ON PRICE EQUILIBRIA OF RANDOM ECONOMIES

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ABSTRACT. We consider an economic system with l commodities. It can be proved, that under reasonable assumptions, there exists an equilibrium price for a deterministic economy. The aim in this paper is to prove, that in a random economy, when the number of random economic agents grows, the set of the equilibrium prices becomes nonempty almost surely. That is, in a stochastic economic system, there exists also an equilibrium price (a.s.), when the number of agents is sufficiently large. Furthermore, we prove a large deviations upper bound result for the random equilibrium set.

1. INTRODUCTION

Let us consider an economic system with $l+1$ commodities and n economic agents. The price of the $l+1$th commodity can be normed to be one, that is $p^{l+1}=1$. It is so called numeraire and it can be thought as money. Let $Z_n^j(p)$ denote the total excess demand on the commodity j at the price $p \in \mathbb{R}^{l+1}$. We are interested in the prices $p^0 \in \mathbb{R}^{l+1}$, which satisfy condition $Z_n^j(p^0)=0$ for all $j=1, \ldots, l+1$. That is, we are interested in the prices, where there is no excess demand in any commodity. We call these prices the equilibrium prices of the economy. In the classical deterministic case, the existence of an equilibrium price was proved by Arrow and Debreu [AD] (for classical presentation, see [Deb]).

If an economic system with total excess demand satisfies Walras' law,

$$Z_n^{l+1}(p) = -\sum_{j=1}^{l} p^j Z_n^j(p)$$

then if $Z_n^j(p)=0$ for all commodities $j=1,\ldots,l$, also $Z_n^{l+1}(p)=0$. We suppose that our economic system satisfies Walras' law and therefore it is sufficient to consider an l dimensional space.

More realistic approach to an economic system is to allow some randomness to be involved. For example, we could assume that the total excess demand is composed of individual excess demands, which are

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random variables for each price p. Pioneers in this area are Hildenbrand [Hil] and Bhattacharya and Majumdar [BM], who studied exchange economies with random agents.

Recently, Nummelin has achieved results in random economies by using large deviations techniques. He has published some of his results for example in [Num1]. This work is based largely on that specific paper the aim being the improvment of some results presented there. We adopt mainly the same notations and therminology as used in [Num1].

Our goal is to prove four results concerning with random economics. First we give the basic set up for an economic system. After that we prove an existence result and a large deviation upper bound result in the case, where the random agents are i.i.d. Then we prove the same results without i.i.d. condition. In the last section we present a large deviations lower bound result and two corollaries of the existence of both large deviations bounds.

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2. Preliminaries

Consider an economic system with l commodities and n random agents. Each agent i in the economy has an excess demand function $\zeta_i(p) \in \mathbb{R}^l$. We assume that $\zeta_i(p)$ is a random variable on some underlying probability space $(\Omega, \mathcal{F}, P)$ for every $i = 1, \ldots, n$ and $p \in D$, where D is called the price domain of the economy. We assume throughout this paper that D is a compact and convex subset of $\mathbb{R}^l_+ \doteq \{(p^1, \ldots, p^l) \in \mathbb{R}^l \mid p^1 \geq 0, \ldots, p^l \geq 0\}$ with nonempty interior. The total excess demand is thus also a random variable and it is denoted by

$$Z_n(p) \doteq \sum_{i=1}^{n} \zeta_i(p).$$

Let us denote $\pi_n^0(\omega) = \{p \in \mathbb{R}^l \mid Z_n(p, \omega) = 0\}$. This means that $\pi_n^0(\omega)$ is the set of equilibrium prices for the realization $\omega \in \Omega$. We say, that π_n^0 is nonempty eventually (a.s.), if there is a finite (a.s.) map $m : \Omega \to \mathbb{N}$ such that $\pi_n^0(\omega) \neq \emptyset$ for all $n \geq m(\omega)$. One of the aims in this paper is to prove, that π_n^0 is nonempty eventually (a.s.) under certain conditions.

Let us denote by $c_n : \mathbb{R}^l \times D \to \mathbb{R}$ the logarithm of the Laplace transform of $Z_n(p)$,

$$c_n(\beta, p) \doteq \log E e^{\beta \cdot Z_n(p)}.$$

The so called free energy function associated with the sequence $(Z_n(p))$ is defined as

$$c(\beta, p) \doteq \limsup_{n\to\infty} n^{-1} c_n(\beta, p).$$

We shall assume in this paper that c is finite and twice continuously differentiable, i.e. a C^2 map $c : \mathbb{R}^l \times D \to \mathbb{R}$. Note that, because $c(\beta, p)$ is a convex function of β for each fixed p, the derivative $(\partial^2 c/\partial\beta^2)(\beta, p)$ is positive definite. It is well known from the theory of large deviations, that there exists the limit

$$z(p) \doteq \lim_{n\to\infty} n^{-1} Z_n(p) = \frac{\partial c}{\partial \beta}(0, p)$$

for all $p \in D$. We call the vector field $z : D \to \mathbb{R}^l$ the expectation economy and the set of equilibrium prices in this economy is denoted by

$$\pi^0 \doteq \{p \in D \mid z(p) = 0\}.$$

For general information about large deviation theory see e.g. [DZ].

3. Existence of an equilibria: an i.i.d. case

We assume in this section, that the random excess demand functions $\zeta_i(p)$ of the agents are i.i.d. for every $p \in D$. We will prove, that under certain conditions, the economy has eventually at least one equilibrium in this case. To prove that we need two lemmas. But first, let us make some simple observations.

If the realization $\zeta_i(p, \omega)$ is continuous on ∂D, then it is uniformly continuous. Therefore, there exists a function $\delta_i(h, \omega)$, $h \geq 0$, such that

$$|\zeta_i(p, \omega) - \zeta_i(q, \omega)| \leq \delta_i(|p - q|, \omega)$$

when $p, q \in \partial D$. We shall refer to the random function $\delta_i(h)$ as the random modulus of continuity (r.m.c.) for $\zeta_i(p)$. Notice that $\delta_i(h)$ is in fact a random variable (viz. measurable), because it can be defined as a supremum of $|\zeta_i(p) - \zeta_i(q)|$ over a countable dense subset of ∂D.

Let all the realizations $\zeta_1(p, \omega)$, ..., $\zeta_n(p, \omega)$ be continuous (a.s.) on ∂D. Now

$$|n^{-1} Z_n(p, \omega) - n^{-1} Z_n(q, \omega)| \leq n^{-1} \sum_{i=1}^{n} \delta_i(|p - q|, \omega) \doteq n^{-1} \tilde{\delta}_n(|p - q|, \omega),$$

so that $n^{-1}\tilde{\delta}_n(h)$ is the r.m.c. for $n^{-1} Z_n(p)$. It follows that the free energy function $c_{\tilde{\delta}}(t, h)$ associated with the sequence $(\tilde{\delta}_n(h))$ is just cumulant generated function: $c_{\tilde{\delta}}(t,h) = c_{\delta_1(h)}(t) \doteq \log Ee^{t\delta_1(h)}$. The function $c_{\delta_1(h)}(t)$ has all derivatives in $\mathring{\mathcal{D}}(c_{\delta_1(h)})$, where

$$\mathcal{D}(c_{\delta_1(h)}) \doteq \{t \in \mathbb{R}^l \mid c_{\delta_1(h)}(t) < \infty\}$$

and $\mathring{\mathcal{D}}(c_{\delta_1(h)})$ is its interior.

Let us denote by $\mathcal{N}(p)$ the set of outward unit normals at the point p, where p is on the boundary of the price domain D. Now, the following lemma gives us a useful result when proving the existence of an equilibrium.

Lemma 3.1. *Suppose that D is a compact convex subset of $\mathbb{R}^l$ having nonempty interior $\mathring{D}$. If $z : D \to \mathbb{R}^l$ is a continuous map and the following boundary condition,*

$$z(p) \cdot u < |z(p)| \quad \text{for all } p \in \partial D, u \in \mathcal{N}(p), \tag{1}$$

is satisfied, then there is $p^0 \in D$ such that $z(p^0) = 0$.

The boundary condition means geometrically, that $z(p)$ does not point into same direction with any of the normals $u \in \mathcal{N}(p)$. For the proof of this lemma, see [Sma, page 337] and also [Num1, page 271].

Another lemma needed is a basic result of large deviations theory (see e.g. [DZ, Exercise 2.3.25]).

Lemma 3.2. *Let (S_n) be a sequence of $\mathbb{R}^l$-valued random variables with free energy function*

$$c(\beta) \doteq \limsup_{n\to\infty} n^{-1} \log Ee^{\beta\cdot S_n}.$$

If the derivative $z \doteq c'(0) \in \mathbb{R}^l$ exists, then for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$P\{|n^{-1}S_n - z| > \varepsilon\} \leq e^{-n\eta(\varepsilon)} \text{ eventually.}$$

We are now ready to prove our first theorem. It states, that under certain assumptions, the random economic system comprising i.i.d. agents has an equilibrium eventually (a.s.).

Theorem 3.3. *Suppose that*

(A1) *$z(p)$ is a continuous map satisfying the boundary condition (1) of Lemma 3.1,*
(A2) *the realizations $\zeta_i(\cdot,\omega) : D \to \mathbb{R}^l$ of the individual excess demand maps are continuous almost surely and*
(A3) *the Laplace transform $Ee^{t_0\delta_1(h_0)}$ is finite for some $t_0 > 0$ and $h_0 > 0$.*

Then there is a constant $\theta > 0$ such that $P\{\pi_n^0 = \emptyset\} \le e^{-n\theta}$ eventually. Consequently, $\pi^0 \neq \emptyset$ eventually (a.s.).

Proof. Let $p \in \partial D$ and $\varepsilon > 0$. It follows directly from assumption (A3) that $0 \in \mathring{\mathcal{D}}(c_{\delta_1(h)})$ for all $h \le h_0$. Now $(\partial c_{\tilde{\delta}}/\partial t)(0,h) = E\delta_1(h) \doteq d(h)$ and Lemma 3.2 holds for the sequence $\tilde{\delta}_n(h)$. Thus we have $\beta(\varepsilon, h) > 0$ such that $P\{\omega \,|\, |n^{-1}\tilde{\delta}_n(h,\omega) - d(h)| > \varepsilon\} \le e^{-n\beta(\varepsilon,h)}$. By the assumption (A3) $E\delta_1(h_0) = d(h_0) < \infty$. Furthermore $\delta_1(h) \le \delta_1(h_0)$ for every $h \le h_0$ and we get $\lim_{h\to 0+} E\delta_1(h) = E(\lim_{h\to 0+}\delta_1(h)) = E0 = 0$ by dominated convergence. Thus $d(h)$ is right continuous at zero and we can choose h small enough to satisfy $d(h) < \varepsilon/3$.

By the compactness of ∂D there is a finite subset $\{q_i, \ldots, q_K\} \subset \partial D$ such that for each $p \in \partial D$ there is some q_i such that $|p - q_i| < h$ with given $h > 0$ and also $|z(p) - z(q_i)| < \varepsilon/3$. Now

$$|n^{-1}Z_n(p,\omega) - n^{-1}Z_n(q_i,\omega)| \le n^{-1}\tilde{\delta}_n(h,\omega).$$

Consequently

$$\begin{aligned}
& P\{\omega \mid \sup_{p\in\partial D} |n^{-1}Z_n(p,\omega) - z(p)| > \varepsilon\} \\
= \ & P\{\omega \mid \exists p \in \partial D \text{ such that } |n^{-1}Z_n(p,\omega) - z(p)| > \varepsilon\} \\
\le \ & P\{\omega \mid \exists q_i \in \partial D \text{ such that } |n^{-1}Z_n(q_i,\omega) - z(q_i)| > \\
& \varepsilon - n^{-1}\tilde{\delta}_n(h,\omega) - \frac{\varepsilon}{3}\} \\
\le \ & \sum_{i=1}^{K} P\{\omega \mid |n^{-1}Z_n(q_i,\omega) - z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3}\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \{\omega \mid |n^{-1}Z_n(q_i,\omega) - z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3}\} \\
= \ & \{\omega \mid |n^{-1}Z_n(q_i,\omega) - z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3} \\
& \quad \text{and } |n^{-1}\tilde{\delta}_n(h,\omega) - d(h)| \le \frac{\varepsilon}{6}\} \\
\cup \ & \{\omega \mid |n^{-1}Z_n(q_i,\omega) - z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3} \\
& \quad \text{and } |n^{-1}\tilde{\delta}_n(h,\omega) - d(h)| > \frac{\varepsilon}{6}\}
\end{aligned}$$

for any i. Suppose that ω satisfies conditions $|n^{-1}Z_n(q_i,\omega) - z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > 2\varepsilon/3$ and $|n^{-1}\tilde{\delta}_n(h,\omega) - d(h)| \le \varepsilon/6$. Because $d(h) < \varepsilon/3$, the latter one gives us $n^{-1}\tilde{\delta}_n(h,\omega) < \varepsilon/2$. Now we use the first condition and obtain

$$|n^{-1}Z_n(q_i,\omega)-z(q_i)| > \frac{2\varepsilon}{3} - n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3} - \frac{\varepsilon}{2} = \frac{\varepsilon}{6}\,.$$

Thus it follows that

$$\begin{aligned}
& P\{\omega \mid |n^{-1}Z_n(q_i,\omega)-z(q_i)| + n^{-1}\tilde{\delta}_n(h,\omega) > \frac{2\varepsilon}{3}\} \\
\leq \;& P\{\omega \mid |n^{-1}Z_n(q_i,\omega)-z(q_i)| > \frac{\varepsilon}{6}\} \\
+ \;& P\{\omega \mid |n^{-1}\tilde{\delta}_n(h,\omega)-d(h)| > \frac{\varepsilon}{6}\} \\
\leq \;& e^{-n\gamma(\varepsilon/6,q_i)} + e^{-n\beta(\varepsilon/6,h)}
\end{aligned}$$

eventually with rates $\gamma(\varepsilon/6,q_i), \beta(\varepsilon/6,h) > 0$. In conclusion

$$P\{\omega \mid \sup_{p\in\partial D} |n^{-1}Z_n(p,\omega)-z(p)| > \varepsilon\} \leq e^{-n\eta(\varepsilon)} \tag{2}$$

with a positive rate $\eta(\varepsilon)$ eventually.

We will now complete the proof by using Lemma 3.1. The proof follows the proof of [Num1, Theorem 1], but we will include some details for the convenience of the reader.

From the boundary condition (1) and compactness, it follows, that there is $\alpha > 0$ such that $|z(p)|-z(p)\cdot u \geq \alpha$ for all $p \in \partial D$, $u \in \mathcal{N}(p)$. If $|Z_n(p)-z(p)| \leq \alpha/3$ for all $p \in \partial D$, we have $n^{-1}Z_n(p)\cdot u \leq z(p)\cdot u+\alpha/3$ for all $p \in \partial D$, $u \in \mathcal{N}(p)$. By triangle inequality also $|z(p)| \leq \alpha/3 + |n^{-1}Z_n(p)|$. It follows that

$$n^{-1}Z_n(p)\cdot u + \frac{2\alpha}{3} \leq z(p)\cdot u + \alpha \leq |z(p)| \leq \frac{\alpha}{3} + |n^{-1}Z_n(p)|$$

and thus

$$|n^{-1}Z_n(p)| - n^{-1}Z_n(p)\cdot u \geq \frac{\alpha}{3}$$

for all $p \in \partial D$, $u \in \mathcal{N}(p)$. Now we apply Lemma 3.1 to the random vector field $n^{-1}Z_n(\cdot)$ and obtain

$$\begin{aligned}
P\{\pi_n^0 \neq \emptyset\} \;&\geq\; P\{\inf_{p\in\partial D, u\in\mathcal{N}(p)} \{|n^{-1}Z_n(p)| - n^{-1}Z_n(p)\cdot u\} > 0\} \\
&\geq\; P\{\sup_{p\in\partial D} |n^{-1}Z_n(p)-z(p)| \leq \frac{\alpha}{3}\} \\
&\geq\; 1-e^{-n\eta(\alpha/3)} \qquad \text{(by equation (2))}.
\end{aligned}$$

Thus $P\{\pi_n^0 = \emptyset\} \leq e^{-n\theta}$ eventually with some $\theta > 0$. It follows now directly from Borel-Cantelli lemma, that π_n^0 is eventually nonempty (a.s.). This completes the proof. □

Remark 3.4. Note, that we have not in fact assumed measurability of the set $\{\pi_n^0 = \emptyset\}$. The result ought to interpreted as follows: there is a measurable set A_n such that $\{\pi_n^0 = \emptyset\} \subset A_n$ and $P(A_n) \leq e^{-n\theta}$ eventually.

4. Convergence of the random equilibria: an i.i.d. case

Now when we have proved the existence of an equilibrium on i.i.d. assumption, we start to study the rate of convergence towards equilibrium in this case. It turns out that if free energy functions c and $c_{\tilde{\delta}}$ satisfy some assumptions, then there exists a large deviations rate function for the economic system.

If $c(\cdot, p)$ is a free energy function, its convex conjugate function $\hat{c}$: $\mathbb{R}^l \times D \to \mathbb{R}_+$ is defined as Legendre transform:

$$\hat{c}(v, \beta) \doteq \sup_{\beta \in \mathbb{R}^l} \{\beta \cdot v - c(\beta, p)\}.$$

In the special case when $v = 0$, we denote

$$I(p) \doteq \hat{c}(0, p) = - \inf_{\beta \in \mathbb{R}^l} c(\beta, p).$$

Moreover, if E is a subset of R^l we denote

$$I(E) = \inf_{p \in E} I(p).$$

We assume in this section that free energy function c satisfies the following assumption:

(A4) the (positive definite) derivative $(\partial^2 c/\partial \beta^2)(\beta, p)$ is strictly positive definite for all β and p.

From the general properties of convex conjugates (see [Roc, Theorem 26.5]) it follows, that under this hypothesis, the range of the derivative map

$$\mathcal{R}(p, (\partial c/\partial \beta)) \doteq \left\{ \frac{\partial c}{\partial \beta}(\beta, p) \mid \beta \in \mathbb{R} \right\}.$$

is an open set in R^l and it equals the interior of the domain of the convex conjucate map $\hat{c}(v, p)$;

$$\mathcal{R}(p, (\partial c/\partial \beta)) = \mathring{\mathcal{D}}(\hat{c}(\cdot, p)),$$

where

$$\mathcal{D}(\hat{c}(\cdot, p)) = \{v \in R^l \mid \hat{c}(v, p) < \infty\}.$$

Notice that if $0 \in \mathcal{R}(p)$, then there is unique $\beta(p)$ such that $I(p) = -c(\beta(p), p)$.

Lemma 4.1. *If, for the free energy function c, the assumption*

(A5) $0 \in \mathcal{R}(p)$ *for all* $p \in D$

holds, then I *is a* C^2*-map on* D*. Furthermore the equilibrium set* π^0 *is compact set* $\{p \in D \mid I(p) = 0\}$ *and* $\lim_{|v|\to 0} \hat{c}(v,p) = I(p)$ *iniformly over* $p \in D$*.*

For the proof, see [Num1].

We are now ready to prove the large deviation upper bound result. Just one extra assumption is still needed. To shorten notations we denote by $\partial_1 f$ the first derivative of a function f with respect to the first argument and by $\partial_1^2 f$ the second derivative taken twice with respect to the first argument.

(A6) $c_{\tilde{\delta}}$ is twice differentiable with respect to t for every $h \le h_0$ for some $h_0 > 0$ and $\sup_{t>0} \partial_1^2 c_{\tilde{\delta}}(t,h) \to 0$ when $h \to 0$.

Theorem 4.2. *Suppose that conditions (A2)-(A6) hold true. Then*

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap \{I \ge \alpha\} \ne \emptyset\} \le -\alpha \text{ for all } \alpha > 0.$$

Proof. Fix $\alpha > 0$ and $\varepsilon > 0$. Let us choose $h > 0$ small enough to satisfy $d(h) < \varepsilon/3$ and $\sup_{t>0} \partial_1^2 c_{\tilde{\delta}}(t,h) \le \varepsilon^2/(18(\alpha+1))$. Let $\{q_1, \dots, q_K\} \subset \{I \ge \alpha\} \cap D$ be a finite set such that for each $p \in \{I \ge \alpha\} \cap D$ there is q_i satisfying $|p - q_i| < h$. Now we can estimate as follows:

$$\begin{aligned}
P\{\pi_n^0 \cap \{I \ge \alpha\} \ne \emptyset\} &= P\{Z_n(p) = 0 \text{ for some } p \in \{I \ge \alpha\}\} \\
&\le P\{|n^{-1} Z_n(q_i)| \le n^{-1}\tilde{\delta}_n(h) \text{ for some } q_i\} \\
&\le P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
&\quad + P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\} \\
&\le K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
&\quad + P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap \{I \ge \alpha\} \ne \emptyset\} \\
\le\ &\max\{\limsup_{n\to\infty} n^{-1} \log K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\}, \\
&\limsup_{n\to\infty} n^{-1} \log P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \ge \varepsilon/3\}\}.
\end{aligned}$$

By Taylor expansion

$$\hat{c}_{\tilde{\delta}}(v,h) = \hat{c}_{\tilde{\delta}}(d(h),h) + \partial_1 \hat{c}_{\tilde{\delta}}(d(h),h)(v - d(h)) + \frac{1}{2}\partial_1^2 \hat{c}_{\tilde{\delta}}(\xi(h),h)(v - d(h))^2,$$

where $\xi(h)$ is between $d(h)$ and v (the derivatives exist by the implicit function theorem). The first two terms in this sum are zeroes. By using the fact that

$$\partial_1^2 \hat{c}_{\tilde{\delta}}(\xi(h), h) = \frac{1}{\partial_1^2 c_{\tilde{\delta}}(t(h), h)}$$

(see e.g [Hol, Exercise I.16]), where $t(h)$ is defined through $\partial_1 c_{\tilde{\delta}}(t(h), h) = \xi(h)$, we get

$$\hat{c}_{\tilde{\delta}}(v, h) = \frac{(v - d(h))^2}{2\partial_1^2 c_{\tilde{\delta}}(t(h), h)}.$$

Now

$$\inf_{|v-d(h)|\geq \varepsilon/3} \hat{c}_{\tilde{\delta}}(v, h) = \inf_{|v-d(h)|\geq \varepsilon/3} \frac{(v - d(h))^2}{2\partial_1^2 c_{\tilde{\delta}}(t(h), h)} = \frac{\varepsilon^2}{18\partial_1^2 c_{\tilde{\delta}}(t(h), h)} \geq \alpha+1$$

and thus

$$\limsup_{n\to\infty} n^{-1} \log P\{|n^{-1}\tilde{\delta}_n(h) - d(h)| \geq \varepsilon/3\} \leq -\alpha - 1.$$

On the other hand

$$\begin{aligned}
& \limsup_{n\to\infty} n^{-1} \log K \max_i P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
= \; & \max_i \limsup_{n\to\infty} n^{-1} \log P\{|n^{-1} Z_n(q_i)| < 2\varepsilon/3\} \\
\leq \; & -\min_i \inf_{|v|\leq 2\varepsilon/3} \hat{c}(v, q_i) \\
\leq \; & -\min_i I(q_i) + \eta(\varepsilon) \text{ [by Lemma 4.1]} \\
\leq \; & -\alpha + \eta(\varepsilon).
\end{aligned}$$

In conclusion,

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha + \eta(\varepsilon)$$

when ε is small enough and the assertion follows after letting $\varepsilon \to 0$. □

Note that if

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha$$

for all $\alpha > 0$, then

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap E \neq \emptyset\} \leq -I(E)$$

for all closed sets $E \subset \mathbb{R}^l$ such that $E \cap D \neq \emptyset$ by choosing $\alpha = I(E)$ (see [Num2, page 242]).

5. The results without i.i.d. assumption

It is, of course, not very realistic to assume independence and identical distributions for excess demand functions of the random agents. In this section we abandon the i.i.d. assumption. Instead of that, we assume more on the free energy function $c_{\tilde{\delta}}$ in order to prove the generalized results.

Let us look first at the existence of equilibrium in the general case. In section 3 we defined the function $\tilde{\delta}(h)$. We can do the same now, but in this case it is not a sum of i.i.d. random variables. In proving Theorem 3.3 we needed the i.i.d. assumption just in the beginning of the proof. Now we make an additional assumption and change the beginning of the proof accordingly.

Theorem 5.1. *Suppose that the assumptions (A1)-(A2) hold. If*

(A7) *the function $d(h) \doteq \partial_1 c_{\tilde{\delta}}(0, h)$ is defined for $h \leq h_0$ for some $h_0 > 0$ and it is right continuous at 0,*

then there is a constant $\theta > 0$ such that $P\{\pi_n^0 = \emptyset\} \leq e^{-n\theta}$ eventually. Consequently, $\pi^0 \neq \emptyset$ eventually (a.s.).

Proof. Let $\varepsilon > 0$. The assumption (A7) ensures, that we can use Lemma 3.2 for all $h \leq h_0$. Thus, due to the continuity from the right of $d(h)$ at zero, we can choose h small enough to satisfy $d(h) < \varepsilon/3$ ($d(0) = 0$). From now on the proof is identical to the proof of Theorem 3.3. □

In the case of Theorem 4.2 we have the same kind of situation. The i.i.d. assumption is used only in the beginning of the proof to ensure, that we can choose h small enough to satisfy $d(h) < \varepsilon/3$. We can use the same argument as in previous theorem.

Theorem 5.2. *Suppose that conditions (A2),(A4)-(A7) hold true. Then*

$$\limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap \{I \geq \alpha\} \neq \emptyset\} \leq -\alpha \text{ for all } \alpha > 0.$$

Proof. Fix $\varepsilon > 0$. Because $c_{\tilde{\delta}}(t, 0) \equiv 0$ it follows that $d(0) = 0$. Now we can choose $h > 0$ small enough to satisfy $d(h) < \varepsilon/3$. The rest of the proof is identical to the proof of Theorem 4.2. □

6. Lower bound and related results

In Theorems 4.2 and 5.2 we proved the large devitions upper bound result for random equilibria. We give now the corresponding lower bound result presented in [Num2, page 246].

Theorem 6.1. *Suppose that for each $p \in D$ the derivatives $n^{-1}Z_n'(p)$, $n \in \mathbb{N}$, exist and are uniformly non-singular almost surely; that means, there is a constans $M(p) < \infty$ such that*

(A8) $\quad |nZ_n'(p)^{-1}| \leq M(p) \quad$ *for all $n \in \mathbb{N}$, a.s.*

Moreover, suppose that the sequence $n^{-1}Z_n'(p)$ is uniformly equicontinous (a.s.):

(A9) $\quad |n^{-1}Z_n'(p) - n^{-1}Z_n'(q)| \leq \varepsilon(|p-q|) \quad$ *for all n, p and q a.s.,*

where $\lim_{h\to 0} \varepsilon(h) = 0$. Then

$$\liminf_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap G \neq \emptyset\} \geq -I(G)$$

for all relatively open (w.r.t. D) sets $G \in \mathbb{R}^l$.

Because $I(\overline{B}) = I(B) = I(\mathring{B})$ for a convex subset B of R^l satisfying $\mathring{B} \cap D \neq \emptyset$ (see [Num2, Theorem 3]), we have the following result ([Num2, Corollary 4.1]), which follows from Theorems 5.2 and 6.1.

Corollary 6.2. *Suppose that $B \subset \mathbb{R}^l$ is a convex set such that $\mathring{B} \cap D \neq \emptyset$. If conditions (A1)-(A2), (A4)-(A9) hold, then there exists a large deviation limit*

$$\lim_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap B \neq \emptyset\} = -I(B).$$

Let us denote the entropy minimizing set of a convex set $B \subset \mathbb{R}^l$ by π_B^0, that is

$$\pi_B^0 \doteq \{q \in \overline{B} \mid I(q) = I(B)\}.$$

We present now the conditional law of large numbers for the random equilibria π_n^0 (see [Num2, Corollary 4.2]). The proof is included to fill in some details.

Corollary 6.3. *Let $B \subset \mathbb{R}^l$ be a convex set such that $\mathring{B} \cap D \neq \emptyset$. Suppose that the conditions (A1)-(A2), (A4)-(A9) hold true. Then*

$$\lim_{n\to\infty} P\{\pi_n^0 \cap B \subset U(\pi_B^0, \varepsilon) \mid \pi_n^0 \cap B \neq \emptyset\} = 1$$

for all $\varepsilon > 0$.

Proof. By Corollary 6.2 there exists the large deviation limit

$$\lim_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap B \neq \emptyset\} = -I(B)$$

Let $B_\varepsilon \doteq \overline{B} \cap U(\pi_B^0, \varepsilon)^C$. Due to the continuity of I, we have

$$\begin{aligned} & \limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap B \cap U(\pi_B^0, \varepsilon)^C \neq \emptyset \mid \pi_n^0 \cap B \neq \emptyset\} \\ \leq \; & \limsup_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap B_\varepsilon \neq \emptyset\} - \lim_{n\to\infty} n^{-1} \log P\{\pi_n^0 \cap B \neq \emptyset\} \\ \leq \; & -I(B_\varepsilon) + I(B) \doteq -\beta < 0. \end{aligned}$$

Now, we can choose such $h > 0$, that $0 \leq P\{\pi_n^0 \cap B \cap U(\pi_B^0, \varepsilon)^C \neq \emptyset \mid \pi_n^0 \cap B \neq \emptyset\} \leq e^{n(-\beta+h)}$ eventually and $-\beta + h < 0$. Thus

$$\lim_{n\to\infty} P\{\pi_n^0 \cap B \subset U(\pi_B^0, \varepsilon) \mid \pi_n^0 \cap B \neq \emptyset\} = 1.$$

□

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