the singular integral need not hold for $n = 2$.

*Proof.* Take $\varepsilon_0 \in (0, 2^{-1})$ such that $\varphi$ is integrable over $D_1 = D \bigcap B$ where $B = B(0, \varepsilon_0)$, and

$$\delta = \sup_{r\in(0,\varepsilon_0)} \frac{1}{|D(r)|} \int\limits_{D(r)} |\varphi(x) - \overline{\varphi_r}|\, dm(x) < \infty$$

where $D(r) = D \bigcap B(r)$, $B(r) = B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$. Further, let $\varepsilon < 2^{-1}\varepsilon_0$, $\varepsilon_k = 2^{-k} e \varepsilon_0$, $A_k = \{x \in \mathbb{R}^n : \varepsilon_{k+1} \le |x| < \varepsilon_k\}$, $B_k = B(\varepsilon_k)$ and let $\varphi_k$ be the mean value of $\varphi(x)$ over $D_k = D \bigcap B_k, k = 1, 2 \ldots.$ Take a natural number $N$ such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and denote $\alpha(t) = (t \log_2 1/t)^{-n}$. Then $D \bigcap A(\varepsilon, \varepsilon_0) \subset \Delta(\varepsilon) = \bigcup\limits_{k=1}^{N} \Delta_k$ where $\Delta_k = D \bigcap A_k$ and

$$\eta(\varepsilon) = \int\limits_{\Delta(\varepsilon)} \varphi(x)\, \alpha(|x|)\, dm(x) \le |S_1| + S_2$$

where

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int\limits_{\Delta_k} (\varphi(x) - \varphi_k)\, \alpha(|x|)\, dm(x)\,,$$

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int\limits_{\Delta_k} \alpha(|x|)\, dm(x)\,.$$

Since $\Delta_k \subset D_k \subset B_k$, $|x|^{-n} \le \Omega_n 2^n/|D_k|$ for $x \in \Delta_k$ where $\Omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and $\log_2 \frac{1}{|x|} > k$ in $\Delta_k$, then

$$|S_1| \le \delta \Omega_n e^n \sum_{k=1}^{N} \frac{1}{k^n} < \delta \Omega_n 2^{n+1}$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^n} < \int_1^\infty \frac{dt}{t^n} = \frac{1}{n-1} \le 1\,.$$

Now,

$$\int_{\Delta_k} \alpha(|x|)\, dm(x) \le \frac{1}{k^n} \int_{A_k} \frac{dm(x)}{|x|^n} = \frac{\omega_n}{k^n}$$

where $\omega_n$ is the $(n-1)$–dimensional area of the unit sphere in $\mathbb{R}^n$. Moreover,

$$|\varphi_k - \varphi_{k-1}| = \frac{1}{|D_k|} \left| \int_{D_k} (\varphi(x) - \varphi_{k-1})\, dm(x) \right| \le$$

$$\le \frac{c^n}{|D_{k-1}|} \int_{D_{k-1}} |(\varphi(x) - \varphi_{k-1})|\, dm(x) \le \delta c^n$$

where c is the constant from the condition of doubling measure and by the triangle inequality

$$\varphi_k = |\varphi_k| \le \varphi_1 + \sum_{l=1}^{k} |\varphi_l - \varphi_{l-1}| \le \varphi_1 + k\delta c^n\,.$$

Hence

$$S_2 = |S_2| \le \omega_n \sum_{k=1}^{N} \frac{\varphi_k}{k^n} \le 2\varphi_1 \omega_n + \delta \omega_n c^n \sum_{k=1}^{N} \frac{1}{k^{(n-1)}}$$

and with the estimate

$$\sum_{k=2}^{\infty} \frac{1}{k^{(n-1)}} < \int_1^\infty \frac{dt}{t^{n-1}} = \frac{1}{n-2} \le 1$$

for $n \ge 3$ the proof of Lemma 2.7 is complete.

Since

$$\sum_{k=2}^{N} \frac{1}{k} < \int_1^N \frac{dt}{t} = \log N < \log_2 N$$

and, for $\varepsilon_0 \in (0, 2^{-1})$ and $\varepsilon < \varepsilon_N$,

$$N < N + \log_2 \left( \frac{1}{2\varepsilon_0} \right) = \log_2 \frac{1}{\varepsilon_N} < \log_2 \frac{1}{\varepsilon}$$

then, for $n \ge 2$,

$$\sum_{k=1}^{N} \frac{1}{k^{(n-1)}} \le \sum_{k=1}^{N} \frac{1}{k} < 1 + \log_2 \log_2 \frac{1}{\varepsilon}\,,$$

and we also come to the proof of (2.10) in Remark 2.9.

We would like to complete the section by simple examples and, in particular, nonnegative functions $\varphi : \mathbb{B}^n \to \mathbb{R}, n \geq 3$, which is of FMO at 0 but not of BMO in each neighborhood of 0.

**Examples.** Note that any degree of $log\frac{1}{|x|}$ is integrable in the unit ball $\mathbb{B}^n$. For $n \geq 3$, the test function

$$\varphi(x) = \left[log\frac{1}{|x|}\right]^{n-1}, \quad x \in \mathbb{B}^n \,, \tag{2.11}$$

does not have finite mean oscillation at 0 (and hence all the more not of BMO) because the necessary condition of the convergence of the singular integral (2.8) does not hold for it, see Lemma 2.8 in [MRSY$_1$].

Now, take an arbitrary consequence of disjoint balls $B_k = B(x_k, r_k) \subset \mathbb{B}^n, k = 1, 2, ...$, such that $x_k \to 0$ and $r_k \to 0$ as $k \to \infty$, and set

$$\varphi^*(x) = c_k \varphi\left(\frac{x - x_k}{r_k}\right), \quad x \in B_k, \tag{2.12}$$

and $\varphi^*(x) = 0$ outside of $\cup B_k$ where $c_k$ can be always chosen in such a way that

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(0,\varepsilon)|} \int\limits_{B(0,\varepsilon)} \varphi^*(x)\, dm(x) < \infty \,. \tag{2.13}$$

Then $\varphi^*$ has finite mean oscillation at 0 by Corollary 2.6 in [MRSY$_1$] but by the construction $\varphi^*$ has not finite mean oscillation at every point $x_k \to 0$. Hence $\varphi^*$ is all the more not of BMO in every neighborhood of 0.

## 3 The main lemma on extension of Q-homeomorphisms.

Let $D \subset \mathbb{R}^n, n \geq 2$, be a domain. $\partial D$ is said to be **strongly accessible** if, for nondegenerate continua E and F in $\overline{D}$,

$$M(\Delta(E, F; D)) > 0 \tag{3.1}$$

and **weakly flat** if, for nondegenerate continua $E$ and $F$ in $\overline{D}$ with $E \cap F \neq 0$,

$$M(\Delta(E, F; D)) = \infty \tag{3.2}$$

where $\Delta(E, F; D)$ is the family of all paths joining $E$ and $F$ in $D$. It is known that every weakly flat boundary is strongly accessible, see Lemma 2.3 in [MRSY$_1$].

A domain $D \subset \mathbb{R}^n, n \geq 2$, is called **locally connected at** $x_0 \in \partial D$ if $x_0$ has an arbitrary small neighborhood $U$ such that $U \cap D$ is connected. Every Jordan domain $D$ in $\mathbb{R}^n$ is locally connected at $\partial D$, see [Wi], p. 66.

**3.3. Lemma.** *Let $f : D \to \mathbb{R}^n, n \geq 2$, be a Q-homeomorphism and let the domain D be locally connected at $x_0 \in \partial D$ and the domain $D' = f(D)$ have a* *strongly accessible boundary. If*

$$\int\limits_{D_{x_0,\varepsilon}} Q(x)\cdot\psi^n_{x_0,\varepsilon}(x)\,dm(x) = o(I^n_{x_0}(\varepsilon)) \tag{3.4}$$

*as* $\varepsilon\to 0$ *where* $D_{x_0,\varepsilon}=\{x\in D:\varepsilon<|x-x_0|<\varepsilon_0\}$, $\varepsilon_0=\delta(x_0)<diam\,D$, *and* $\psi_{x_0,\varepsilon}(t)$ *are nonnegative measurable (by Lebesgue) functions on* $(0,\infty)$ *such that*

$$I_{x_0}(\varepsilon)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi_{x_0,\varepsilon}(t)\,dt<\infty,\quad \varepsilon\in(0,\varepsilon_0)\,, \tag{3.5}$$

*then* $f$ *can be extended to* $x_0$ *by continuity.*

*Proof.* We must show that the cluster set $E=C(x_0,f)=\{y\in\overline{\mathbb{R}^n}:y=\lim\limits_{k\to\infty}f(x_k),x_k\to x_0,x_k\in D\}$ is a singleton. Note $E$ is a continuum because $D$ is locally connected at $x_0$. Let us assume that the continuum $E$ is not degenerate.

Let $\Gamma_\varepsilon$ be a family of all paths joining the spheres $S_\varepsilon=\{x\in\mathbb{R}^n:|x-x_0|=\varepsilon\}$ and $S_0=\{x\in\mathbb{R}^n:|x-x_0|=\varepsilon_0\}$ in $D_{x_0,\varepsilon}$. Let $\psi^*_{x_0,\varepsilon}$ be a Borel function such that $\psi^*_{x_0,\varepsilon}(t)=\psi_{x_0,\varepsilon}(t)$ for a.e. $t\in(\varepsilon,\varepsilon_0)$. Such functions $\psi^*_{x_0,\varepsilon}$ exist by the Lusin theorem, see e.g. 2.3.5. in [Fe] and [Sa], p. 69. Then the function

$$\rho_\varepsilon(x)=\begin{cases}\psi^*(|x-x_0|)/I_{x_0}(\varepsilon), & \text{if } x\in D_{x_0,\varepsilon},\\ 0, & \text{if } x\in\mathbb{R}^n\backslash D_{x_0,\varepsilon},\end{cases}$$

is admissible for $\Gamma_\varepsilon$ and hence

$$M(f\Gamma_\varepsilon)\le\int\limits_D Q(x)\cdot{\rho_\varepsilon}^n(|x|)\ \ dm(x)\,,$$

i.e., $M(f\Gamma_\varepsilon)\to 0$ as $\varepsilon\to 0$ in view of (3.4).

On the other hand, $M(f\Gamma_\varepsilon)\ge M_0=M(\Delta(fS_0,E;D')$ and by the strong accessibility of $\partial D'$ we have that $M_0>0$. The contradiction disproves the above assumption.

**3.6. Corollary.** *If in addition* $D$ *is locally connected at* $\partial D$ *and the condition (3.4) holds at every point* $x_0\in\partial D$, $Q\in L^1(D)$ *and* $\partial D'$ *is weakly flat, then* $f$ *is extended to a homeomorphism* $\overline{f}:\overline{D}\to\overline{D'}$.

The latter is a direct consequences of Lemma 3.6 in [MRSY$_1$] and Lemma 3.3.

# 4 Consequences for quasiextremal distance domains

A domain $D\subset\mathbb{R}^n, n\ge 2$, is called a **quasiextremal distance domain**, abbr. a **QED domain**, if

$$M(\Delta(E,F;\overline{\mathbb{R}^n})\le K\cdot M(\Delta(E,F;D)) \tag{4.1}$$

for some $K \geq 1$ and for all pairs of disjoint continua $E$ and $F$ in $D$, see [GM]. It is known that the equality (4.1) also holds in a QED domain for every pair of disjoint continua $E$ and $F$ in $\overline{D}$, see Theorem 2.8 in [HK$_1$], p. 173, cf. Lemma 6.11 in [MV], p. 35. The latter implies (4.1) for nondegenerate intersecting continua $E$ and $F$ in $\overline{D}$, too. Hence QED domains have weakly flat boundaries, cf. Lemma 3.1 in [HK$_2$], p. 196. Every QED domain is **quasiconvex**, i.e., each pair of points $x_1$ and $x_2 \in D$ can be joined by a rectifiable arc $\gamma$ in $D$ such that

$$s(\gamma) \leq a \cdot |x_1 - x_2|, \tag{4.2}$$

see Lemma 2.7 in [GM], p. 184. Hence D is locally connected at $\partial D$, cf. also Lemma 2.11 in [GM], p. 187, and [HK$_2$], p. 190.

A domain $D \subset \mathbb{R}^n, n \geq 2$, is said to be a **uniform domain** if the inequalities (4.2) and

$$\min_{i=1,2} \; s(\gamma(x_i, x)) \leq b \cdot d(x, \partial D) \tag{4.3}$$

hold for some $\gamma$ and for all $x \in \gamma$ where $\gamma(x_i, x)$ is the part of $\gamma$ between $x_i$ and $x$, see [MS]. Every uniform domain is a QED domain but there exist QED domains which are not uniform. Bounded convex domains provide simple examples of uniform domains.

**4.4. Theorem.** *Let $f$ be a $Q$-homeomorphism between QED domains $D$ and $D'$ in $\mathbb{R}^n, n \geq 2$. If the condition (3.4) holds at a point $x_0 \in \partial D$, then there is a limit $f(x)$ as $x \to x_0$.*

**4.5. Corollary.** *If under the conditions of Theorem 4.4, $Q \in L^1(D)$ and (3.4) holds at every point $x_0 \in \partial D$, then $f$ admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

In particular, taking in (3.4) $\psi(t) = 1/t$ and $\psi(t) = 1/t log\frac{1}{t}$, we have the following.

**4.6. Theorem.** *Let $f$ be a $Q$-homeomorphism between QED domains $D$ and $D'$ in $\mathbb{R}^n, n \geq 2$. If at every point $x \in \partial D$*

$$q(r) = O\left(\left[log\frac{1}{r}\right]^{n-1}\right) \tag{4.7}$$

*as $r \to 0$ where $q(r)$ is a mean value of $Q(y)$ over the intersection of the sphere $|y - x| = r$ with the domain $D$, then $f$ is extended to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$.*

**4.8. Corollary.** *In particular, the conclusion holds if*

$$Q(y) = O\left(\left[log\frac{1}{|y-x|}\right]^{n-1}\right) \tag{4.9}$$

*as $y \to x$ for every $x \in \partial D$.*

Similarly, choosing in (3.4) $\psi(t) = 1/tq^{\beta}(t)$ we have the next consequence of Theorem 4.4

**4.10. Corollary.** *The same is true if*

$$\int_0^{\varepsilon_0} \frac{dr}{rq^{\beta}(r)} = \infty \tag{4.11}$$

*for some $\beta \geq 1/(n-1)$ and, in particular, if*

$$\int_0^{\varepsilon_0} \frac{dr}{rq(r)} = \infty\,. \tag{4.12}$$

**4.13. Corollary.** *The same is valid if, for some $\beta \geq 1/(n-1)$ and $\alpha \geq 1$, at every point $x \in \partial D$*

$$\int_0^{\varepsilon_0} \frac{dr}{rq_{\alpha}^{\beta}(r)} = \infty \tag{4.14}$$

*where*

$$q_{\alpha}(r) = q_{\alpha}(x,r) = \left( \frac{1}{|S(r)|} \int_{S(r)} Q^{\alpha}(y) \right)^{\frac{1}{\alpha}}, \tag{4.15}$$

$$S(r) = S_D(x,r) = \{y \in D : |y-x| = r\}. \tag{4.16}$$

Indeed, by the Jensen inequality $q_{\alpha}(r) \geq q_1(r) = q(r)$, see e.g. [Zi], p. 20.

**4.17. Corollary.** *The conclusion is valid if, for some $\beta \geq 1/(n-1)$ and $\alpha \geq 1$, and at every point $x \in \partial D$*

$$\int_0^{\varepsilon_0} \frac{dr}{rK_{\alpha}^{\beta}(r)} = \infty \tag{4.18}$$

*where*

$$K_{\alpha}(r) = \sup_{t \in [r,\varepsilon_0]} Q_{\alpha}(t) \tag{4.19}$$

*and*

$$Q_{\alpha}(t) = Q_{\alpha}(x,t) = \left( \frac{1}{|B(t)|} \int_{B(t)} Q^{\alpha}(x)\, dm(x) \right)^{\frac{1}{\alpha}}, \tag{4.20}$$

$$B(t) = B_D(x,t) = \{y \in D : |y-x| < t\} \tag{4.21}$$

Really, by the theorem on the differentiability of indefinite integral, see e.g. [Sa], p. 118, applied to $\int_0^t \varphi(\rho)\, d\rho$ where $\varphi(\rho) = \int_{S(\rho)} Q^{\alpha}(x)$ and $|B(t)| = \int_0^t |S(\rho)| d\rho$ we have the inequality

$$q_{\alpha}(r) \leq K_{\alpha}(r) \tag{4.22}$$

for a.e. $r \in (0, \varepsilon_0)$ and hence (4.18) implies (4.14).

Combining Theorem 4.4 with Lemma 2.7 we obtain, in particular, the following Theorem.

**4.23. Theorem.** *Let $f$ be a $Q$-homeomorphism between bounded convex domains $D$ and $D'$ in $\mathbb{R}^n, n \geq 2$. If $Q(x) \in L^1(D)$ has finite mean oscillation at every point $x_0 \in \partial D$, then $f$ has a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

**4.24. Corollary.** *If $f$ is a $Q$-homeomorphism of the unit ball $\mathbb{B}^n, n \geq 2$, onto itself where $Q \in L^1(\mathbb{B}^n)$ has finite mean oscillation at every point $x_0 \in \partial\mathbb{B}^n$, then $f$ admits a homeomorphic extension $\overline{f} : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$.*

# 5 On singular null-sets for extremal distances

A closed set $X \subset \mathbb{R}^n, n \geq 2$, is called a **null-set for extremal distances,** abbr. a **NED set**, if

$$M(\Delta(E, F; \mathbb{R}^n)) = M(\Delta(E, F; \mathbb{R}^n \backslash X)) \tag{5.1}$$

for every pair of disjoint continua $E$ and $F \subset \mathbb{R}^n \backslash X$.

**5.2. Remark.** It is known that, if $X \subset \mathbb{R}^n$ is a NED set, then

$$|X| = 0 \tag{5.3}$$

and $X$ does not locally disconnect $\mathbb{R}^n$, i.e.,

$$dim X \leq n - 2. \tag{5.4}$$

Conversely, if $X \subset \mathbb{R}^n$ is closed and

$$\Lambda_{n-1}(X) = 0, \tag{5.5}$$

then $X$ is a NED set, see [Va].

Here $\Lambda_{n-1}(X)$ denotes the $(n-1)$-dimensional Hausdorff measure of $X$ in $\mathbb{R}^n$. Below we also use the designation of the **cluster set**

$$f(X) = C(X, f) = \{y \in \overline{\mathbb{R}^n} : y = \lim_{k \to \infty} f(x_k),\ x_k \to x_0 \in X,\ x_k \in D\} \tag{5.6}$$

for sets $X \subset \partial D$ and mappings $f : D \to \overline{\mathbb{R}^n}$.

**5.7. Proposition.** *Let $f : D \backslash X \to \overline{\mathbb{R}^n}, n \geq 2$, be a $Q$-homeomorphism and let $X \subset D$ and $f(X)$ be NED sets and $Q$ be integrable in a neighborhood of the set $X$. If the condition (3.4) holds at every point $x_0 \in X$, then $f$ has a homeomorphic extension to $D$.*

**5.8. Theorem.** *If $Q \in L^1_{loc}(D)$ has finite mean oscillation at each point of a NED set $X \subset D$, then every $Q$-homeomorphism $f : D \backslash X \to \overline{\mathbb{R}^n}$ with a NED set $C(X, f)$ has a homeomorphic extension to $D$.*

**5.9. Corollary.** *In particular, this holds if*

$$\Lambda_{n-1}(X) = \Lambda_{n-1}(f(X)) = 0 \tag{5.10}$$

*and $Q \in L^1_{loc}(D)$ has finite mean oscillation at every point of a closed set $X \subset D$.*

**5.11. Corollary.** *If all points of a closed set $X \subset D$ with the condition (5.10) are points of Lebesgue for the function $Q \in L^1_{loc}(D)$, then the Q-homeomorphism $f : D\backslash X \to \overline{\mathbb{R}^n}$ admits a homeomorphic extension to $D$.*

**5.12. Corollary.** *If the singular integral*

$$\int\limits_U \frac{Q(y)-Q(x)}{|y-x|^n}\, dm(y) \tag{5.13}$$

*is convergent for every $x$ of a closed set $X \subset D$ over a neighborhood $U$ of the set $X$, then under the condition (5.10) every Q-homeomorphism $f : D\backslash X \to \overline{\mathbb{R}^n}$ has homeomorphic extension to $D$.*

**5.14. Corollary.** *If at least one of the conditions (4.7), (4.9), (4.11), (4.12), (4.14), (4.18) holds at every point $x$ of a closed set $X \subset D$, then under the condition (5.10) every Q-homeomorphism $f : D\backslash X \to \overline{\mathbb{R}^n}$ admits a homeomorphic extension to $D$.*

In the other words, using the known term of the theory of analytic functions, the given types of singularities $X$ of a Q-homeomorphism $f : D\backslash X \to \overline{\mathbb{R}^n}$ are **unessentional**, i.e., $f$ is extended to $X$ by continuity as a homeomorphism of $D$. The term is all the more justified for the exceptional NED sets in view of (5.3) and (5.4).

The results of the work have also a number of the corresponding consequences for wide classes of mappings with finite distortion.

**5.15. Remark.** In particular, all the above theorems in Sections 3-5 hold for homeomorphisms $f$ of the Sobolev class $W^{1,n}_{loc}$ with $f^{-1} \in W^{1,n}_{loc}$ if

$$K_I(x,f) \leq Q(x) \tag{5.16}$$

where $K_I(x,f)$ is the so-called inner dilatation of the mapping $f$ at a point $x$, see Theorems 4.6 and 7.10 in [MRSY$_2$].

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