On existence and uniqueness of degenerate Beltrami equation

Olli Martio and Vladimir Miklyukov

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Abstract

Existence and uniqueness of solutions of degenerate Beltrami equation are studied under the condition that the dilatation K has an upper bound in $W^{1,2}$.

1 Introduction and main results

Let $w : D \to \mathbf{R}^2$ be a homeomorphism of a domain $D \subset \mathbf{R}^2$ onto a domain $\mathcal{D} = w(D) \subset \mathbf{R}^2$ and let $\mu : D \to \mathbf{R}^2$ be a measurable function such that $|\mu(z)| < 1$ for a.e. $z \in D$. We say that w is a μ-homeomorphism if $w \in W^{1,1}_{\text{loc}}(D)$ and

$$w_{\overline{z}} = \mu(z)\, w_z \tag{1.1}$$

for a.e. $z \in D$. Write

$$K_\mu(z) = (1 + |\mu(z)|)/(1 - |\mu(z)|),$$

then $K_\mu(z) \in [1, \infty]$ and $K_\mu(z) < \infty$ for a.e. $z \in D$.

By the theorem of Gehring and Lehto [1] a μ-homeomorphizm w is a.e. differenciable and (1.1) yields

$$\|w'(z)\|^2 \le K_\mu(z)\, J(z, w) \quad a.e. \tag{1.2}$$

where $\|w'(z)\|$ is the operator norm

$$\|w'(z)\| = \sup_{|h|=1} |w'(z)\, h|$$

of the linear map $w'(z)$ and $J(z, w) = \det w'(z)$ is the Jacobian of $w'(z)$.

We recall that a mapping w is q-quasiconformal, $1 \le q < \infty$, if w is a μ-homeomorphism and $K_\mu(z) \le q$ for a.e. $z \in D$. For $q = 1$ the mapping w is conformal. Note that if w is q-quasiconformal, then $w \in W^{1,2}_{\text{loc}}(D)$ and w^{-1} is q-quasiconformal as well.

A measurable function $p : D \to [1, \infty]$ is said to be locally $W^{1,2}$ – majorized in D if for every subdomain $U \subset\subset D$ there exists a function $K_U \in W^{1,2}(U)$ such that

$$p(z) \le K_U(z)$$

a.e. in U. Note that if p is locally $W^{1,2}$ – majorized then the set $\{z : \lim_{\xi \to z} p(\xi) = \infty\}$ is of 2-capacity 0 (for example, see [2, Theorem **6.2.1**]).

1.3. Theorem. *Suppose that $\mu : D \to \mathbf{R}^2$ is a measurable function such that $|\mu| < 1$ a.e. and that K_μ is locally $W^{1,2}$ – majorized in D. Then there exists a μ-homeomorphism $w : D \to \mathbf{R}^2$ onto a domain $\mathcal{D} = w(D)$ with $w \in W^{1,2}_{\mathrm{loc}}(D)$ and $w^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$,*

Moreover, the mapping w is unique up to conformal mappings on the w-plane.

Homeomorphic $W^{1,2}_{\mathrm{loc}}$ solutions of the Beltrami equation (1.1) with $K_\mu(z) \le q$ were first systematically studied by Morrey [3]. Bers [4] and Belinskii [5] considered mappings with locally bounded distortion coefficients $K_\mu(z)$.

The proof for Theorem 1.3 follows from the following result on μ-homeomorphisms of a disk. We let $B = B(0,1)$ denote the unit disk and the space $\overset{\circ}{W}{}^{1,2}(B)$ is the closure of $C_0^\infty(B)$ in $W^{1,2}(B)$.

1.4. Theorem. *Let $\mu : B \to \mathbf{R}^2$ be a measurable function such that $|\mu| < 1$ a.e. and that $K_\mu(z) \le K(z)$ a.e. in D where $K \in W^{1,2}(B)$. Then there exists a μ-homeomorphism w of B onto B such that $w \in W^{1,2}_{\mathrm{loc}}(B)$ and $w^{-1} \in W^{1,2}_{\mathrm{loc}}(B)$. Moreover, w is unique up to conformal transformations on the w-plane. If, in addition,*

$$K_\mu(z) \le K_0 + K(z) \quad \text{a.e. in} \quad D, \tag{1.5}$$

where $1 \le K_0 \equiv \mathrm{const} < \infty$ and $K \in \overset{\circ}{W}{}^{1,2}(B)$, then $w \in W^{1,2}(B)$ and $w^{-1} \in W^{1,2}(B)$.

The case (1.5) of Theorem 1.4 was proved by Miklyukov and Suvorov [6]. Because $W^{1,2}$ functions belong to BMO and VMO classes (see [7, §**I.2**]) then quasiconformal mappings with locally $W^{1,2}$ majorized chractheristics are BMO-qc and, respectively, VMO-qc mappings which are being studied in V. Ryazanov, U. Srebro, and E. Yakubov [8], O. Martio, V. Ryazanov, U. Srebro and E. Yakubov [9]. For other results concerning degenerate quasiconformal mappings, see Pesin [10], Dzuraev [11], Lehto [12], Kruglikov [13], Yakubov [14], David [15], Iwaniec and Martin [16, §10], Gutlyanskii, Martio, Sugawa, and Vuorinen [17] and others. For the existence we employ the standard method of cutting the dilatation μ. However, the uniqueness is the essential feature of the paper because without uniqueness the mappings are difficult to handle.

2 Proof of Theorem 1.4

For $n = 1, 2, \ldots$ we set

$$\mu_n(z) = \begin{cases} \mu(z) \ \text{ if } \ |\mu(z)| \le 1 - \frac{1}{n}, \\[2ex] 0 \ \text{ if } \ |\mu(z)| > 1 - \frac{1}{n}. \end{cases}$$

There exists a mapping $w_n(z)$ from B onto B with dilatation μ_n [18, §1, Section V]. We may assume that $w_n(0) = 0$ for all $n = 1, 2, \ldots$. We call such a sequence w_n is a *canonical approximating sequence* for μ.

Because $w_n = (u_n, v_n) \in W^{1,2}(B)$ for every fixed $n = 1, 2, \ldots$ we can find a sequence

$$\{w_{nk}\} = \{(u_{nk}, v_{nk})\}_{k=1}^{\infty} \subset W^{1,2}(B) \cap C^{\infty}(B)$$

such that $w_{nk} \to w_n$ in $W^{1,2}(B)$ or in other words

$$\lim_{k\to\infty} \int_B \left(|\nabla(u_{nk} - u_n)|^2 + |\nabla(v_{nk} - v_n)|^2 \right) dxdy = 0, \tag{2.6}$$

and

$$\lim_{k\to\infty} \int_B \left(|u_{nk} - u_n|^2 + |v_{nk} - v_n|^2 \right) dxdy = 0. \tag{2.7}$$

We may also assume that $u_{nk} \to u_n$ and $v_{nk} \to v_n$ pointwise in B. This is needed later.

Fix n and k. Let ψ a nonnegative continuous function with a compact support $\operatorname{supp}\psi \subset B$. Then

$$\left| \int_B \psi\, J(z, w_n)\, dxdy - \int_B \psi\, J(z, w_{nk})\, dxdy \right| =$$

$$= \left| \int_B \psi\, (\partial_x u_n \partial_y v_n - \partial_y u_n \partial_x v_n)\, dxdy - \int_B \psi\, (\partial_x u_{nk} \partial_y v_{nk} - \partial_y u_{nk} \partial_x v_{nk})\, dxdy \right| \le$$

$$\le \int_B \psi\, |(\partial_x u_n - \partial_x u_{nk})\, \partial_y v_n - (\partial_y u_n - \partial_y u_{nk})\, \partial_x v_n|\, dxdy +$$

$$+ \int_B \psi\, |\partial_x u_{nk}\, (\partial_y v_n - \partial_y v_{nk}) - \partial_y u_{nk}\, (\partial_x v_n - \partial_x v_{nk})|\, dxdy\,.$$

Thus, from the inequality

$$|ab + cd| \le \sqrt{a^2 + c^2}\, \sqrt{b^2 + d^2}\,,$$

it follows that

$$\left| \int_B \psi\, J(z, w_n)\, dxdy - \int_B \psi\, J(z, w_{nk})\, dxdy \right| \le$$

$$\le \int_B \psi\, |\nabla(u_n - u_{nk})|\, |\nabla v_n|\, dxdy + \int_B \psi\, |\nabla u_{nk}|\, |\nabla(v_n - v_{nk})|\, dxdy\,.$$

Now from the Cauchy inequality we obtain

$$\left| \int_B \psi \, J(z, w_n) \, dxdy - \int_B \psi \, J(z, w_{nk}) \, dxdy \right| \le$$

$$\le \left(\int_B |\nabla (u_n - u_{nk})|^2 \, dxdy \right)^{1/2} \left(\int_B \psi^2 \, |\nabla v_n|^2 dxdy \right)^{1/2}$$

$$+ \left(\int_B |\nabla (v_n - v_{nk})|^2 \, dxdy \right)^{1/2} \left(\int_B \psi^2 \, |\nabla u_{nk}|^2 dxdy \right)^{1/2} .$$

Note that

$$\int_B \psi^2 \, |\nabla v_n|^2 dxdy \le M^2 \int_B |\nabla v_n|^2 \, dxdy$$

where $M = \max_B \psi$ and that

$$\int_B \psi^2 \, |\nabla u_{nk}|^2 dxdy \le 2M^2 \int_B |\nabla (u_{nk} - u_n)|^2 dxdy + 2M^2 \int_B |\nabla u_n|^2 dxdy \, .$$

Hence it follows from (2.6) that

$$\int_B \psi \, J(z, w_n) \, dxdy = \lim_{k \to \infty} \int_B \psi \, J(z, w_{nk}) \, dxdy \, . \tag{2.8}$$

Next we fix a sequence $\{K_l\}$ of $C^1(B)$ functions with

$$||K_l - K||_{W^{1,2}(B)} \to 0 \quad \text{as} \quad l \to \infty. \tag{2.9}$$

Let $U \subset\subset B$ be a domain with a rectifiable boundary ∂U. We choose a Lipschitz function $\phi : B \to \mathbf{R}$ with a compact support $\operatorname{supp} \phi \subset B$ such that $\phi = 1$ on U. Let V be a subdomain of B with a smooth boundary ∂V and such that

$$U \subset\subset \operatorname{supp} \phi \subset\subset V \subset\subset B.$$

We set $\psi = \phi^2 \, K_l$. Then ψ has a compact support in B. The function ψ belongs to the Lipschitz class, so that for $k, l = 1, 2, \ldots$ the Green formula and partial integration yield

$$\int\limits_V \psi\, J(z,w_{nk})\, dxdy = \int\limits_V \psi\, \left(\partial_x u_{nk}\partial_y v_{nk} - \partial_y u_{nk}\partial_x v_{nk}\right)\, dxdy =$$

$$= \int\limits_{\partial V} \psi\, u_{nk}\, \left(\partial_x v_{nk}\, dx + \partial_y v_{nk}\, dy\right) - \int\limits_V \psi\, u_{nk}\, \left(\partial_x v_{ky} - \partial_y v_{kx}\right)\, dxdy -$$

$$- \int\limits_V u_{nk}\, \left(\partial_x \psi \partial_y v_{nk} - \partial_y \psi \partial_x v_{nk}\right)\, dxdy = - \int\limits_B u_{nk}\, \left(\partial_x \psi \partial_y v_{nk} - \partial_y \psi \partial_x v_{nk}\right)\, dxdy =$$

$$= -2 \int\limits_B \phi\, u_{nk}\, K_l\, \left(\partial_x \phi \partial_y v_{nk} - \partial_y \phi \partial_x v_{nk}\right)\, dxdy -$$

$$- \int\limits_B u_{nk}\, \phi^2\, \left(K_{lx}\partial_y v_{nk} - K_{ly}\partial_x v_{nk}\right)\, dxdy\, .$$

This leads to the estimate

$$\begin{aligned} \int\limits_B \phi^2\, K_l\, J(z,w_{nk})\, dxdy \quad &\leq 2 \int\limits_B \phi\, |u_{nk}|\, |K_l|\, |\nabla \phi|\, |\nabla v_{nk}|\, dxdy + \\ &+ \int\limits_B |u_{nk}|\, \phi^2\, |\nabla K_l|\, |\nabla v_{nk}|\, dxdy\, . \end{aligned}$$

Letting $l \to \infty$ and using (2.9) for a fixed $k = 1, 2, \ldots$, we obtain

$$\begin{aligned} \int\limits_B \phi^2\, K\, J(z,w_{nk})\, dxdy \quad &\leq 2 \int\limits_B \phi\, |u_{nk}|\, |K|\, |\nabla \phi|\, |\nabla v_{nk}|\, dxdy + \\ &+ \int\limits_B |u_{nk}| \phi^2\, |\nabla K|\, |\nabla v_{nk}|\, dxdy\, . \end{aligned} \tag{2.10}$$

Next we employ the length and area principle (see, for example, Suvorov [19, §6, Chapter III, Part I]). Let z', $z'' \in U$ be a pair of points with

$$|z'' - z'| < \min\{1, \operatorname{dist}^2(U, \partial B)\}. \tag{2.11}$$

We consider the family of circles $\{S^1(z',r)\}$ with center at z' and radius r,

$$r' = |z'' - z'| < r < r'' = |z'' - z'|^{1/2}.$$

For $f_k = u_{nk}$ or $f_k = v_{nk}$, $k = 1, 2, \ldots$, we have the standard oscillation estimate

$$\int\limits_{r'}^{r''} \operatorname{osc}^2\left(f_k, S^1(z',r)\right) \frac{dr}{r} \leq c_1\, I(f_k)\, ,$$

where c_1 is a constant and

$$I(f_k) = \int\limits_B |\nabla f_k|^2 \, dxdy \, .$$

This yields

$$\inf_{r\in(r',r'')} \mathrm{osc}\,(f_k, S^1(z',r)) \leq c_1^{1/2}\, I^{1/2}(f_k) \log^{-1/2} \frac{1}{|z''-z'|^{1/2}} \, .$$

Now the mappings $w_{nk} = (u_{nk}, v_{nk})$ are homeomorphisms and thus each of functions u_{nk}, v_{nk} is monotone in the sense of Lebesgue. The condition (2.11) implies that the circles $S^1(z',r)$ lie inside B and hence,

$$\mathrm{osc}\,(f_k, B^2(z',r')) \leq \mathrm{osc}\,(f_k, S^1(z',r)) \, , \quad r \in (r', r'') \, .$$

From the previous inequalities it follows that

$$|f_k(z'') - f_k(z')| \leq c_1^{1/2}\, I^{1/2}(f_k) \log^{-1/2} \frac{1}{|z''-z'|^{1/2}} \, ,$$

and hence

$$|w_{nk}(z'') - w_{nk}(z')| \leq C(w_{nk}) \log^{-1/2} \frac{1}{|z''-z'|^{1/2}} \tag{2.12}$$

for any $z', z'' \in U$ satisfying (2.11), and

$$C(w_{nk}) = 2^{1/2}\, c_1^{1/2} \, \max\{I^{1/2}(u_{nk}), I^{1/2}(v_{nk})\} \, .$$

For fixed n the integrals $I(u_{nk}), I(v_{nk})$ are uniformly bounded by (2.6). Since the mappings w_{nk} are uniformly bounded as well, it follows from (2.12) that the sequence of mappings $\{w_{nk}\}$ is equicontinuous on U. Thus $w_{nk} \to w_n$ as $k \to \infty$ uniformly on each subdomain U of B compactly contained in B.

For $k \to \infty$ in (2.10) the Fatou lemma implies

$$\begin{aligned}
\int\limits_B \phi^2\, K\, J(z, w_n)\, dxdy \quad &\leq \liminf{}_{k\to\infty} \int\limits_B \phi^2\, K\, J(z, w_{nk})\, dxdy \leq \\
&\leq 2 \lim{}_{k\to\infty} \int\limits_B \phi\, |u_{nk}|\, K\, |\nabla\phi|\; |\nabla v_{nk}|\; dxdy + \\
&+ \lim{}_{k\to\infty} \int\limits_B |u_{nk}|\, \phi^2\; |\nabla K|\; |\nabla v_{nk}|\, dxdy
\end{aligned}$$

and now (2.8) together with (2.6) yields

$$\begin{aligned}
\int\limits_B \phi^2\, K\, J(z, w_n)\, dxdy \quad &\leq 2 \int\limits_B \phi\, |u_n|\, K\, |\nabla\phi|\, |\nabla v_n|\, dxdy + \\
&+ \int\limits_B |u_n|\, \phi^2\; |\nabla K|\; |\nabla v_n|\, dxdy \, .
\end{aligned} \tag{2.13}$$

Next we observe that $K_\mu \le K$ implies

$$|\nabla u_n|^2 + |\nabla v_n|^2 \le 2K_{\mu_n}\, J(z,w_n) \le 2K\, J(z,w_n)\,. \tag{2.14}$$

Combining (2.13) and the Cauchy inequality

$$|ab| \le \frac{\epsilon^2}{2} a^2 + \frac{1}{2\epsilon^2} b^2\,, \tag{2.15}$$

we have

$$\int\limits_B \phi^2 \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dxdy \le 2 \int\limits_B \phi^2 K\, J(z,w_n)\, dxdy \le$$

$$\le 4 \int\limits_B \phi\, |u_n|\, K\, |\nabla\phi|\, |\nabla v_n|\, dxdy\; + 2\int\limits_B |u_n|\, \phi^2\; |\nabla K|\; |\nabla v_n|\, dxdy \le$$

$$\le 3\epsilon^2 \int\limits_B \phi^2 |\nabla v_n|^2 dxdy + \frac{2}{\epsilon^2}\int\limits_B |u_n|^2\, K^2 |\nabla\phi|^2\, dxdy + \frac{1}{\epsilon^2}\int\limits_B |u_n|^2\, \phi^2 |\nabla K|^2\, dxdy.$$

We choose $\epsilon > 0$ such that $3\epsilon^2 < 1$, for example $\epsilon = 1/2$. Then,

$$\int\limits_B \phi^2 \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dxdy \le$$

$$\le c_2(\epsilon) \left(\int\limits_B |u_n|^2\, K^2 |\nabla\phi|^2\, dxdy + \int\limits_B |u_n|^2\, \phi^2 |\nabla K|^2\, dxdy \right),$$

where $c_2(\epsilon) = 2\,/(\epsilon^2(1-3\epsilon^2))$.

Since $\phi \equiv 1$ on U we have proved

2.16. Lemma. *Under the conditions of Theorem* 1.4, *for each subdomain* $U \subset\subset B$ *and each Lipschitz function* $\phi : B \to \mathbf{R}$, $\phi \equiv 1$ *on* $\overline{U}$ *and* $\operatorname{supp}\phi \subset B$, *we have*

$$\begin{aligned}\int\limits_U \left(|\nabla u_n|^2 \right. &+ \left. |\nabla v_n|^2\right) dxdy \le \\ &\le c_3 \int\limits_{\operatorname{supp}\phi} |u_n|^2 \left(K^2 + |\nabla K|^2\right) dxdy\,,\end{aligned} \tag{2.17}$$

where

$$c_3 = c_2\, \operatorname*{ess\,sup}_{z\in B} \max\{\phi^2(z), |\nabla\phi(z)|^2\}\,.$$

Next we consider the convergence of w_n as $n \to \infty$. From (2.17) and the inequality $|w_n| < 1$ it follows that

$$\int_U \left(|\nabla u_n|^2 + |\nabla v_n|^2\right)\, dxdy \le c_3 \int_B \left(K^2 + |\nabla K|^2\right)\, dxdy\,.$$

Similarly as in (2.12) for $z', z'' \in U$ with $|z'' - z'| < \min\{1, \operatorname{dist}^2(U, \partial B)\}$, we thus have

$$|w_n(z'') - w_n(z')| \le c_4 \log^{-1/2} \frac{1}{|z'' - z'|^{1/2}}, \tag{2.18}$$

where c_4 is independent from $n = 1, 2, \ldots$.

The estimate (2.18) means that the sequence $\{w_n\}_{n=1}^{\infty}$ is equicontinuous on every compact subset of B. By the standard diagonal process we can now choose a subsequence $\{w_{n'}\}$ which converges locally uniformly to some continuous mapping $w_0 : B \to \mathbf{R}^2$. We shall show that the *limit mapping w_0 is a homeomorphism.*

Let $\mu_{n'}(w)$ be the dilatation coefficient of the inverse mapping to $w = w_{n'}(z)$. Since $K_{\mu_{n'}(w)}(w_{n'}(z)) = K_{\mu_{n'}}(z)$ we see that

$$\int_B K_{\mu_{n'}(w)}(w)\, J(w, w_{n'}^{-1})\, dudv = \int_B K_{\mu_{n'}}(z)\, dxdy \le$$

$$\le \pi^{1/2} \left(\int_B K^2_{\mu_{n'}}(z)\, dxdy\right)^{1/2} \le \pi^{1/2} \left(\int_B K^2(z)\, dxdy\right)^{1/2} < \infty\,.$$

Now the quasiconformality of the inverse mapping $(x_{n'}, y_{n'})$ to $w_{n'}$ yields a.e. in B,

$$x^2_{n'u} + x^2_{n'v} + y^2_{n'u} + y^2_{n'v} \le 2K_{\mu_{n'}}(w)\, J(w, w_{n'}^{-1})\,,$$

and hence

$$\int_B \left(|\nabla x_{n'}|^2 + |\nabla y_{n'}|^2\right)\, dudv \le 2\pi^{1/2} \left(\int_B K^2(z)\, dxdy\right)^{1/2} < \infty\,. \tag{2.19}$$

As in (2.18) it follows from (2.19) that the family $\{w_{n'}^{-1}\}$ of inverse mappings is uniformly continuous on any compact subset of B and since $w_{n'}(0) = 0$ for each n', the limit mapping w_0 of $w_{n'}$ is a homeomorphism.

Sequences $\{w_{n'}\}$ and $\{w_{n'}^{-1}\}$ converge to w and w^{-1}, respectively. Using (2.17) and (2.19) we conclude that $w \in W^{1,2}_{\mathrm{loc}}(B)$ and $w^{-1} \in W^{1,2}(\mathcal{D})$, $\mathcal{D} = w(B)$.

Next we employ [17, Proposition 3.1]:

2.20. Lemma. *Let a measurable function $\mu : D \to \mathbf{R}^2$ satisfies $|\mu| < 1$ a.e. Suppose that a homeomorphism $w : D \to \mathbf{R}^2$ is a uniform limit of the canonical approximating sequence w_n for μ. If $w \in W^{1,2}_{\mathrm{loc}}(D)$, then w is a μ-homeomorphism a.e. in D.*

The domain $\mathcal{D} = w(B)$ is simply connected, contains the point $w = 0$ and is contained in B. If $\mathcal{D} \neq B$ then we use an auxiliary conformal mapping $f : \mathcal{D} \to B$, $f(0) = 0$, and consider the mapping $\tilde{w} = f \circ w$. Since f is conformal, f is locally bi-Lipschitz and thus $\tilde{w} \in W^{1,2}_{\text{loc}}(B)$. It easily follows that $\tilde{w}^{-1} \in W^{1,2}_{\text{loc}}(B)$.

It remainds to consider uniqueness. Assume that $f : B \to B$ and $g : B \to B$ are two μ-homeomorphisms. Because $f \in W^{1,2}_{\text{loc}}(B)$ and $g^{-1} \in W^{1,2}_{\text{loc}}(B)$ then the superpostion $f \circ g^{-1}$ belongs to $W^{1,1}_{\text{loc}}(B)$ and a.e. in B has the dilatation $\mu \equiv 0$. Therefore, the mapping $f \circ g^{-1}$ is conformal [16, Corollary **5.3.1**].

Finally, we prove that the condition $K_\mu(z) \leq K_0 + K(z)$ with $K \in \overset{\circ}{W}{}^{1,2}(B)$ implies that $w,\, w^{-1} \in W^{1,2}(B)$. For this observe that $K_{\mu_{\tilde{w}}} = K_\mu$, $\tilde{w} = f \circ w$, and hence we can consider the mapping w instead $\tilde{w}$. Now this and (2.14) yield

$$\int\limits_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dxdy \leq 2 \int\limits_B K_\mu\, J(z, w_n)\, dxdy \leq$$

$$\leq 2K_0 \int\limits_B J(z, w_n)\, dxdy + 2 \int\limits_B K\, J(z, w_n)\, dxdy.$$

The estimate (2.13) can now be used with $\phi \equiv 1$ because $K \in \overset{\circ}{W}{}^{1,2}(B)$. It follows that

$$\begin{aligned} \int\limits_B (|\nabla u_n|^2 + |\nabla v_n|^2)\, dxdy \quad &\leq 2K_0\pi + 2 \int\limits_B K\, J(z, w_n)\, dxdy \leq \\ &\leq 2K_0\pi + 2 \int\limits_B |u_n|\; |\nabla K|\; |\nabla v_n|\, dxdy \end{aligned}$$

and from $|u_n| < 1$ and from (2.15), we obtain

$$\begin{aligned} \int\limits_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dxdy \quad &\leq 2K_0\pi + \epsilon^2 \int\limits_B |\nabla K|^2\; dxdy + \\ &+ \tfrac{1}{\epsilon^2} \int\limits_B |\nabla v_n|^2\, dxdy\, . \end{aligned}$$

Choose ϵ such that $\frac{1}{\epsilon^2} < 1$. Now

$$\left(1 - \frac{1}{\epsilon^2}\right) \int\limits_B \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) dxdy \leq 2K_0\pi + \epsilon^2 \int\limits_B |\nabla K|^2\; dxdy$$

and thus the limit mapping w belongs to $W^{1,2}(B)$. Because $K_{\mu_{w^{-1}}}(w(z)) = K_\mu(z)$ it is easy to see that $w^{-1} \in W^{1,2}(B)$ as well. The theorem follows. $\Box$

3 Proof of Theorem 1.3

Let $\{D_n\}_{n=1}^{\infty}$ be an expanding sequence of subdomains of D which exhaust D and $\overline{D}_n$ is compactly contained in D, that is

$$D_1 \subset\subset D_2 \ldots \subset\subset D_n \ldots, \quad \text{and} \quad \cup_{n=1}^{\infty} D_n = D\,.$$

Set $R_n = \max_{z\in\overline{D}_n} |z|$ and define

$$\mu_n(z) = \begin{cases} \mu(z) \quad \text{for} \quad z \in D_n, \\ \\ 0 \quad \text{for} \quad z \in \mathbf{R}^2 \setminus D_n\,. \end{cases}$$

It is easy to see that Theorem 1.4 holds also for the dilatation μ given in any disk $|z| < R$. Fix a point $z_1 \in D_1$, $z_1 \neq 0$. The condition $K_\mu \leq K_U$ a.e. in $U \subset\subset D$ enables us to use Theorem 1.4. Then there exists a homeomorphism $w = w_n(z)$, $w_n(0) = 0$, from the disk $|z| < R_n$ onto some disk $|w| < r_n$ having the dilatation μ_n almost everywhere on $|z| < R_n$ and $w \in W^{1,2}_{\rm loc}(B(R_n))$, $w^{-1} \in W^{1,2}_{\rm loc}(B(r_n))$ Using an additional similarity of $|w| < r_n$ we choose r_n so that $w_n(z_1) = z_1$.

In each fixed domain D_{n_0} the mappings $w_n(z)$, $n > n_0$, can be represented in the form $w_n = \phi_n \circ w_{n_0}(z)$ where ϕ_n are conformal mappings in $\tilde{D}_{n_0} = w_{n_0}(D_{n_0})$. Because $\phi_n(0) = 0$ and $\phi_n(z_1) = z_1$, Theorem **4.1** [18] guarantees that there is a subsequence $\{\phi_{n'}\}$ locally uniformly convergent in $\tilde{D}_{n_0}$ to a conformal mapping $\phi : \tilde{D}_{n_0} \to \mathbf{R}^2$. Therefore, $\{w_n\}$ contains a subsequence $\{w_{n'}\}$ of quasiconformal mappings which converges locally uniformly on D_{n_0} to the homeomorphism $w_0 = \phi \circ w_{n_0}$. By Lemma 2.20 the mapping w_0 has the dilatation μ a.e. in D_{n_0}.

Using the well-known diagonal process we find a subsequence $\{w_{n''}\}$ uniformly inside D convergent to some homeomorphism $w : D \to \mathbf{R}^2$ with the dilatation μ. Clearly, the mapping w satisfies both conditions $w \in W^{1,2}_{\rm loc}(D)$ and $w^{-1} \in W^{1,2}_{\rm loc}(\mathcal{D})$. This mapping is unique in the indicated class up to conformal transformations. $\square$

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O. Martio:
Department of Mathematics
P.O.Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki,
FINLAND
Email: `martio@cc.helsinki.fi`

V.M. Miklyukov:
Department of Mathematics
VolgGU, 2Prodolnaya 30
Volgograd, 400062
RUSSIA
Email: `miklyuk@vlink.ru`