On an implicit function theorem

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2000 Mathematics Subject Classification: Primary 26B35, Secondary 26B10

Abstract. A nonsmooth variant of the implicit function theorem is proved.

We shall prove a nonsmooth variant of the well-known implicit function theorem for Lipschitz mappings. For other nonsmooth variants, see Pourciau [1], Warga [2], Cristea [3], Zhuravlev and Igumnov [4].

Let $m, n \geq 1$ be integers. Denote by M_n the linear space of the $n \times n$ matrices with real elements, by I_n the unit matrix of M_n. Let $B^n(x, r)$ be the ball in $\mathbf{R}^n$ with center at the point x and radius $r > 0$.

If $F(x, y)$ is any locally Lipschitz vector-function of variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$ and (x, y) is a point of differentiability of F, then let $F'(x, y)$ be its Jacobi matrix, $F'_x(x, y)$ be the Jacobi matrix with respect to x for any fixed y and $F'_y(x, y)$ be the Jacobi matrix with respect to y for any fixed x.

For an arbitrary matrix $C \in M_n$ we put

$$\|C\| = \sup_{|h|=1} |Ch|.$$

If $C(x) : D \subset \mathbf{R}^m \to M_n$ is a matrix function, then set

$$\|C\|_D = \operatorname{ess\,sup}_{x \in D} \|C(x)\|.$$

For $P \subset \mathbf{R}^m$ let $K : P \subset \mathbf{R}^m \to M_n$ be an arbitrary matrix function. We set

$$\operatorname{osc}(K, P) = \operatorname{ess\,sup}_{x,y \in P} |K(x) - K(y)|.$$

We shall prove the following result.

Theorem. *Let* $x_0 \in \mathbf{R}^n$, $y_0 \in \mathbf{R}^m$ *and* $a = (x_0, y_0) \in \mathbf{R}^{n+m}$. *Let* $D = B^n(x_0, r') \times B^m(y_0, r'')$ *be a domain and* $F : D \to \mathbf{R}^m$ *be a locally Lipschitz mapping. Suppose that*

$$\mu \equiv \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,(1 + \|F'\|_D) < 1. \tag{1}$$

Then there exist $\rho = \rho(\mu, r', r'') > 0$ *and a (unique) Lipschitz mapping*

$$G(x) : B^n(x_0, \rho) \to \mathbf{R}^m, \quad G(x_0) = y_0,$$

such that

$$F(x, G(x)) = F(x_0, y_0) \text{ for all } x \in B^n(x_0, \rho).$$

Moreover, we can put

$$\rho = \frac{r^*}{L}, \quad r^* = \min\{r', r''\}, \quad L = (1 + \|F'_x\|_D)/\,(1 - \mu),$$

and G *satisfies the Lipschitz condition with a constant*

$$\text{Lip }(G, B^n(x_0, \rho)) \le \sqrt{L^2 - 1}.$$

For the proof we need a simple indication of locally Lipschitz mappings to be one-to-one on convex regions.

Lemma. *Let* $D \subset \mathbf{R}^n$ *be a convex domain and let* $f : D \to \mathbf{R}^n$ *be a locally Lipschitz mapping. If*

$$\|f'(x) - I_n\|_D \equiv \Omega < 1, \tag{2}$$

then f *is homeomorphic in* D. *Moreover, for arbitrary points* $x', x'' \in D$ *we have*

$$(1 - \Omega)\,|x'' - x'| \le |f(x'') - f(x')| \le (1 + \Omega)\,|x'' - x'|. \tag{3}$$

Proof. Let E be the set of the points $x \in D$ in which f is differentiable. Since f is locally Lipschitz then by the Stepanoff Theorem we have $\mathcal{H}^n(D \setminus E) = 0$.

By (2) a.e. in D the estimate

$$|f'(x) - I_n| \leq \Omega < 1 \tag{4}$$

holds.

Fix arbitrary points $x', x'' \in D$ and denote by $l(x', x'')$ the line segment joining x' and x''. Since the region D is convex then $l(x', x'')$ lies inside D. Let $l(\tilde{x}', \tilde{x}'')$ be a line segment with endpoints $\tilde{x}'$ and $\tilde{x}''$ formed by a parallel translation of $l(x', x'')$. For almost all such segments sufficiently close to $l(x', x'')$ we have

$$\mathcal{H}^1(l(\tilde{x}', \tilde{x}'') \setminus E) = 0. \tag{5}$$

Let $l(\tilde{x}'_k, \tilde{x}''_k)$ be a sequence of segments having (5) and such that

$$\tilde{x}'_k \to x', \quad \tilde{x}''_k \to x''.$$

Because f is locally Lipschitz then f is absolutely continuous on $l(\tilde{x}'_k, \tilde{x}''_k)$ and a.e. along $l(\tilde{x}'_k, \tilde{x}''_k)$ the derivative f' exists. Integrating we find

$$|(f(x''_k) - x''_k) - (f(x'_k) - x'_k)| =$$

$$= \left| \int_0^1 (f'(x'_k + t(x''_k - x'_k)) - I)(x''_k - x'_k)\, dt \right| \leq$$

$$\leq \int_0^1 |f'(x'_k + t(x''_k - x'_k)) - I|\, |x''_k - x'_k|\, dt.$$

Then by (4) we obtain

$$|(f(x''_k) - x''_k) - (f(x'_k) - x'_k)| \leq \Omega\, |x''_k - x'_k|.$$

Setting $k \to \infty$ we arrive at the estimate

$$|(f(x'') - x'') - (f(x') - x')| \leq \Omega\, |x'' - x'| \qquad (x', x'' \in D). \tag{6}$$

Let $\phi(x) = f(x) - x$. For arbitrary points $x', x'' \in D$ we have

$$f(x'') - f(x') = (\phi(x'') - \phi(x')) - (x'' - x').$$

Thus,

$$|f(x'') - f(x')| \leq |\phi(x'') - \phi(x')| + |x'' - x'|.$$

Using (6) we can write

$$|f(x'') - f(x')| \le (1 + \Omega)\, |x'' - x'|.$$

Analogously,

$$|f(x'') - f(x')| \ge |x'' - x'| - |\phi(x'') - \phi(x')|$$

and next,

$$|f(x'') - f(x')| \ge (1 - \Omega)\, |x'' - x'|.$$

Thus (3) holds and Lemma is proved. □

Proof of Theorem. Consider the mapping $\Phi : D \to \mathbf{R}^n \times \mathbf{R}^m$ defined by

$$(x, y) \xrightarrow{\Phi} (X, Y) = (x_1, \ldots, x_n, F_1(x, y), \ldots, F_m(x, y)).$$

We need to prove that $\Phi(x, y)$ satisfies the assumptions of Lemma . The Jacobi matrix of Φ has the form

$$\Phi'(x, y) = \begin{pmatrix} I_n & Z^n_m \\ F'_x(x, y) & F'_y(x, y) \end{pmatrix},$$

where Z^n_m is the zero $n \times m$ matrix.

Consider the $(n + m) \times (n + m)$ matrix

$$Q(x, y) = \begin{pmatrix} I_n & Z^n_m \\ -F'_x(x, y) & I_m \end{pmatrix}.$$

It is easy to see that

$$|Q(x_1, y_1) - Q(x_2, y_2)|_D \le |F'_x(x_1, y_1) - F'_x(x_2, y_2)|_D \le \operatorname{osc}(F'_x, D).$$

We observe now that

$$\begin{aligned} Q(x, y)\Phi'(x, y) - I_{m+n} &= \begin{pmatrix} I_n & Z^n_m \\ Z^m_n & F'_y(x, y) \end{pmatrix} - I_{n+m} = \\ &= \begin{pmatrix} Z^n_n & Z^n_m \\ Z^m_n & F'_y(x, y) - I_m \end{pmatrix}. \end{aligned}$$

Hence,

$$\|Q(x,y)\Phi'(x,y) - I_{m+n}\|_D = \|F'_y(x,y) - I_m\|_D. \tag{7}$$

For an arbitrary fixed point $(x^*, y^*) \in D$ we define the map

$$\Psi(x,y) = Q(x^*, y^*)\Phi(x,y) : D \to \mathbf{R}^n \times \mathbf{R}^m. \tag{8}$$

Using (7) we find

$$||\Psi'(x,y) - I_{m+n}||_D = \|Q(x^*, y^*)\Phi'(x,y) - I_{n+m}\|$$

$$= \|Q(x,y)\Phi'(x,y) - I_{m+n} + (Q(x^*, y^*) - Q(x,y))\Phi'(x,y))\|_D$$

$$\leq \|Q(x,y)\Phi'(x,y) - I_{m+n}\|_D + \|Q(x^*, y^*) - Q(x,y)\|_D\|\Phi'(x,y)\|_D$$

$$\leq \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,\|\Phi'(x,y)\|_D.$$

Taking into consideration that

$$\begin{aligned}\|\Phi'\| &= \max_{|h|=1} |\Phi' h| \leq \\ &\leq \max_{|h|=1}(|h| + |(F'_x + F'_y)h|) \leq 1 + \|F'\|,\end{aligned}$$

we obtain

$$\|\Psi'(x,y) - I_{m+n}\|_D \leq \|F'_y - I_m\|_D + \operatorname{osc}(F'_x, D)\,(1 + \|F'\|_D).$$

Thus by (1) for the fixed point (x^*, y^*) we have

$$||\Psi'(x,y) - I_{m+n}||_D \leq \mu < 1. \tag{9}$$

The domain $D = B^n(x_0, r') \times B^m(y_0, r'')$ is convex and we may use Lemma . By the inequality (9) we conclude that the map

$$\Psi(x,y) = Q(x^*, y^*)\Phi(x,y)$$

is homeomorphic. By Horn, Johnson [5, Corollary **5.6.16**] from (9) it follows also that the matrix $\Psi'(x,y)$ is nondegenerate. Therefore from this relation it follows that matrices $\Phi'(x,y)$ and $Q \equiv Q(x^*,y^*)$ are nondegenerate. Thus the map $\Phi = Q^{-1}\Psi : D \to \mathbf{R}^{n+m}$ is also homeomorphic.

For the evaluation of $\rho = \rho(\mu, r', r'')$ we shall need some special information about Ψ. Using (9) and (3) we find

$$\begin{aligned}(1-\mu)\,|(x-x_0, y-y_0)| \ &\le |\Psi(x,y) - \Psi(x_0,y_0)| \le \\ &\le (1+\mu)\,|(x-x_0,y-y_0)|.\end{aligned}$$

Then by $\Psi = Q\Phi$ we may write

$$\begin{aligned}\frac{1-\mu}{|Q|}\,|(x-x_0,y-y_0)| \ &\le |\Phi(x,y) - \Phi(x_0,y_0)| \le \\ &\le (1+\mu)\,|Q^{-1}|\,|(x-x_0,y-y_0)|.\end{aligned} \tag{10}$$

However,

$$\begin{pmatrix} I_n & Z_m^n \\ F_x'(x_0,y_0) & I_m \end{pmatrix} = \begin{pmatrix} I_n & Z_m^n \\ Z_n^m & I_m \end{pmatrix} + \begin{pmatrix} Z_n^n & Z_m^n \\ F_x'(x_0,y_0) & Z_m^m \end{pmatrix}$$

and hence $|Q| \le 1 + \|F_x'\|_D$.

Since the ball $B^{n+m}(a, r^*)$ is contained in D from (10) we find

$$B' \equiv B^{n+m}\left(\Phi(a), r^*(1-\mu)/|Q|\right) \subset \Phi(B^{n+m}(a,r^*)). \tag{11}$$

Further,

$$B'' \equiv B^{n+m}(\Phi(a), r^*(1-\mu)/(1+\|F_x'\|_D)) \subset B' \subset \Phi\,(D)\,. \tag{12}$$

By (3) the mapping Ψ^{-1} satisfies the Lipschitz condition in $\Psi(D)$ with a constant

$$\mathrm{Lip}\left(\Psi^{-1}, \Psi(D)\right) \le \frac{1}{1-\mu}.$$

The mapping $\Phi(x,y)$ had been defined such that its inverse map has the form

$$x = X, \quad y = \Theta(X,Y).$$

Moreover, $\Phi^{-1} = \Psi^{-1}Q$ where Q is the nondegenerate linear transformation and Ψ^{-1} is the Lipschitz map. Hence, Φ^{-1} satisfies the Lipschitz condition in $\Phi(D)$ with a constant

$$\mathrm{Lip}\,\Big(\Phi^{-1}, \Phi(D)\Big) \le |Q|\,\mathrm{Lip}\,\Big(\Psi^{-1}, \Phi(D)\Big) \le L.$$

For the function $\Theta(X, Y)$ we have

$$\mathrm{Lip}\,(\Theta, \Phi(D)) \le \sqrt{L^2 - 1}.$$

Next we observe that

$$(X, Y) = \Phi(\Phi^{-1}(X, Y)) = (X, F(X, \Theta(X, Y))).$$

From this relation it follows that

$$F(X, \Theta(X, Y)) = Y. \tag{13}$$

From (11) it follows that B' lies in $\Phi(D)$. The intersection Π of the ball B' and the plane $Y_1 = F_1(x_0, y_0), \dots, Y_m = F_m(x_0, y_0)$ is a connected set with the codimension m containing the point $(X_0, Y_0) = (x_0, F(a))$. Denote by j the orthogonal projection from $\mathbf{R}^n \times \mathbf{R}^m$ onto $\mathbf{R}^n$. For any set $A \subset \mathbf{R}^n \times \mathbf{R}^m$ we have

$$j(A) = \{x \in \mathbf{R}^n : (x, y) \in A\}.$$

By the definition of Φ we may write

$$\begin{aligned} j(\Phi(A')) \quad &= \Phi(j(A')) \quad \forall\ A' \subset D, \\ j(\Phi^{-1}(A'')) \quad &= \Phi^{-1}(j(A'')) \quad \forall\ A'' \subset \Phi(D). \end{aligned} \tag{14}$$

The equation of the connected piece of the surface $\Phi^{-1}(\Pi)$ containing $a = (x_0, y_0)$ can be rewritten in the nonparametric form. Namely, let

$$(X, Y) = (x, \Theta(x, Y_0)), \quad x \in \Phi^{-1}(j(B')).$$

We put $G(x) = \Theta(x, Y_0)$.

By (13) we now find

$$F(x, G(x)) = Y_0 = F(x_0, y_0),$$

where

$$G(x_0) = \Theta(x_0, Y_0) = \Theta(X_0, Y_0) = y_0.$$

Uniqueness of the map G follows from the bijectivity of $\Phi(x, y)$. In fact, if $(x, y_1), (x, y_2) \in D$ and $F(x, y_1) = F(x, y_2)$ then $\Phi(x, y_1) = \Phi(x, y_2)$. Thus $y_1 = y_2$.

The relation (12) states that the ball B'' is contained in B' and guarantees (together with (14)) that the ball $B^n(x_0, \rho)$ lies in $\Phi^{-1}(j(B'))$. This implies the necessary bound for ρ. Theorem is proved completely. $\square$

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