SOBOLEV CAPACITY ON THE SPACE $W^{1,p(\cdot)}(\mathbb{R}^n)$

PETTERI HARJULEHTO, PETER HÄSTÖ,
MIKA KOSKENOJA, AND SUSANNA VARONEN

A . We define Sobolev capacity on the generalized Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^n)$. It is a Choquet capacity provided that the variable exponent $p : \mathbb{R}^n \to [1, \infty)$ is bounded away from 1 and ∞. We discuss the relation between the Hausdorff dimension and the Sobolev capacity. As another application we study quasicontinuous representatives in the space $W^{1,p(\cdot)}(\mathbb{R}^n)$.

1. I

In the beginning of the last decade Kováčik and Rákosník introduced variable exponent Lebesgue and Sobolev spaces as a new method for dealing with non-linear Dirichlet boundary value problems with nonstandard growth and coercivity assumption, see [KR] for details. Another area where these spaces have found applications is the study of electrorheological fluids, see the papers by Diening alone [Die1] and with Růžička [DR] on the role of variable exponent in this context. The same spaces appear also in the study of variational integrals with non-standard growth, see [Zhi], [Mar] and [AM].

In fact, generalized Lebesgue and Sobolev spaces are special cases of so-called Orlicz-Musielak spaces, and in this form their investigation goes back a bit further, to Hudzik [Hud] and Musielak [Mus]. It seems, however, that there is good reason to study the particular spaces introduced by Kováčik and Rákosník [KR], and in recent years several papers have appeared along this line of investigation; it is now known that generalized Lebesgue and Sobolev spaces satisfy several of the properties of their classical equivalents. Let us mention some recent advances. Edmunds and Rákosník [ER2] showed that the Sobolev embedding theorem holds provided that the variable exponent is Lipschitz continuous; Pick and Růžička [PR] showed that in some cases the Hardy-Littlewood maximal operator is not bounded whereas Diening [Die2] showed that in other cases it is (unfortunately, there is a gap between the necessary and sufficient conditions here).

Sobolev capacity for fixed exponent spaces has found a great number of uses, see for instance the monographs by Maz'ya [Maz], Evans and Gariepy [EG], and Heinonen, Kilpeläinen, and Martio [HKM]. Nevertheless, this tool has not previously been considered in connection with variable exponent Sobolev spaces. Our purpose, then, is to generalize the Sobolev capacity to the variable exponent case, and, more importantly, show that this generalization makes sense, at least when $1 < \operatorname{ess\,inf} p \leqslant \operatorname{ess\,sup} p < \infty$: in Corollary 3.3 we show that the Sobolev $p(\cdot)$-capacity is an outer measure and in Corollary 3.4 we show that it is a Choquet capacity. We also derive several results that illustrate the utility of the Sobolev capacity. In Section 4 we show that the capacity is

2000 *Mathematics Subject Classification.* 46E35, 31B15.

Key words and phrases. Sobolev capacity, variable exponent, generalized Sobolev space, Hausdorff dimension, quasicontinuous representative.

related to the Hausdorff dimension of a set. In Section 5 we show that every Sobolev function has a quasicontinuous representative in the variable exponent case provided that $C^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. This allows us to derive an estimate of the capacity of level sets of variable exponent Sobolev functions.

2. D

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center x and radius r by $B(x, r)$. We will next introduce variable exponent Sobolev spaces in $\mathbb{R}^n$; note that we nevertheless use the standard definitions of the spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$ in the fixed exponent case $p \geqslant 1$ with open $\Omega \subset \mathbb{R}^n$.

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function (called the *variable exponent* on $\mathbb{R}^n$). Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$. We define the *generalized Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ to consist of all measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_{\mathbb{R}^n} |\lambda u(x)|^{p(x)}\, dx < \infty$ for some $\lambda > 0$. The function $\varrho_{p(\cdot)} : L^{p(\cdot)}(\mathbb{R}^n) \to [0, \infty)$ is called the *modular* of the space $L^{p(\cdot)}(\mathbb{R}^n)$. One can define a norm, the so-called *Luxemburg norm*, on this space by the formula $\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}$, see [KR]. With regards to the relationship between the modular and the norm, Kováčik and Rákosník [KR, (2.11)] showed that $\varrho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}$ if $\|u\|_{p(\cdot)} \leqslant 1$.

The *generalized Sobolev space* $W^{1,p(\cdot)}(\mathbb{R}^n)$ is the space of measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that u and the distributional gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$ are in $L^{p(\cdot)}(\mathbb{R}^n)$. The function $\varrho_{1,p(\cdot)} : W^{1,p(\cdot)}(\mathbb{R}^n) \to [0, \infty)$ is defined as $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(\nabla u)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space.

The corresponding local spaces $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ and $W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ are defined as follows: a function u is in $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ if $\int_G |u(x)|^{p(x)}\, dx < \infty$ for every bounded open subset G of $\mathbb{R}^n$, and a function u is in $W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ if both u and ∇u are in $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$.

Let us conclude this section by proving some simple but quite useful results.

2.1. Lemma. *We have* $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $W^{1,p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Follows directly from [KR, Theorem 2.8] since $p(x) \geqslant 1$ for every $x \in \mathbb{R}^n$. □

We show next that the generalized Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^n)$ is a lattice. This property is well known in the usual fixed exponent case, see [EG, Section 4.7] or [HKM, Section 1].

2.2. Theorem. *If* $u, v \in W^{1,p(\cdot)}(\mathbb{R}^n)$, *then* $\min(u, v)$ *and* $\max(u, v)$ *are in* $W^{1,p(\cdot)}(\mathbb{R}^n)$ *with*

$$\nabla \max(u, v)(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \geqslant v\} \\ \nabla v(x) & \text{for a.e. } x \in \{v \geqslant u\} \end{cases} \tag{2.3}$$

and

$$\nabla \min(u, v)(x) = \begin{cases} \nabla u(x) & \text{for a.e. } x \in \{u \leqslant v\} \\ \nabla v(x) & \text{for a.e. } x \in \{v \leqslant u\}. \end{cases} \tag{2.4}$$

In particular, $|u|$ *belongs to* $W^{1,p(\cdot)}(\mathbb{R}^n)$ *and* $|\nabla|u|| = |\nabla u|$ *a.e.*

Proof. It is clear that it suffices to consider the claims for $\max(u, v)$ since $\min(u, v) = -\max(-u, -v)$. Fix a point $x \in \mathbb{R}^n$. By Lemma 2.1 we know that $u|_{B(x,1)} \in W^{1,1}(B(x, 1))$ and so it follows from [HKM, Theorem 1.20] that

$$\nabla \max(u, v)(y) = \begin{cases} \nabla u(y) & \text{for a.e. } y \in \{B(x, 1) : u \geqslant v\} \\ \nabla v(y) & \text{for a.e. } y \in \{B(x, 1) : v \geqslant u\}. \end{cases} \tag{2.5}$$

We may cover $\mathbb{R}^n$ by a countable set of unit radius balls; since a countable union of measure zero sets has measure zero as well, we see that (2.3) holds.

We next note that $\varrho_{p(\cdot)}(\max(u, v)) \leqslant \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(v)$ and, using inequality (2.5), $\varrho_{p(\cdot)}(\nabla \max(u, v)) \leqslant \varrho_{p(\cdot)}(\nabla u) + \varrho_{p(\cdot)}(\nabla v)$. It thus follows that $\max(u, v) \in W^{1,p(\cdot)}(\mathbb{R}^n)$. □

2.6. Lemma. *Let $p^+ < \infty$ and $u_j, v_j \in W^{1,p(\cdot)}(\mathbb{R}^n)$ for $j = 1, 2, \ldots$. Assume further that the sequence $\big(\varrho_{p(\cdot)}(u_j)\big)_{j=1}^{\infty}$ is bounded. If $\varrho_{p(\cdot)}(u_j - v_j) \to 0$ as $j \to \infty$, then*

$$|\varrho_{p(\cdot)}(u_j) - \varrho_{p(\cdot)}(v_j)| \to 0 \quad \text{as } j \to \infty.$$

Proof. We have

$$|v_j|^{p(x)} = |v_j - u_j + u_j|^{p(x)} \leqslant 2^{p^+}|v_j - u_j|^{p(x)} + 2^{p^+}|u_j|^{p(x)}$$

and so it follows that $\varrho_{p(\cdot)}(u_j - v_j) + \varrho_{p(\cdot)}(u_j) \geqslant 2^{-p^+}\varrho_{p(\cdot)}(v_j)$. This means that $\big(\varrho_{p(\cdot)}(u_j)\big)_j$ is also a bounded sequence; let us choose $c > 0$ such that $\varrho_{p(\cdot)}(u_j) \leqslant c$ as well as $\varrho_{p(\cdot)}(v_j) \leqslant c$ for every j.

For each $M > 0$ we have

$$\begin{aligned} \varrho_{p(\cdot)}(u_j) - \varrho_{p(\cdot)}(v_j) &= \int_{\mathbb{R}^n} |u_j(x)|^{p(x)} - |v_j(x)|^{p(x)}\,dx \\ &= \int_{\mathbb{R}^n} |u_j(x) - v_j(x) + v_j(x)|^{p(x)} - |v_j(x)|^{p(x)}\,dx \\ &\leqslant \int_{\mathbb{R}^n} (1 + M)^{p(x)-1}|u_j(x) - v_j(x)|^{p(x)} + ((1 + \tfrac{1}{M})^{p(x)-1} - 1)|v_j(x)|^{p(x)}\,dx \\ &= (1 + M)^{p^+-1}\varrho_{p(\cdot)}(u_j - v_j) + ((1 + \tfrac{1}{M})^{p^+-1} - 1)\varrho_{p(\cdot)}(v_j), \end{aligned}$$

where the inequality follows from [MZ, Lemma 1.1]. Swapping u_j and v_j gives a similar inequality, and combining the inequalities gives

$$|\varrho_{p(\cdot)}(u_j) - \varrho_{p(\cdot)}(v_j)| \leqslant (1 + M)^{p^+-1}\varrho_{p(\cdot)}(u_j - v_j) + ((1 + \tfrac{1}{M})^{p^+-1} - 1)\big(\varrho_{p(\cdot)}(u_j) + \varrho_{p(\cdot)}(v_j)\big).$$

Let $\varepsilon > 0$ be given. Since $\varrho_{p(\cdot)}(u_j) + \varrho_{p(\cdot)}(v_j) \leqslant 2c$, we can choose M so that

$$((1 + \tfrac{1}{M})^{p^+-1} - 1)\big(\varrho_{p(\cdot)}(u_j) + \varrho_{p(\cdot)}(v_j)\big) \leqslant \tfrac{\varepsilon}{2}.$$

We can then choose j so large that

$$(1 + M)^{p^+-1}\varrho_{p(\cdot)}(u_j - v_j) \leqslant \tfrac{\varepsilon}{2}$$

and so we get $|\varrho_{p(\cdot)}(u_j) - \varrho_{p(\cdot)}(v_j)| \leqslant \varepsilon$. Since ε was arbitrary, this means that $|\varrho_{p(\cdot)}(u_j) - \varrho_{p(\cdot)}(v_j)| \to 0$, which was to be shown. □

3. S $p(\cdot)$-

For $E \subset \mathbb{R}^n$ we denote

$$S_{p(\cdot)}(E) = \{u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geqslant 1 \text{ in an open set containing } E\}.$$

The *Sobolev $p(\cdot)$-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\right) dx.$$

In case $S_{p(\cdot)}(E) = \emptyset$, we set $C_{p(\cdot)}(E) = \infty$. Functions $u \in S_{p(\cdot)}(E)$ are said to be *$p(\cdot)$-admissible* for the set E. Note that the full notation used above can be abbreviated as $C_{p(\cdot)}(E) = \inf \varrho_{1,p(\cdot)}(u)$ where the infimum is taken over all $p(\cdot)$-admissible functions for the set E.

The Sobolev $p(\cdot)$-capacity enjoys all relevant properties of general capacities; specifically, it will be seen that $C_{p(\cdot)}(E)$ defines a Choquet capacity under mild conditions on the exponent p. We start with some properties that hold for arbitrary measurable exponents $p : \mathbb{R}^n \to [1, \infty)$.

3.1. Theorem. *The set function $E \mapsto C_{p(\cdot)}(E)$ has the following properties:*

(i) $C_{p(\cdot)}(\emptyset) = 0$.

(ii) *If $E_1 \subset E_2$, then $C_{p(\cdot)}(E_1) \leqslant C_{p(\cdot)}(E_2)$.*

(iii) *If E is a subset of $\mathbb{R}^n$, then*

$$C_{p(\cdot)}(E) = \inf_{\substack{E \subset U \\ U \text{ open}}} C_{p(\cdot)}(U).$$

(iv) *If E_1 and E_2 are subsets of $\mathbb{R}^n$, then*

$$C_{p(\cdot)}(E_1 \cup E_2) + C_{p(\cdot)}(E_1 \cap E_2) \leqslant C_{p(\cdot)}(E_1) + C_{p(\cdot)}(E_2).$$

(v) *If $K_1 \supset K_2 \supset \ldots$ are compact, then*

$$\lim_{i \to \infty} C_{p(\cdot)}(K_i) = C_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i\right).$$

Proof. Assertion (i) is clear by the definition, since the constant function $u \equiv 0$ belongs to $S_{p(\cdot)}(\emptyset)$.

To prove (ii), let $E_1 \subset E_2$. Then $S_{p(\cdot)}(E_1) \supset S_{p(\cdot)}(E_2)$, and hence by definition

$$C_{p(\cdot)}(E_1) = \inf_{u \in S_{p(\cdot)}(E_1)} \varrho_{p(\cdot)}(u) \leqslant \inf_{u \in S_{p(\cdot)}(E_2)} \varrho_{p(\cdot)}(u) = C_{p(\cdot)}(E_2).$$

To prove (iii), let $E \subset \mathbb{R}^n$ be arbitrary. Obviously,

$$C_{p(\cdot)}(E) \leqslant \inf_{\substack{E \subset U \\ U \text{ open}}} C_{p(\cdot)}(U).$$

Fix $\varepsilon > 0$. There exists a function $u \in S_{p(\cdot)}(E)$ such that $E \subset \operatorname{int}\{u \geqslant 1\}$ and

$$C_{p(\cdot)}(\operatorname{int}\{u \geqslant 1\}) \leqslant \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\right) dx \leqslant C_{p(\cdot)}(E) + \varepsilon,$$

and the claim follows.

To prove (iv), let $\varepsilon > 0$. As above, choose $u_1 \in S_{p(\cdot)}(E_1)$ such that $E_1 \subset \operatorname{int}\{u_1 \geqslant 1\}$ and

$$\int_{\mathbb{R}^n} \left(|u_1(x)|^{p(x)} + |\nabla u_1(x)|^{p(x)}\right) dx \leqslant C_{p(\cdot)}(E_1) + \varepsilon.$$

Choose also $u_2 \in S_{p(\cdot)}(E_2)$ such that $E_2 \subset \operatorname{int}\{u_2 \geqslant 1\}$ and

$$\int_{\mathbb{R}^n} \left(|u_2(x)|^{p(x)} + |\nabla u_2(x)|^{p(x)}\right) dx \leqslant C_{p(\cdot)}(E_2) + \varepsilon.$$

We have $\max(u_1, u_2) \in S_{p(\cdot)}(E_1 \cup E_2)$ and $\min(u_1, u_2) \in S_{p(\cdot)}(E_1 \cap E_2)$ and, by Theorem 2.2,

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla \max(u_1, u_2)(x)|^{p(x)}\, dx + \int_{\mathbb{R}^n} |\nabla \min(u_1, u_2)(x)|^{p(x)}\, dx \\ &= \int_{\mathbb{R}^n} |\nabla u_1(x)|^{p(x)}\, dx + \int_{\mathbb{R}^n} |\nabla u_2(x)|^{p(x)}\, dx. \end{aligned}$$

Consequently

$$C_{p(\cdot)}(E_1 \cup E_2) + C_{p(\cdot)}(E_1 \cap E_2) \leqslant \varrho_{1,p(\cdot)}(u_1) + \varrho_{1,p(\cdot)}(u_2) \leqslant C_{p(\cdot)}(E_1) + C_{p(\cdot)}(E_2) + 2\varepsilon,$$

from which (iv) follows as ε tends to zero.

To prove (v), let $K_1 \supset K_2 \supset \dots$ be compact. Since $\bigcap_{i=1}^{\infty} K_i \subset K_j$ for each $j = 1, 2, \dots$, property (ii) gives

$$C_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i\right) \leqslant \lim_{i\to\infty} C_{p(\cdot)}(K_i).$$

To prove the opposite inequality, choose any open set U with $\bigcap_i K_i \subset U$. Because every K_i is compact (so that $\bigcap_i K_i$ is compact, as well), there is a positive integer k such that $K_i \subset U$ for all $i \geqslant k$. Thus

$$\lim_{i\to\infty} C_{p(\cdot)}(K_i) \leqslant C_{p(\cdot)}(U),$$

and by property (iii)

$$\lim_{i\to\infty} C_{p(\cdot)}(K_i) \leqslant C_{p(\cdot)}\left(\bigcap_{i=1}^{\infty} K_i\right).$$

□

Properties (i), (ii), and (iii) from the previous theorem yield that the Sobolev $p(\cdot)$-capacity is an outer capacity. In order to get the remaining Choquet property (that is, (vi) in the next theorem) we need an extra assumption for the variable exponent:

3.2. Theorem. *If $1 < p^- \leqslant p^+ < \infty$, then the set function $E \mapsto C_{p(\cdot)}(E)$ has the following additional properties:*

(vi) *If $E_1 \subset E_2 \subset \dots$ are subsets of $\mathbb{R}^n$, then*

$$\lim_{i\to\infty} C_{p(\cdot)}(E_i) = C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right).$$

(vii) *If $E_i \subset \mathbb{R}^n$ for $i = 1, 2, \dots$, then*

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} C_{p(\cdot)}(E_i).$$

Proof. To prove (vi), denote $E = \bigcup_{i=1}^{\infty} E_i$. Note first that (ii) implies that

$$\lim_{i\to\infty} C_{p(\cdot)}(E_i) \leqslant C_{p(\cdot)}(E).$$

We will prove the opposite inequality. We may assume that $\lim_{i\to\infty} C_{p(\cdot)}(E_i) < \infty$. Let $u_i \in S_{p(\cdot)}(E_i)$ and $\varrho_{1,p(\cdot)}(u_i) \leqslant C_{p(\cdot)}(E_i) + 2^{-i}$ for every $i = 1, 2, \dots$. Since $W^{1,p(\cdot)}(\mathbb{R}^n)$ is reflexive [KR, Corollary 2.7] and the sequence (u_i) is bounded in $W^{1,p(\cdot)}(\mathbb{R}^n)$, there is a subsequence of (u_i) which converges weakly to a function $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$. By the Mazur lemma there is a sequence (v_j) converging strongly to u such that every v_j is a convex combination of the u_i's, $i \geqslant j$. Since $E_j \subset E_{j+1} \subset \ldots$, it follows that $E_j \subset \operatorname{int}\{v_j \geqslant 1\}$, and we obtain

$$\varrho_{1,p(\cdot)}(v_j) \leqslant \sup_{i \geqslant j} \varrho_{1,p(\cdot)}(v_i) \leqslant \sup_{i \geqslant j} \Big(C_{p(\cdot)}(E_i) + 2^{-i} \Big) \leqslant \lim_{i\to\infty} C_{p(\cdot)}(E_i) + 2^{-j}.$$

By considering a subsequence if necessary, we may assume that $\|v_{j+1} - v_j\|_{1,p(\cdot)} \leqslant 2^{-j}$. We set

$$w_j = v_j + \sum_{i=j}^{\infty} |v_{i+1} - v_i|,$$

and observe that $w_j \in W^{1,p(\cdot)}(\mathbb{R}^n)$. Since $w_j \geqslant \sup_{i \geqslant j}\{v_i\}$, we see that $w_j \geqslant 1$ on the open set

$$\bigcup_{i=j}^{\infty} \operatorname{int}\{v_i \geqslant 1\} \supset E,$$

so $w_j \in S_{p(\cdot)}(E)$. This yields $C_{p(\cdot)}(E) \leqslant \varrho_{1,p(\cdot)}(w_j)$ for $j = 1, 2, \ldots$. We also find that

$$\|w_j - v_j\|_{1,p(\cdot)} \leqslant \sum_{i=j}^{\infty} \|v_{i+1} - v_i\|_{1,p(\cdot)} \leqslant \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1},$$

and hence

$$\varrho_{1,p(\cdot)}(w_j - v_j) \leqslant \|w_j - v_j\|_{1,p(\cdot)} \to 0 \quad \text{as } j \to \infty.$$

Since

$$\varrho_{1,p(\cdot)}(v_j) \leqslant \lim_{i\to\infty} C_{p(\cdot)}(E_i) + 2^{-j},$$

Lemma 2.6 yields

$$C_{p(\cdot)}(E) \leqslant \varrho_{1,p(\cdot)}(w_j) \to \lim_{i\to\infty} C_{p(\cdot)}(E_i) \quad \text{as } j \to \infty.$$

This completes the proof of (vi).

It remains to prove (vii). From (iv) it follows by induction that

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{k} E_i \right) \leqslant \sum_{i=1}^{k} C_{p(\cdot)}(E_i)$$

for any finite family of subsets $E_1, E_2, \ldots, E_k$ in $\mathbb{R}^n$. Since $\bigcup_{i=1}^{k} E_i$ increases to $\bigcup_{i=1}^{\infty} E_i$, (vi) implies (vii). This completes the proof of Theorem 3.2. □

By the definition of outer measure, properties (i), (ii), and (vii) of the $p(\cdot)$-capacity yield:

3.3. Corollary. *If $1 < p^- \leqslant p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure.*

A set function which satisfies the capacity properties (i), (ii), (v), and (vi) is called a *Choquet capacity*, see [Cho]. We therefore have the following result:

3.4. Corollary. *Let $1 < p^- \leqslant p^+ < \infty$. Then the set function $E \mapsto C_{p(\cdot)}(E)$, $E \subset \mathbb{R}^n$, is a Choquet capacity. In particular, all Suslin sets $E \subset \mathbb{R}^n$ are capacitable, this is,*

$$C_{p(\cdot)}(E) = \inf_{\substack{E \subset U \\ U \text{ open}}} C_{p(\cdot)}(U) = \sup_{\substack{K \subset E \\ K \text{ compact}}} C_{p(\cdot)}(K).$$

We can derive a weak form of the subadditivity (property (vii)) even if we dispense with the lower bound assumption on the variable exponet p.

3.5. Lemma. *Suppose that $p^+ < \infty$. If every E_i is a subset of $\mathbb{R}^n$ with $C_{p(\cdot)}(E_i) = 0$, $i = 1, 2, \ldots$, then*

$$C_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right) = 0.$$

Proof. Fix $0 < \varepsilon < 1$. Since $\varrho_{1,p(\cdot)}(u_i) \to 0$ if and only if $\|u_i\|_{1,p(\cdot)} \to 0$ [KR, (2.28)], we find a sequence (u_i) with $u_i \in S_{p(\cdot)}(E_i)$ and $\|u_i\|_{1,p(\cdot)} \leqslant \varepsilon 2^{-i}$. Define $v_i = u_1 + \ldots + u_i$. Then (v_i) is a Cauchy sequence and since $W^{1,p(\cdot)}(\mathbb{R}^n)$ is complete [KR, Theorem 2.5], there exists $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $v_i \to v$.

Let us show that $v \in S_{p(\cdot)}\left(\bigcup_{i=1}^{\infty} E_i\right)$, at least after a redefinition of v in a set of measure zero. Let $U_i = \operatorname{int}\{u_i \geqslant 1\}$. We show that $v \geqslant 1$ almost everywhere in an open set $\bigcup_{i=1}^{\infty} U_i \supset \bigcup_{i=1}^{\infty} E_i$. Suppose this were not so. Then there exists a set N and an index i such that $v|_N < 1$ and $|N \cap U_i| > 0$. It follows that

$$\int_{N \cap U_i} |1 - v(x)|^{p(x)}\, dx > 0.$$

But $v_j(x) \geqslant 1$ for $x \in N \cap U_i$ for all $j > i$, which contradicts the fact that $v_j \to v$. Therefore $v \in S_{p(\cdot)}\left(\bigcup_{i=1}^{n} E_i\right)$. We find that

$$\|v\|_{1,p(\cdot)} \leqslant \limsup_{i\to\infty} \|v_i\|_{1,p(\cdot)} \leqslant \sum_{i=1}^{\infty} \|u_i\|_{1,p(\cdot)} \leqslant \varepsilon.$$

Therefore

$$C_{p(\cdot)}\left(\bigcup_{1=1}^{\infty} E_i\right) \leqslant \varrho_{1,p(\cdot)}(v) \leqslant \|v\|_{1,p(\cdot)} \leqslant \varepsilon$$

which proves the claim, since ε was arbitrary. □

We say that the variable exponent $p : \mathbb{R}^n \to [1, \infty)$ satisfies the *density condition* if $C^\infty(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$, see Section 5 for details and further references. If p satisfies the density condition, we achieve the following result which allows us to consider only a limited class of $p(\cdot)$-admissible functions in the definition of the Sobolev $p(\cdot)$-capacity.

3.6. Lemma. *Let $p : \mathbb{R}^n \to [1, \infty)$ satisfy the density condition. If K is compact, then*

$$C_{p(\cdot)}(K) = \inf_{u \in S^{\infty}_{p(\cdot)}(K)} \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\right) dx,$$

where $S^{\infty}_{p(\cdot)}(K) = S_{p(\cdot)}(K) \cap C^\infty(\mathbb{R}^n)$.

Proof. Let $u \in S_{p(\cdot)}(K)$. We choose a sequence of functions $\varphi_j \in C^\infty(\mathbb{R}^n)$ converging to u in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Let U be an open bounded neighborhood of K such that $u \geqslant 1$ in U. Let $\psi \in C^\infty(\mathbb{R}^n), 0 \leqslant \psi \leqslant 1$, be such that $\psi = 1$ in $\mathbb{R}^n \setminus U$ and $\psi = 0$ in an open neighborhood of K. Then it is seen that the functions $\psi_j = 1 - (1 - \varphi_j)\psi$ converge to u in $W^{1,p(\cdot)}(\mathbb{R}^n)$. This establishes the assertion since $\psi_j \in S^{\infty}_{p(\cdot)}(K)$. □

4. $p(\cdot)$-C H

In this section we study how the variable Sobolev capacity relates to the Hausdorff measures. The following lemma follows easily from the definition of the capacity.

4.1. Lemma. *Every measurable set $E \subset \mathbb{R}^n$ satisfies $|E| \leqslant C_{p(\cdot)}(E)$.*

Proof. If $u \in S_{p(\cdot)}(E)$, then there is an open set $O \supset E$ such that $u \geqslant 1$ in O and hence

$$|E| \leqslant |O| \leqslant \int_{\mathbb{R}^n} |u(x)|^{p(x)} dx \leqslant \int_{\mathbb{R}^n} |u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} dx.$$

We obtain the claim by taking the infimum over all $p(\cdot)$-admissible functions for E. □

The s-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^n$ is denoted by $\mathcal{H}^s(E)$, see [EG, Section 2.1] or [Mat].

4.2. Theorem. *If $E \subset \mathbb{R}^n$ with $C_{p(\cdot)}(E) = 0$, then*
(i) $\mathcal{H}^s(E) = 0$ *for all $s > n - p^-$, and*
(ii) $\mathcal{H}^{n-1}(E) = 0$ *for compact E with $p^- = 1$.*

Proof. Let $r > 0$, and let $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $i = 1, 2, \dots$, be such that $E \subset \operatorname{int}\{u_i \geqslant 1\}$ and

$$\int_{\mathbb{R}^n} |u_i(x)|^{p(x)} + |\nabla u_i(x)|^{p(x)} dx < 2^{-i}.$$

Let ϕ be a Lipschitz function with $\phi = 1$ in $B(0, r)$, $\phi = 0$ in $\mathbb{R}^n \setminus B(0, 2r)$ and $0 \leqslant \phi \leqslant 1$. Then by the Hölder inequality $\phi u_i \in W_0^{1,p^-}(B(0, 2r)) \subset W^{1,p^-}(\mathbb{R}^n)$. We thus obtain $C_{p^-}(E \cap B(0, r)) = 0$. It then follows by subadditivity that $C_{p^-}(E) = 0$. But we know from [EG, Theorem 4, p. 156, and Theorem 3, p. 193] that this implies (i) and (ii), respectively. □

4.3. Corollary. *Suppose that $p^- > n$, and let $E \subset \mathbb{R}^n$. If $C_{p(\cdot)}(E) = 0$, then $E = \emptyset$.*

Proof. Since $\mathcal{H}^0$ is a counting measure [EG, Theorem 2, p. 63], the claim follows directly from Theorem 4.2(i). □

4.4. Theorem. *Suppose that $p^+ < n$, and let $E \subset \mathbb{R}^n$. If $\mathcal{H}^{n-p^+}(E) = 0$, then $C_{p(\cdot)}(E) = 0$.*

Proof. It follows from [EG, Theorem 4, p. 156] that $C_{p^+}(E) = 0$. Thus there exists a sequence $v_j \in S_{p^+}(E)$ such that $\varrho_{1,p^+}(v_j) \to 0$ as $j \to \infty$. Let ϕ be a Lipschitz function with $\phi = 1$ in $B(0, r)$, $\phi = 0$ in $\mathbb{R}^n \setminus B(0, 2r)$, and $0 \leqslant \phi \leqslant 1$. Then $\phi v_j \in W_0^{1,p^+}(B(0, 2r)) \subset W^{1,p^+}(\mathbb{R}^n)$ and by [KR, Theorem 2.8] we obtain

$$\|\phi v_j\|_{1,p(\cdot)} \leqslant (1 + |B(0, 2r)|)\|\phi v_j\|_{1,p^+}.$$

This yields $\phi v_j \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and so it follows that $\phi v_j \in S_{p(\cdot)}(E \cap B(0, r))$. Since $\varrho_{1,p(\cdot)}(\phi v_j) \to 0$ as $j \to \infty$, we obtain $C_{p(\cdot)}(E \cap B(0, r)) = 0$, which implies the result by subadditivity, Lemma 3.5. □

Recall that the Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is defined in terms of the Hausdorff measure by

$$\dim_{\mathcal{H}}(A) = \inf\{s > 0 : \mathcal{H}^s(A) = 0\},$$

see [Mat] for details. We define a local version of the Hausdorff dimension, which allows us to take into account the variability of the exponent to derive a sharper version of Theorem 4.2(i).

The *local Hausdorff dimension of a set* $A \subset \mathbb{R}^n$ is defined to be a function $\dim_{\mathcal{H}}(A,\cdot) : \mathbb{R}^n \to [0,\infty)$ given for $x \in \mathbb{R}^n$ by

$$\dim_{\mathcal{H}}(A,x) = \lim_{r\to 0} \inf\{s > 0 : \mathcal{H}^s(A \cap B(x,r)) = 0\}.$$

Before stating our result on the local Hausdorff dimension of zero capacity sets, let us illustrate the utility of this concept; the next result shows that $\dim_{\mathcal{H}}(A,\cdot)$ allows us to estimate the Hausdorff dimension of arbitrary subsets of A.

4.5. Lemma. *For* $B \subset A \subset \mathbb{R}^n$ *we have* $\dim_{\mathcal{H}}(B) \leqslant \sup_{x\in B} \dim_{\mathcal{H}}(A,x)$.

Proof. Fix $s > \sup_{x\in B} \dim_{\mathcal{H}}(A,x)$. It follows from the definition of the local dimension that for every $x \in B$ there exists an $r_x > 0$ such that $\dim_{\mathcal{H}}(A\cap B(x,r_x)) < s$ so that $\mathcal{H}^s(A\cap B(x,r_x)) = 0$. From the open cover $\bigcup_{x\in B} B(x,r_x)$ we can choose a countable subcovering of B, $\bigcup_{i=1}^{\infty} B(x_i,r_{x_i})$. It follows by subadditivity that $\mathcal{H}^s(A\cap\bigcup_{i=1}^{\infty} B(x_i,r_{x_i})) = 0$, and since $B \subset A \cap \bigcup_{i=1}^{\infty} B(x_i,r_{x_i})$, it follows by monotony that $\mathcal{H}^s(B) = 0$. Therefore $\dim_{\mathcal{H}}(B) < s$ and so it follows that $\dim_{\mathcal{H}}(B) \leqslant \sup_{x\in B} \dim_{\mathcal{H}}(A,x)$. □

The following example shows that the inequality in the previous lemma can be sharp and that the local Hausdorff dimension needs not be continuous.

4.6. Example. We write $e_1 = (1,0)$ and $e_2 = (0,1)$. Let $A = [-e_1,e_1] \cup B(e_2,1)$ and $B = [-e_1,e_1]$. Then clearly $\dim_{\mathcal{H}}(A,x) = 1$ for $x \in [-e_1,e_1] \setminus \{0\}$ but $\dim_{\mathcal{H}}(A,0) = 2$, a discontinuity. Moreover, $\dim_{\mathcal{H}}(B) = 1 < 2 = \sup_{x\in B} \dim_{\mathcal{H}}(A,x)$.

4.7. Theorem. *Let* $p : \mathbb{R}^n \to [1,\infty)$ *be continuous. If* $E \subset \mathbb{R}^n$ *with* $C_{p(\cdot)}(E) = 0$*, then* $p(x) \leqslant n$ *and* $\dim_{\mathcal{H}}(E,x) \leqslant n - p(x)$ *for every* $x \in E$.

Proof. Suppose that $p(x) > n$ for some $x \in E$. Then there exists a ball $B(x,r)$ such that $p(y) > (n + p(x))/2$ for all $y \in B(x,r)$. Let ϕ be a Lipschitz function with $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in $B(x,\frac{r}{2})$, and $\phi = 0$ outside of $B(x,r)$. We define a new variable exponent $q : \mathbb{R}^n \to [1,\infty)$ given by

$$q(y) = \begin{cases} p(y) & \text{for } y \in B(x,r), \\ \frac{n+p(x)}{2} & \text{otherwise.} \end{cases}$$

Now, if $u \in S_{p(\cdot)}(E)$, then $\phi u \in S_{q(\cdot)}(E \cap B(x,\frac{r}{2}))$ and, moreover, $\varrho_{1,q(\cdot)}(\phi u) \leqslant c\,\varrho_{1,p(\cdot)}(u)$, where the constant $c > 0$ does not depend on u. It follows that

$$C_{q(\cdot)}(E \cap B(x,\tfrac{r}{2})) \leqslant c\, C_{p(\cdot)}(E) = 0.$$

Since $\operatorname{ess\,inf} q(x) = (n+p(x))/2 > n$, it follows from Theorem 4.2(iii) that $E\cap B(x,\frac{r}{2}) = \emptyset$. But $E \cap B(x,\frac{r}{2})$ contains x; this contradiction shows that the assumption $p(x) > n$ was false.

To prove the second statement of the theorem, fix $x \in E$ and $\varepsilon > 0$. Let $r > 0$ be such that $|p(y) - p(x)| < \varepsilon$ for all $y \in B(x,r)$. Like above we define a new variable exponent

$$q(y) = \begin{cases} p(y) & \text{for } y \in B(x,r), \\ p(x) - \varepsilon & \text{otherwise;} \end{cases}$$

arguing as above it follows by Theorem 4.2(i) that $\mathcal{H}^s(E \cap B(x,r/2)) = 0$ for $s > n - p(x) + \varepsilon$ and so $\dim_{\mathcal{H}}(E \cap B(x,r)) \leqslant n - p(x) + \varepsilon$. Letting $r \to 0$ we see that $\dim_{\mathcal{H}}(E,x) \leqslant n - p(x) + \varepsilon$. Since ε was arbitrary, this further implies that $\dim_{\mathcal{H}}(E,x) \leqslant n - p(x)$, which was to be shown. □

5. $p(\cdot)$-Q

The following two results are analogous to the results by Kinnunen and Martio stated on metric measure spaces in the fixed exponent case [KM, Theorem 3.3 and Corollary 3.7]. Our proofs are easy modifications of their proofs.

We say that a claim holds $p(\cdot)$-*quasieverywhere* if it holds everywhere except in a set of $p(\cdot)$-capacity zero.

5.1. Lemma. *Let* $1 < p^- \leqslant p^+ < \infty$. *For each Cauchy sequence of functions in* $C(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$ *there is a subsequence which converges pointwise* $p(\cdot)$-*quasieverywhere in* $\mathbb{R}^n$. *Moreover, the convergence is uniform outside a set of arbitrary small* $p(\cdot)$-*capacity.*

Proof. Let (u_i) be a Cauchy sequence in $C(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$. We assume without loss of generality, by considering a subsequence if necessary, that $\|u_i - u_{i+1}\|_{1,p(\cdot)} \leqslant 4^{-i}$ for every $i = 1, 2, \ldots$. We denote

$$E_i = \{x \in \mathbb{R}^n : |u_i(x) - u_{i+1}(x)| > 2^{-i}\}.$$

for $i = 1, 2, \ldots$ and

$$F_j = \bigcup_{i=j}^{\infty} E_i.$$

Using Theorem 2.2 it is easy to show that $v = 2^i |u_i - u_{i+1}| \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and by assumption we have $\|v\|_{1,p(\cdot)} \leqslant 2^{-i}$. It was shown in [KR, (2.11)] that $\varrho_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}$ if $\|u\|_{p(\cdot)} \leqslant 1$. It follows that

$$C_{p(\cdot)}(E_i) \leqslant \varrho_{1,p(\cdot)}(v) \leqslant 2^i \|u_i - u_{i+1}\|_{1,p(\cdot)}.$$

The subadditivity of the $p(\cdot)$-capacity, Theorem 3.2(vii), implies that

$$C_{p(\cdot)}(F_j) \leqslant \sum_{i=j}^{\infty} C_{p(\cdot)}(E_i) \leqslant \sum_{i=j}^{\infty} 2^{-i} \leqslant 2^{1-j}.$$

Hence we obtain

$$C_{p(\cdot)}\left(\bigcap_{j=1}^{\infty} F_j\right) \leqslant \lim_{j\to\infty} C_{p(\cdot)}(F_j) = 0.$$

Since (u_i) converges pointwise in $\mathbb{R}^n \setminus \bigcap_{j=1}^{\infty} F_j$, we have proved the first claim of the lemma. Moreover, we have

$$|u_l(x) - u_k(x)| \leqslant \sum_{i=l}^{k-1} |u_i(x) - u_{i+1}(x)| \leqslant \sum_{i=l}^{k-1} 2^{-i} < 2^{1-l}$$

for every $x \in \mathbb{R}^n \setminus F_j$ and every $k > l > j$. Therefore the convergence is uniform in $\mathbb{R}^n \setminus F_j$. □

The variable exponent $p : \mathbb{R}^n \to [1, \infty)$ is said to satisfy the *density condition* if the set $C^\infty(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. The extent of the validity of the density condition is still an open problem, however, Edmunds and Rákosník [ER2, Theorem 1] have proven that the following condition is sufficient: for every $x \in \mathbb{R}^n$ there exists a number $h(x) > 0$ and a vector $\xi(x) \in \mathbb{R}^n \setminus \{0\}$ such that

(i) $h(x) < |\xi(x)| \leqslant 1$, and
(ii) $p(x) \leqslant p(x + y)$ for a.e. $x \in \mathbb{R}^n$ and $y \in \bigcup_{0<t\leqslant 1} B(t\xi(x), th(x))$.

We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists an open set O with $C_{p(\cdot)}(O) < \varepsilon$ such that u is continuous in $\mathbb{R}^n \setminus O$. The following result gives a sufficient setting to guarantee that a $p(\cdot)$-quasicontinuous representative exists.

5.2. Theorem. *Let $p : \mathbb{R}^n \to [1, \infty)$ satisfy the density condition with $1 < p^- \leqslant p^+ < \infty$. For each $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ there is a $p(\cdot)$-quasicontinuous function $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = v$ almost everywhere in $\mathbb{R}^n$.*

Proof. Let $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$. It follows from the density condition that there exist functions $u_i \in C^\infty(\mathbb{R}^n) \cap W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u_i \to u$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$. It follows from Lemma 5.1 that the sequence converges uniformly outside a set of arbitrarily small capacity. But uniform convergence implies continuity of the limit and so we get a function continuous outside a set of arbitrarily small capacity, as was to be shown. □

Next we show that every quasicontinuous Sobolev function satisfies a weak type capacity inequality; the proofs follow the ideas from the book by Malý and Ziemer [MZ, Lemmata 2.21 and 2.22].

5.3. Lemma. *Let $p^+ < \infty$, and let E be a subset of $\mathbb{R}^n$. Suppose that $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is a nonnegative $p(\cdot)$-quasicontinuous function such that $u \geqslant 1$ on E. Then for every $\varepsilon > 0$ there exists a function $v \in S_{p(\cdot)}(E)$ such that $\varrho_{1,p(\cdot)}(u - v) < \varepsilon$.*

Proof. Let $0 < \delta < 1$, and let $O \subset \mathbb{R}^n$ be an open set such that u is continuous in $\mathbb{R}^n \setminus O$ and $C_{p(\cdot)}(O) < \delta$. Moreover, let $w \in S_{p(\cdot)}(O)$ be such that $\varrho_{1,p(\cdot)}(w) < \delta$, and write $v = (1 + \delta)u + |w|$. Using Theorem 2.2 it is easy to show that $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$. The set

$$G = \{x \in \mathbb{R}^n \setminus O : u(x) > 1\} \cup O$$

is open, contains E, and $v \geqslant 1$ on G, thus $v \in S_{p(\cdot)}(E)$. Using the power mean inequality $|w + \delta u|^{p(\cdot)} \leqslant 2^{p^+-1}(|w|^{p(\cdot)} + |\delta u|^{p(\cdot)})$ we obtain

$$\begin{aligned}\varrho_{1,p(\cdot)}(u - v) &= \int_{\mathbb{R}^n} |w(x) + \delta\, u(x)|^{p(x)} + |\nabla(w(x) + \delta\, u(x))|^{p(x)}\, dx \\ &\leqslant 2^{p^+-1}(\varrho_{1,p(\cdot)}(w) + \delta^{p^-} \varrho_{1,p(\cdot)}(u)) < 2^{p^+-1}\left(\delta + \delta^{p^-} \varrho_{1,p(\cdot)}(u)\right).\end{aligned}$$

Letting $\delta \to 0$ completes the proof of Lemma 5.3. □

5.4. Theorem. *Let $p^+ < \infty$. If $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is a $p(\cdot)$-quasicontinuous function and $\lambda > 0$, then*

$$C_{p(\cdot)}(\{x \in \mathbb{R}^n : |u(x)| > \lambda\}) \leqslant \int_{\mathbb{R}^n} \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} + \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} dx.$$

Proof. By Theorem 2.2, $|u| \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and $|\nabla|u|| = |\nabla u|$. By Lemma 5.3, there is a sequence $v_j \in S_{p(\cdot)}(\{x \in \mathbb{R}^n : \frac{|u(x)|}{\lambda} > 1\})$ such that

$$\varrho_{1,p(\cdot)}\Big(\tfrac{|u|}{\lambda} - v_j\Big) \to 0 \quad \text{as } j \to \infty.$$

Hence we obtain by Lemma 2.6 that

$$\varrho_{1,p(\cdot)}(v_j) \to \varrho_{1,p(\cdot)}\Big(\tfrac{|u|}{\lambda}\Big) \quad \text{as } j \to \infty.$$

□

R

[AM] Emilio Acerbi and Giuseppe Mingione: Regularity results for a class of functionals with non-standard growth, *Arch. Ration. Mech. Anal.* **156** (2001), 121–140.

[Cho] Gustave Choquet: Theory of capacities, *Ann. Inst. Fourier (Grenoble)* **5** (1953-1954), 131–295.

[Die1] Lars Diening: Theoretical and Numerical Results for Electrorheological Fluids, Ph.D. Thesis, University of Freiburg, Germany, 2002.

[Die2] Lars Diening: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.*, to appear.

[DR] Lars Diening and Michael Růžička: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, Preprint.

[ER1] David E. Edmunds and Jiři Rákosník: Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992), 229–236.

[ER2] David E. Edmunds and Jiři Rákosník: Sobolev embedding with variable exponent, *Studia Math.* **143** (2000), 267–293.

[EG] Lawrence C. Evans and Ronald F. Gariepy: *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[HKM] Juha Heinonen, Tero Kilpeläinen, and Olli Martio: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1993.

[Hud] Henryk Hudzik: The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces $W^k_M(\Omega)$, *Comment. Math. Prace Mat.* **21** (1980), 315–324.

[KM] Juha Kinnunen and Olli Martio: The Sobolev capacity on metric spaces, *Ann. Acad. Sci. Fenn. Ser. A I. Math.* **21** (1996), 367–382.

[KR] Ondrej Kováčik and Jiři Rákosník: On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41(116)** (1991), 592–618.

[MZ] Jan Malý and William P. Ziemer: *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, Volume 51, American Mathematical Society, Providence, RI, 1997.

[Mar] Paolo Maracellini: Regularity and existance of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* **50** (1991), no. 1, 1–30.

[Mat] Pertti Mattila: *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability*, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.

[Maz] Vladimir G. Maz'ya: *Sobolev Spaces*, Spinger-Verlag, Berlin, 1985.

[Mus] Julian Musielak: *Orlicz Spaces and Modular Spaces*, Springer-Verlag, Berlin, 1983.

[PR] Luboš Pick and Michael Růžička: An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, *Expo. Math.* **19** (2001), 369–371.

[Rak] Jiři Rákosník: Sobolev inequality with variable exponent, *Function Spaces, Differential Operators and Nonlinear Analysis*, editors Vesa Mustonen and Jiři Rákosník, Mathematical Institute of the Academy of Sciences of the Czech Republic, Praha, 2000, pp. 220–228.

[Zhi] Vasiliĭ V. Zhikov: Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk. SSSR Ser. Mat.* **50** (1986), 675–710, 877.

D M
P.O. B 4 (Y 5)
FIN-00014 U H
F

E-mail addresses:
petteri.harjulehto@helsinki.fi
peter.hasto@helsinki.fi
mika.koskenoja@helsinki.fi
susanna.varonen@helsinki.fi