POLAR SETS ON METRIC SPACES

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Abstract. We show that if X is a proper metric measure space equipped with a doubling measure supporting a Poincaré inequality, then subsets of X with zero p-capacity are precisely the p-polar sets; that is, for every relatively compact set of zero p-capacity there exists a p-superharmonic function on a domain in X containing that set whose set of singularities contains the given set. In addition, we prove that if X is a p-hyperbolic metric space, then the p-superharmonic function can be required to be p-superharmonic on the entire space X.

0. Introduction

Sets of zero capacity in potential theory play the same role played by sets of zero measure in the study of L^p-spaces. Given a metric measure space X, functions in the equivalence classes in the Sobolev space $N^{1,p}(X)$ agree up to sets of zero p-capacity. The purpose of this note is to characterize sets of zero p-capacity in terms of the singular sets of p-superharmonic functions.

A polar set of a metric space is a set $E \subset X$ so that there exists a p-superharmonic function which takes on the value of infinity at every point in E. It was shown by Kilpeläinen in [K1] and [K2] that when X is an Euclidean space and E is a subset of a domain in this Euclidean space with zero p-capacity, then there exists a p-superharmonic function on the domain taking on the value of infinity at each point of E. It is known that polar sets have zero p-capacity (see [HKM]); hence, the collection of all sets of zero p-capacity in X and the collection of all polar sets in X are one and the same collection. We extend this result to the setting of metric measure spaces that are equipped with a doubling measure supporting a $(1,q)$-Poincaré inequality for some q with $1 \le q < p$.

It should be noted here that unlike in the setting of Riemannian manifolds, in the general setting of metric measure spaces p-superharmonic functions need not be associated in any way with a corresponding differential equation. Hence the technique of sweeping of measures developed in [K1] and [K2] is not applicable in the generality considered in this paper. However, we do follow the general idea of the proof given in [K1] in proving the main result of this paper, Proposition 3.2. Proposition 3.2, which states that zero p-capacity sets are p-polar, together with Proposition 2.2, which states that p-polar sets have zero p-capacity, give a characterization of zero p-capacity sets in terms of p-polar sets. Observe that the p-superharmonic functions which identify a closed p-polar set E also act as a barrier

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function for for the points in E, considered as boundary points of the open set $\Omega\setminus E$ whenever Ω is a domain containing E. Thus the study of zero p-capacity sets in terms of p-polar sets is of interest in potential theory.

This note is organized as follows. The first section gathers together the necessary background such as the definition of Newtonian (Sobolev) spaces, capacity, and superharmonic functions. In the second section we prove that p-polar sets are of zero p-capacity. The third section is focused toward proving that zero p-capacity sets are p-polar by using the tools developed in [KM2] and the general idea of the proof from [K1]. It is also interesting to know when is it true that given a set of zero p-capacity there is a function that is p-superharmonic on the whole space X so that the given set is a subset of the singularity set of such superharmonic function. In the fourth section we give a partial answer to this question by demonstrating that whenever X is a p-hyperbolic metric space such global p-superharmonic functions can be constructed. It is also interesting to know whether a zero p-capacity set is precisely the set of all singularities of a given p-superharmonic function. Sections 5 through 7 are devoted to answering this question under a modification of what is meant by a p-harmonic function.

1. Preliminaries

We assume that (X, d, μ) is a metric measure space endowed with a metric d and a non-trivial Borel-regular measure μ. We assume in addition that X is connected and proper, that is, closed and bounded subsets of X are compact. While the results in this note remain valid if we assume that X is locally compact rather than proper, we assume properness for the sake of simplifying notation. The proof of the results in this note can be extended to the locally compact case with some mild modifications.

In this section we recall basic definitions and describe the general setup of our study.

1.1. Upper gradients. Let u be a real-valued function on X. A non-negative Borel measurable function $g\colon X \to [0,\infty]$ is said to be an *upper gradient* of u if for all non-constant compact rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds. \tag{1.2}$$

See [C] and [Sh2] for a discussion on upper gradients.

Let $1 \le p < \infty$. A property is said to hold for *p-almost all* paths, if the set of non-constant paths for which the property fails is of zero p-modulus (see, for example, [Sh2] for the definition of the p-modulus). If (1.2) holds for p-almost all paths γ, then g is said to be a *p-weak upper gradient* of u.

1.3. Newtonian spaces. Let $1 \le p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all functions $u \in L^p(X)$ that have a p-weak upper gradient $g \in L^p(X)$. This space is equipped with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients of u. When $p > 1$, by the uniform convexity of $L^p(X)$ we have that whenever $u \in N^{1,p}(X)$ there is a function

g_u in $L^p(X)$-convex hull formed by the set of all p-weak upper gradients of u, called the minimal p-weak upper gradient of u (or the minimal p-integrable p-weak upper gradient), so that g_u is a p-weak upper gradient of u and $\|g_u\|_{L^p(X)} = \inf \|g\|_{L^p(X)}$ where the infimum is over all p-weak upper gradients g of u; see [C] or [Sh1]. Indeed, in this case, if g is another p-integrable p-weak upper gradient of u, then $g_u \le g$ almost everywhere in X.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of the Newtonian spaces we refer to [Sh2]. We recall here some facts for the future reference. It can be shown that $N^{1,p}(X)$ is a Banach space. The functions in $N^{1,p}(X)$ are absolutely continuous on p-almost every path, which means that $u \circ \gamma$ is absolutely continuous on $[0, \mathrm{length}(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in X.

The *p-capacity* of a set $E \subset X$ is defined by

$$\mathrm{cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, whose restriction to E is bounded below by 1. Capacity is the natural measure for exceptional sets of Sobolev functions. It is easy to see that sets of zero capacity are also of measure zero, but the converse is not true in general. See [KM] for more properties of the capacity.

Let E be an arbitrary subset of X. Following the method of [KKM], we define $\widetilde{N}^{1,p}_0(E)$ to be the set of functions $u\colon E \to [-\infty, \infty]$ for which there exists a function $\widetilde{u} \in \widetilde{N}^{1,p}(X)$ such that $\widetilde{u} = u$ μ-almost everywhere in E and

$$\mathrm{cap}_p(\{x \in X \setminus E\colon \widetilde{u}(x) \neq 0\}) = 0.$$

Next we define an equivalence relation on $\widetilde{N}^{1,p}_0(E)$ by saying that $u \sim v$ if $u = v$ μ-almost everywhere on E. Finally we let $N^{1,p}_0(E) = \widetilde{N}^{1,p}_0(E)/\sim$, equipped with the norm

$$\|u\|_{N^{1,p}_0(E)} = \|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)},$$

be the *Newtonian space with zero boundary values.* The norm is unambiguously defined by [Sh1] and the obtained space is a Banach space consisting of equivalence classes, each class containing a representative that also belongs to $N^{1,p}(X)$.

1.4. Poincaré inequalities. A metric measure space X is said to be *doubling* if there is a constant $c_d \ge 1$ so that

$$\mu(B(z, 2r)) \le c_d \mu(B(z, r)) \tag{1.5}$$

for every open ball $B(z, r)$ in X. The constant c_d in (1.5) is called the doubling constant of μ.

Let $1 \le q < \infty$. In the following discussion, whenever $A \subset X$ is a measurable set, u_A denotes the quantity $\frac{1}{\mu(A)} \int_A u \, d\mu = \fint_A u \, d\mu$.

The space X is said to *support a weak* $(1,q)$*-Poincaré inequality* if there are constants $c_0 > 0$ and $\tau \ge 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \le c_0 r \Big(\fint_{B(z,\tau r)} g^q \, d\mu \Big)^{1/q} \tag{1.6}$$

for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,r)$ and for all upper gradients g of u. If $\tau = 1$, then X is said to support a (strong) $(1,q)$-Poincaré inequality.

A result of [HaK] shows that in a doubling measure space a weak $(1,q)$-Poincaré inequality implies a weak (t,q)-Poincaré inequality for some $t > q$ possibly with possibly different constants τ and c_0. More precisely, there exist constants $c > 0$ and $\tau' \ge 1$ such that

$$\Big(\fint_{B(z,r)} |u - u_{B(z,r)}|^t \, d\mu \Big)^{1/t} \le cr \Big(\fint_{B(z,\tau' r)} g^q \, d\mu \Big)^{1/q}, \tag{1.7}$$

where $1 \le t < Qq/(Q-q)$ if $q < Q$ and $t \ge 1$ if $q \ge Q$, for all balls $B(z,r)$ in X and for all integrable functions u in $B(z,r)$ and all upper gradients g of u. Moreover, in (1.6) and (1.7) we can replace the upper gradient g with any p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$, because of the result in [KoM] which states that every p-weak upper gradient in $L^p_{\mathrm{loc}}(X)$ can be approximated in $L^p(X)$ by an upper gradient in $L^p_{\mathrm{loc}}(X)$. Indeed, given any p-weak upper gradient $g_1 \in L^p_{\mathrm{loc}}(X)$ of u and any $\epsilon > 0$ we can find an upper gradient g_2 so that $\|g_2 - g_1\|_{L^p(X)} < \epsilon$.

1.8. Sobolev inequalities. Next we prove a Sobolev type inequality for functions which vanish on a large set. The paper [B] has a better capacitary version of this inequality, but for our purposes it suffices to consider the more easily proved version below (see [KSh, Lemma 2.1])

1.9. Lemma. *Let X be a doubling metric measure space supporting a weak $(1,q)$-Poincaré inequality for some $1 < q < p$. Suppose that $u \in N^{1,p}(X)$, and denote the minimal p-weak upper gradient g_u. Let $A = \{x \in B(z,r) : |u(x)| > 0\}$. If $\mu(A) \le \gamma \mu(B(z,r))$ for some γ with $0 < \gamma < 1$, then there is a constant $c > 0$ so that*

$$\Big(\fint_{B(z,r)} |u|^t \, d\mu \Big)^{1/t} \le cR \Big(\fint_{B(z,\tau' r)} g_u^q \, d\mu \Big)^{1/q},$$

where t and τ' are as in (1.7)*. The constant c depends only on γ and the constants c and τ' of* (1.7)*.*

We observe that Lemma 1.9 gives a Sobolev inequality for Sobolev functions with zero boundary values. To be more precise, there exists $c > 0$ so that for every ball $B(z,r)$ with $0 < r \le \operatorname{diam}(X)/3$ and every $u \in N^{1,p}_0(B(z,r))$ we have

$$\Big(\fint_{B(z,r)} |u|^t \, d\mu \Big)^{1/t} \le cR \Big(\fint_{B(z,r)} g_u^q \, d\mu \Big)^{1/q}. \tag{1.10}$$

This follows easily from Lemma 1.9 after noting that $g_u = 0$ almost everywhere on $X \setminus B(z,r)$ and by observing that there must be a point on the sphere $\partial B(z,2r)$. If there is no such point, then it is easy to construct a function which violates the Poincaré inequality.

1.11. Minimizers. Now we are ready to formulate the minimization problem for the p-Dirichlet integral in a metric measure space. By $N^{1,p}_{\mathrm{loc}}(\Omega)$ we mean the space of all functions $u \in L^p_{\mathrm{loc}}(\Omega)$ that have an upper gradient in $L^p_{\mathrm{loc}}(\Omega)$, where $L^p_{\mathrm{loc}}(\Omega)$ is the space of all measurable functions that are p-integrable on compact subsets of Ω.

Suppose that $\Omega \subset X$ is open and let $1 < p < \infty$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be p-harmonic on Ω if for every relatively compact open subset Ω' of Ω with $\overline{\Omega'} \subset \Omega$ and for every $v \in N^{1,p}(\Omega')$ with $u - v \in N^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu,$$

where g_u and g_v are the minimal weak upper gradients of u and v respectively. p-harmonic functions on metric measure spaces have been studied in [C], [Sh1], and [KSh].

Let $\vartheta \in N^{1,p}(\Omega)$ and $\psi \colon \Omega \to [-\infty, \infty]$ be an arbitrary function. Following [KM2], we define

$$\mathcal{K}_{\psi,\vartheta}(\Omega) = \{v \in N^{1,p}(\Omega) \colon v - \vartheta \in N^{1,p}_0(\Omega), v \ge \psi \ \mu\text{-a.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ is a *solution of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem with the obstacle ψ and the boundary values ϑ*, if

$$\int_{\Omega} g_u^p \, d\mu \le \int_{\Omega} g_v^p \, d\mu$$

for every $v \in \mathcal{K}_{\psi,\vartheta}(\Omega)$. Here g_u and g_v are the minimal p-weak upper gradients of u and v in Ω, respectively. In other words, the solution of the obstacle problem minimizes the p-Dirichlet energy among all functions which lie above the obstacle function ψ and have the boundary values ϑ. If $\psi = -\infty$, then the obstacle does not hinder and a solution of the $\mathcal{K}_{-\infty,\vartheta}(\Omega)$-obstacle problem is said to be a *minimizer with boundary values ϑ in Ω.*

It is an easy observation that whenever ϑ is in $N^{1,p}(X)$, the solution to the $\mathcal{K}_{\psi,\vartheta}$ obstacle problem is also in $N^{1,p}(X)$.

1.12. Definition. A function $u : \Omega \to (-\infty, \infty]$ is said to be *p-superharmonic* in Ω if

(i) u is lower semicontinuous and is not identically infinite on any component of Ω, and

(ii) for all relatively compact open subsets $V \subset\subset \Omega$ and for all functions $v \in N^{1,p}(X)$ that are p-minimizers in V, whenever $v \le u$ p-q.e. in $\Omega \setminus V$ we have $v \le u$ p-q.e. in V.

Remark. The second condition above can be replaced with the condition that for all relatively compact subsets $V \subset\subset \Omega$ and for all functions $v \in N^{1,p}(X) \cap \mathrm{Lip}(X)$ that are p-minimizers in V, whenever $v \le u$ p-q.e. in $\Omega \setminus V$ we have $v \le u$ p-q.e. in V, because of the fact that Lipschitz functions are dense in $N^{1,p}(X)$ (see [Sh1]) together with [Sh3,—]. If we can approximate general open sets by regular open sets for the Dirichlet problem, this definition coincides with the definition of p-superharmonic functions given in [HKM, Chapter 7].

Definition 1.1 is slightly different from the definition of p-superharmonicity given in [KM2], the difference being in the requirement of comparison with harmonic functions. Let Ω be an open subset of X. Suppose that $V \subset\subset \Omega$ is open and $v \in N^{1,p}(V)$. Let $h_V(v) = h(v)$ be the unique continuous p-minimizer in V with $v - h(v) \in N_0^{1,p}(V)$. Observe that every p-minimizer is locally Hölder continuous after redefinition on a set of measure zero by [KSh]. The definition given in [KM2] replaces the requirement (ii) of Definition 1.1 with the condition

(ii') for every relatively compact open set $V \subset\subset \Omega$ and for every function $v \in C(\overline{V}) \cap N^{1,p}(V)$, whenever $v \le u$ on ∂V we have that $h(v) \le u$ on V.

The object of the next lemma is to prove that the two definitions are indeed equivalent.

1.13. Lemma. *A lower semicontinuous function on Ω that is not identically infinite on any component of Ω is p-superharmonic if and only if it satisfies the condition* (ii') *above.*

Proof. First suppose that u is a p-superharmonic function, let $V \subset\subset \Omega$ and let $v \in C(\overline{V}) \cap N^{1,p}(V)$ such that $v \le u$ on ∂V. For each $\epsilon > 0$ there is a compactly supported Lipschitz function φ_ϵ on X so that $|v - \varphi_\epsilon| \le \epsilon$ on ∂V. Note that $\varphi_\epsilon \in N^{1,p}(X)$. Since $\min\{\varphi_\epsilon + \epsilon - v, 0\} \in N_0^{1,p}(V)$, and consequently $\min\{h(\varphi_\epsilon) + \epsilon - h(v), 0\} \in N_0^{1,p}(V)$, we have by the comparison theorem (see Lemma 3.4 in [KM2]) that

$$h(\varphi_\epsilon) - \epsilon \le h(v) \le h(\varphi_\epsilon) + \epsilon$$

on V. We observe that the function

$$h = \begin{cases} h(\varphi_\epsilon) & \text{on } V, \\ \varphi_\epsilon & \text{on } X \setminus V, \end{cases}$$

belongs to $N^{1,p}(X)$. Moreover, since $u \ge v \ge \varphi_\epsilon - \epsilon$ on ∂V and u is p-superharmonic, we have $u \ge h(\varphi_\epsilon) - \epsilon$ on V. Hence $u \ge h(v) - 2\epsilon$ on V. Letting $\epsilon \to 0$ we see that $u \ge h(v)$ on V.

Now suppose that u satisfies the condition (ii'). Then u is a p-superharmonic function in the sense of [KM2], and hence by [KM2, Corollary 7.8] the functions $u_i = \min\{u, i\}$, $i = 1, 2, \ldots$, are superminimizers on Ω. Let $V \subset\subset \Omega$ and $v \in N^{1,p}(X)$ so that v is a p-minimizer on V and $v \le u$ p-q.e. on $\Omega \setminus V$. By [Sh1, Lemma 4.9] the sequence $v_i = \min\{v, i\}$ is in $N^{1,p}(X)$ and $v_i \to v$ in $N^{1,p}(X)$. Thus by [Sh3, Theorem 1.1], $h(v_i)$ converges locally uniformly to $v = h(v)$ in V. Since $v \le u$ p-q.e. on ∂V, we have $v_i \le u_i$ p-q.e. on ∂V. From this it follows that $h(v_i) \le u_i \le u$ on V, see the discussion following the proof of Corollary 4.8 in [Sh3]. Letting $i \to \infty$, we see that $h(v) \le u$ on V, and hence u is a p-superharmonic function on Ω. □

From now on we assume without further notice that the metric measure space X is connected, proper, and is equipped with a doubling Borel regular measure for which the measure of every open set is positive and that the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$. We assume throughout this paper that Ω is a bounded open set in X with the property that $\mathrm{cap}_p(X \setminus \Omega) > 0$.

2. Polar sets are zero capacity sets.

2.1. Definition. A set $E \subset \Omega$ is said to be *p-polar* if there is a p-superharmonic function in Ω so that $u = \infty$ on E.

2.2. Proposition. *Let u be a p-superharmonic function on Ω, and let*

$$E = \{x \in \Omega \colon u(x) = \infty\}.$$

Then E has zero p-capacity.

Proof. Let F be any compact subset of E. It suffices to show that the p-capacity of F is zero.

Suppose that the p-capacity of F is positive. Let B be a relatively compact subdomain of Ω that contains F, and let v be the p-potential of the condenser $(F; B)$; that is, $v = 1$ on F, $v = 0$ on $X \setminus B$, $0 \le v \le 1$, v is p-minimizing on $B \setminus F$, and v is in $N^{1,p}(X)$. Since p-superharmonic functions are lower semicontinuous and hence are locally bounded from below, we may assume that u is non-negative on B. As $B \setminus F$ is a relatively compact open subset of Ω and $kv \le u$ on $\Omega \setminus (B \setminus F)$ for every $k = 1, 2, \ldots$, and u is a p-superharmonic function, we have $kv \le u$ on $B \setminus F$ for every $k = 1, 2, \ldots$ We can choose B so that there is a point $y \in B$ with $u(y)$ finite. Then y is in $B \setminus F$. Hence $0 \le kv(y) \le u(y) < \infty$ for every positive k. Hence $v(y) = 0$.

As the p-capacity of F is positive, u is a non-constant p-minimizing function on $B \setminus F$, and hence by the strong comparison theorem, $v(y) > 0$, thus providing the contradiction. □

2.3. Definition. A function $f \colon \Omega \to [-\infty, \infty]$ is *p-quasicontinuous* if for every $\epsilon > 0$ there exists an open set E such that $\mathrm{cap}_p(E) < \epsilon$ and the restriction of f to $\Omega \setminus E$ is continuous.

It should be noted that functions in $N^{1,p}(X)$ and $N^{1,p}_{\mathrm{loc}}(X)$ are p-quasicontinuous; see Remark 4.4 of [Sh2].

2.4. Corollary. *Let u be a p-superharmonic function on Ω. Then u is p-quasicontinuous.*

Proof. By [KM2, Corollary 7.8] the functions $u_i = \min\{u, i\}$, $i = 1, 2, \ldots$, are p-superminimizers in Ω, and in particular, they belong to $N^{1,p}_{\mathrm{loc}}(\Omega)$. Hence the functions u_i are p-quasicontinuous and the result follows from Proposition 2.2 and the fact that the p-capacity is a Choquet capacity; see [KM3,—]. Indeed, the sets $U_i = \{x \in \Omega : u(x) > i\}$ are open and $\mathrm{cap}_p(U_i) \to 0$ as $i \to \infty$. □

3. Zero capacity sets are polar sets.

3.1. Lemma. *Let $E \subset\subset \Omega$ such that $\mathrm{cap}_p(E) = 0$. Then there exists a sequence of functions $\varphi_i \in N^{1,p}(X)$ so that $\varphi_i = 0$ on $X \setminus \Omega$, $0 \le \varphi_i \le 1$, $\varphi_i = 1$ on E, φ_i is a p-superharmonic p-superminimizer on Ω, and $\varphi_i \to 0$ in $N^{1,p}(X)$ and φ_i monotonically decreases to zero p-q.e. in Ω.*

Proof. Since $\mathrm{cap}_p(E) = 0$, there exists a sequence of open sets $U_i \subset\subset \Omega$ with $E \subset U_i \subset U_{i-1}$ so that $\mathrm{cap}_p(U_i) \le 2^{-i}$. Let $\widetilde{\varphi}_i$ be the p-potential of the condenser $(U_i; \Omega)$, and let φ_i be the lower semicontinuous regularization of $\widetilde{\varphi}_i$ given by [KM2, Proposition 7.4]. The functions φ_i satisfy the conditions of the lemma. □

3.2. Proposition. *Let $\Omega \subset X$ be a bounded open set, and let $\psi_i \in N^{1,p}(X)$ be a p-q.e. monotonic decreasing sequence of non-negative functions so that $\psi_i \to \psi$ in $N^{1,p}(X)$. Let u_i be the solution to the $\mathcal{K}_{\psi_i,\psi_i}(\Omega)$-obstacle problem. Then there exists a function $u \in N^{1,p}(X)$ so that u_i monotonically decreases p-q.e. in Ω to u and u is the solution to the $\mathcal{K}_{\psi,\psi}(\Omega)$-obstacle problem.*

The above proposition is a complementary result to [KM2, Theorem 6.1]. To prove this proposition we need the following lemma. This lemma is from [KM2, Lemma 3.5].

3.3. Lemma. *Let $\psi \in N^{1,p}(X)$ be a non-negative function and let v be a solution to the $\mathcal{K}_{\psi,\psi}(\Omega)$-obstacle problem If $u \in N^{1,p}(X)$ is a p-superminimizer on Ω with $\min\{u, v\} \in \mathcal{K}_{\psi,\psi}(\Omega)$, then $u \geq v$ p-q.e. on Ω. Moreover, if $\widetilde{u}$ and $\widetilde{v}$ are the lower semicontinuous regularizations of u and v respectively, we have $\widetilde{u} \geq \widetilde{v}$ everywhere on Ω.*

Proof. For the proof of the first part, see [KM2, Lemma 3.5]. While their proof involves comparing u and v outside of a relatively compact subdomain of Ω, as u and v are in $N^{1,p}(X)$ the relative compact open subsets are unnecessary. Moreover, by [KM2, Theorem 7.14] we have

$$\widetilde{v}(x) = \operatorname{ess}\liminf_{y\to x} \widetilde{v}(y) \leq \operatorname{ess}\liminf_{y\to x} \widetilde{u}(y) = \widetilde{u}(x)$$

for every $x \in \Omega$. □

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. Since $u_i \geq \psi_i \geq \psi_{i+1}$, therefore $u_i \in \mathcal{K}_{\psi_{i+1},\psi_i}(\Omega)$. As u_i is a p-superminimizer on Ω and u_{i+1} is the solution to the obstacle problem with obstacle ψ_{i+1} and boundary data $\psi_{i+1} \leq \psi_i$, by Lemma 3.3 above we have $u_i \geq u_{i+1}$. Without loss of generality we may assume that u_i is lower semicontinuous in Ω; see [KM2, Theorem 5.1]. Hence $(u_i)_i$ is a monotone decreasing sequence. Let $u = \lim_{i\to\infty} u_n$. As $u_i \in L^p(\Omega)$ and $u \geq 0$, we have $u \in L^p(\Omega)$. Also, as $\psi_i \in \mathcal{K}_{\psi_i,\psi_i}(\Omega)$, we have

$$\int_\Omega g_{u_i}^p \, d\mu \leq \int_\Omega g_{\psi_i}^p \, d\mu.$$

Hence by [KaS, Lemma 3.1] we see that $u \in N^{1,p}(X)$ with

$$\int_\Omega g_u^p \, d\mu \leq \liminf_{i\to\infty} \int_\Omega g_{u_i}^p \, d\mu \leq \liminf_{i\to\infty} \int_\Omega g_{\psi_i}^p \, d\mu \leq \int_\Omega g_\psi^p \, d\mu.$$

Since $u_i \geq \psi_i \geq \psi$, we conclude that $u \geq \psi$ and hence $u \in \mathcal{K}_{\psi,\psi}(\Omega)$ because $u_i - \psi_i \in N_0^{1,p}(\Omega)$ and ψ_i converges to ψ in $N^{1,p}(X)$. Let v be the solution to the $\mathcal{K}_{\psi,\psi}(\Omega)$-obstacle problem. Then

$$\int_\Omega g_v^p \, d\mu \leq \int_\Omega g_u^p \, d\mu.$$

Also by Lemma 3.3 again, $u_i \geq v$, and hence $u \geq v$. Let $\varphi_i = \max\{v, \psi_i\}$. Then as $\psi \leq \psi_i$, $\varphi_i \in \mathcal{K}_{\psi_i,\psi_i}(\Omega)$, and hence

$$\int_\Omega g_{u_i}^p \, d\mu \leq \int_\Omega g_{\varphi_i}^p \, d\mu.$$

As ψ_i monotonically decreases to ψ p-q.e. in Ω and $v \geq \psi$, therefore $\varphi_i \to v$ pointwise p-q.e. in Ω. Moreover, as $\psi_i \to \psi$ in $N^{1,p}(X)$, we see that $\varphi_i \to v$ in $N^{1,p}(X)$. Therefore,

$$\liminf_{i\to\infty} \int_\Omega g_{u_i}^p \, d\mu \leq \int_\Omega g_v^p \, d\mu,$$

and hence,

$$\int_\Omega g_u^p \, d\mu \leq \int_\Omega g_v^p \, d\mu.$$

Thus we see that

$$\int_\Omega g_u^p \, d\mu = \int_\Omega g_v^p \, d\mu.$$

By the uniqueness of solutions to obstacle problems (see for example [KM2, Theorem 3.2] or [HKM, Theorem 3.21]) we have $u = v$; that is, u is the solution to the $\mathcal{K}_{\psi,\psi}(\Omega)$-obstacle problem. □

Now we state and prove the main theorem of this section.

3.4. Theorem. *Let $x_0 \in \Omega$ and E be a relatively compact subset of $\Omega \setminus \{x_0\}$ such that $\operatorname{cap}_p(E) = 0$. Then there exists a non-negative p-superharmonic function u on Ω so that $u(x_0) < \infty$, $u = 0$ on $X \setminus \Omega$, and*

$$E \subset \{x \in \Omega \colon u(x) = \infty\}.$$

This theorem was proved in the setting of $X = \mathbf{R}^n$ by Kilpeläinen; see [K1, Theorem 4.1]. The proof of this theorem follows the lead of [K1]; however, we avoid their use of Perron-Wiener-Brelot solutions by using Proposition 3.2 instead.

Proof. Let φ_i be the functions given by Lemma 3.1. There exists an index i_1 so that $\varphi_{i_1} \leq \frac{1}{2}$ in a neighbourhood of x_0 and $\varphi_{i_1} = 1$ on a neighbourhood of E. Let $\widetilde{v}_1$ be the lower semicontinuous regularization of the p-superminimizer φ_{i_1}; see [KM2, Theorem 5.1]. Note that as φ_{i_1} is already a p-superharmonic function as well, we have $\widetilde{v}_1 = \varphi_{i_1}$. We inductively construct functions $\widetilde{v}_n$ as follows.

Suppose $\widetilde{v}_1, \dots, \widetilde{v}_{n-1}$ have already been chosen so that they are p-superharmonic p-superminimizers in Ω, $\widetilde{v}_i = 0$ on $X \setminus \Omega$, $\widetilde{v}_i = i$ on a neighbourhood of E, and $\widetilde{v}_i(x_0) \leq \sum_{k=1}^{i} 2^{-k}$, and $\widetilde{v}_i \in N^{1,p}(X)$.

Let $v_{n,j}$ be the solution to the obstacle problem on Ω with obstacle $\widetilde{v}_{n-1} + \varphi_j$ and boundary data 0. Since $\varphi_j \to 0$ in $N^{1,p}(X)$ and φ_j monotonically decreases to 0 p-q.e., we see that $\widetilde{v}_{n-1} + \varphi_j$ monotonically decreases to $\widetilde{v}_{n-1}$ p-q.e. and in $N^{1,p}(X)$. Now by Proposition 3.2, $v_{n,j}$ monotonically decreases to $\widetilde{v}_{n-1}$ p-q.e., and hence its lower semicontinuous regularization monotonically decreases to $\widetilde{v}_{n-1}$ everywhere in Ω; see [KM2, Theorem 5.1]. Hence there is an index i_n so that the lower semicontinuous regularization of v_{n,i_n} is bounded by $\sum_{k=1}^{n} 2^{-k}$ at x_0, and as $v_{n,i_n} = n$ on a neighbourhood of E, the lower semicontinuous regularization of v_{n,i_n} also takes on the value of n in a neighbourhood of E. Set $\widetilde{v}_n$ to be this lower semicontinuous regularization of v_{n,i_n}.

The sequence $(\widetilde{v}_n)$ has the following properties:

- $\widetilde{v}_n \geq 0$ and is p-superharmonic in Ω,
- $\widetilde{v}_n = 0$ on $X \setminus \Omega$,
- $\widetilde{v}_n \in N^{1,p}(X)$,

- $\widetilde{v}_n = n$ on E,
- $\widetilde{v}_{n+1} \geq \widetilde{v}_n$ pointwise on Ω, and
- $\widetilde{v}_n(x_0) \leq \sum_{k=1}^{n} 2^{-k}$.

Let $u = \lim_{n\to\infty} \widetilde{v}_n$. By [KM2, Lemma 7.1] and by Lemma 1.13 above, since $u(x_0) \leq \sum_{k=1}^{\infty} 2^{-k}$ is finite, u is p-superharmonic on Ω. Moreover, it is easy to see that $u = \infty$ on E. □

Remark. In the Euclidean setting, in the proof of Lemma 3.1 we can approximate Ω by regular open sets $\Omega_j \subset \Omega$ and the open sets U_j by regular open sets $\widetilde{U}_j$; this technique has been used extensively in [HKM] to study the Perron-Wiener-Brelot solutions. In this case, we can ensure that the functions φ_i are all continuous, and then it follows from [KM2, Theorem 5.5] that the functions $\widetilde{v}_i$ constructed in the above proof are all continuous. Unfortunately, in the general setting of metric spaces considered in this note, such approximations are not known to be possible.

4. The existence of global p-superharmonic maps defining p-polar sets in a p-hyperbolic metric space.

Following [Ho], we say that a metric space is *p-hyperbolic* if there exists a compact set $C \subset X$ so that the relative capacity $\mathrm{Cap}_p(C\colon X)$ is positive. Here

$$\mathrm{Cap}_p(C\colon X) = \lim_{i\to\infty} \mathrm{Cap}_p(C\colon \Omega_i),$$

where $\{\Omega_i\}_i$ is an increasing sequence of bounded domains in X such that $X = \cup_i \Omega_i$. It is easily seen that a metric space is p-hyperbolic if and only if it is not possible to find a sequence of compactly supported functions $f_i \in N^{1,p}(X)$ so that $f_i \to 1$ in $N^{1,p}_{\mathrm{loc}}(X)$ while their energies $\int_X g_{f_i}^p \, d\mu$ remain bounded.

4.1. Theorem. *If X is a p-hyperbolic metric measure space equipped with a doubling measure supporting a $(1,q)$-Poincaré inequality for some $1 \leq q < p$, and if $E \subset X$ is a compact set so that $\mathrm{cap}_p(E) = 0$, then there exists a non-negative function u on X so that u is p-superharmonic in X and $E \subset \{x \in X\colon u(x) = \infty\}$.*

The proof of this theorem follows along the same lines as the proof of Theorem 3.4, once an analog of Lemma 3.1 and Proposition 3.2 are proven in the setting $\Omega = X$.

4.2. Definition. In what follows, the family $N^{1,p}_{\mathrm{loc},0}(X)$ denotes the collection of all functions f on X for which there exists a sequence of compactly supported functions $f_i \in N^{1,p}(X)$ so that $f_i \to f$ in $N^{1,p}_{\mathrm{loc}}(X)$ and the corresponding sequence of energies $\int_X g_{f_i}^p \, d\mu$ is bounded independent of i. Weak limits of the functions g_{f_i} can be easily seen to be p-weak upper gradients of f in X. *From the fact that $f_i \to f$ in $N^{1,p}_{\mathrm{loc}}(X)$ we can infer that $g_{f_i} \to g_f$ in $L^p(X)$ and that $g_f \in L^p(X)$.*

It is easy to see that if $\lambda > 0$ and $f, h \in N^{1,p}_{\mathrm{loc},0}(X)$ then the functions $\min\{\lambda, f\}$ and $\max\{f, h\}$ are in $N^{1,p}_{\mathrm{loc},0}(X)$. Furthermore, $N^{1,p}_{\mathrm{loc},0}(X)$ is a vector space endowed with a family of seminorms inherited from $N^{1,p}_{\mathrm{loc}}(X)$.

4.3. Definition. Let $f \in N^{1,p}_{\mathrm{loc}}(X)$, and let $\mathcal{K}_f$ denote the collection of all functions $u \in N^{1,p}_{\mathrm{loc}}(X)$ such that $u - f \in N^{1,p}_{\mathrm{loc},0}(X)$ and $u \geq f$ p-q.e. in X. By *a solution to* *the obstacle problem* $\mathcal{K}_f$ we mean a function $u \in \mathcal{K}_f$ so that

$$\int_X g_u^p \, d\mu \le \int_X g_v^p \, d\mu$$

whenever $v \in \mathcal{K}_f$. Observe that

$$\int_X g_u^p \, d\mu \le \int_X g_f^p \, d\mu.$$

4.4. Lemma. *Let X be a p-hyperbolic metric space. Given $E \subset X$ so that $\operatorname{cap}_p(E) = 0$ there exists a sequence of functions $\varphi_i \in N^{1,p}_{\mathrm{loc},0}(X)$ so that $\varphi_i = 1$ on a neighbourhood of E, φ_i is a p-superharmonic p-superminimizer on X, $\int_X g_{\varphi_i}^p \, d\mu \to 0$ as $i \to \infty$, and φ_i monotonically decreases to 0 pointwise p-q.e. in X.*

The proof of Lemma 4.4 depends on the following lemma.

4.5. Lemma. *Let X be a p-hyperbolic space equipped with a doubling measure supporting a $(1,p)$-Poincaré inequality. Given a bounded function $f \in N^{1,p}_{\mathrm{loc}}(X)$ with finite energy there is one, and only one, solution to the obstacle problem $\mathcal{K}_f$ in the class $N^{1,p}_{\mathrm{loc},0}(X)$. Such a solution is p-superharmonic and is a p-supersolution in X.*

Proof. We denote the bound of f by M. Let Ω_i be an increasing sequence of bounded domains in X so that $X = \cup_i \Omega_i$. As $\overline{\Omega_i}$ is a compact set and $N^{1,p}_{\mathrm{loc},0}(X) \subset N^{1,p}_{\mathrm{loc}}(X)$, we see that the solution u_i to the obstacle problem $\mathcal{K}_{f,f}(\Omega_i)$ exists and lies in $N^{1,p}_{\mathrm{loc},0}(X)$. By Lemma 3.3 we see that $f \le u_i \le u_{i+1} \le M$ p-q.e. on X.

On the other hand, by the energy minimization property of u_i, we have

$$\int_X g_{u_i}^p \, d\mu = \int_{\Omega_i} g_{u_i}^p \, d\mu + \int_{X \setminus \Omega_i} g_f^p \, d\mu \le \int_X g_f^p \, d\mu < \infty,$$

where we know that the last integral in the above chain of inequalities is finite by assumption. Hence the sequence u_i has the property that $\int_X g_{u_i}^p \, d\mu$ is bounded independent of i. Thus the limit function $u = \lim_{i\to\infty} u_i$ lies in the collection $N^{1,p}_{\mathrm{loc}}(X)$. Observe that $u_i \in N^{1,p}_{\mathrm{loc}}(X)$ and that $\|u_i\|_{N^{1,p}(\Omega_k)}$ are bounded above by a constant independent of i. Therefore, as $u - f = \lim_i (u_i - f)$ with $u_i - f$ of compact support, we see that $u \in \mathcal{K}_f$.

Suppose $v \in \mathcal{K}_f$. Then as $v \ge f$, there exists a sequence of non-negative functions $w_i \in N^{1,p}(X)$ with compact support so that $v = \lim_i (w_i + f)$ and (because of Mazur's lemma) so that

$$\int_X g_v^p \, d\mu = \lim_i \int_X g_{w_i+f}^p \, d\mu.$$

Fix $\epsilon > 0$ and choose w_i so that

$$\int_X g_v^p \, d\mu \ge \int_X g_{w_i+f}^p \, d\mu - \epsilon.$$

Since $X = \cup_j \Omega_j$, we can find Ω_j so that the compact support of w_i is contained in Ω_j; thus $w_i + f \in \mathcal{K}_{f,f}(\Omega_j)$ for sufficiently large j. Now by the minimal energy property of u_j, we see that

$$\int_X g_{u_j}^p \, d\mu \le \int_X g_{w_i+f}^p \, d\mu.$$

Hence for sufficiently large j we have that

$$\int_X g_{u_j}^p \, d\mu \le \int_X g_v^p \, d\mu + \epsilon.$$

Letting $j \to \infty$ and then $\epsilon \to 0$, we see that

$$\int_X g_u^p \, d\mu \le \int_X g_v^p \, d\mu,$$

that is, u is a solution to the obstacle problem $\mathcal{K}_f$.

Since the monotone increasing sequence of functions $(u_i)_{i \ge j}$ are p-superharmonic and are p-supersolutions in Ω_j, by [KM2] we see that u is itself such a function on Ω_j for every j; that is, u is p-superharmonic and is a p-supersolution in X.

It now remains to show that no other solution to the obstacle problem exists. Suppose v is another solution to the obstacle problem $\mathcal{K}_f$. Then as $v \ge f$ p-q.e. on X, we see by an application of Lemma 3.3 to the domains Ω_j that $v \ge u_j$ for each j. It follows that $v \ge u$ p-q.e. on X. As $v \not\equiv u$ and X supports a $(1,p)$-Poincaré inequality (and hence is an MEC_p-space, see [Sh2]), we have $g_{v-u} \not\equiv 0$ or else $v - u$ is a non-zero constant function. By the results in [Sh2], the support of g_{v-u} is contained in the support of $v - u$ itself. As $u \not\equiv v$, we see that $\mu(\{x \in X : u(x) < v(x)\}) > 0$. Note that the set $\{x \in X : u(x) < v(x)\}$ is a countable union of sets of the form $\{x \in X : u(x) < a < v(x)\}$ where $a \in \mathbf{R}$ is a positive rational number. Let a be a positive rational number such that $\mu(\{x \in X : u(x) < a < v(x)\}) > 0$, and define

$$h(x) = \begin{cases} v(x), & \text{if } v(x) \le a, \\ a, & \text{if } u(x) < a < v(x), \\ u(x), & \text{if } u(x) \ge a. \end{cases}$$

It is easily seen that $h \in \mathcal{K}_f$, and hence by the minimality of the energy of the solutions,

$$\int_X g_u^p \, d\mu = \int_X g_v^p \, d\mu \le \int_X g_h^p \, d\mu = \int_{\{v \le a\}} g_v^p \, d\mu + \int_{\{u \ge a\}} g_u^p \, d\mu.$$

By the above inequality,

$$\int_{\{v \le a\}} g_v^p \, d\mu = \int_{\{v \le a\}} g_u^p \, d\mu \quad \text{and} \quad \int_{\{u \ge a\}} g_v^p \, d\mu = \int_{\{u \ge a\}} g_u^p \, d\mu.$$

Hence

$$\int_X g_h^p \, d\mu = \int_X g_u^p \, d\mu - \int_{\{u < a < v\}} g_u^p \, d\mu \ge \int_X g_u^p \, d\mu,$$

which leads us to the conclusion that $g_u = 0$ almost everywhere in the set $\{x \in X\colon u(x) < a < v(x)\}$, and a similar argument shows that $g_v = 0$ almost everywhere on this set. Thus $g_{v-u} \le g_v + g_u = 0$ almost everywhere on the set $\{x \in X\colon u(x) < a < v(x)\}$. Thus $g_{v-u} = 0$ almost everywhere on the set $\{x \in X\colon u(x) < v(x)\}$. Hence it follows that $u + c \equiv v$ for some constant $c > 0$. On the other hand, as $u - f, v - f \in N^{1,p}_{\mathrm{loc},0}(X)$ we have $v - u \in N^{1,p}_{\mathrm{loc},0}(X)$. Hence as X is p-hyperbolic, we have $c = 0$, yielding the uniqueness result for the solution to the obstacle problem $\mathcal{K}_f$. □

Proof of Lemma 4.4. By Lemma 3.1 we obtain a sequence of compactly supported functions $\phi_i \in N^{1,p}(X)$ so that $\phi_i = 1$ on E,

$$\int_X g_{\phi_i}^p \, d\mu \le 2^{-ip},$$

and ϕ_i monotonically decreases to zero p-q.e. in X. Let φ_i be the solution to the $\mathcal{K}_{\phi_i}$-obstacle problem constructed in the proof of Lemma 4.5. As ϕ_i is compactly supported, $\varphi_i \in N^{1,p}_{\mathrm{loc},0}(X)$ with

$$\int_X g_{\varphi_i}^p \, d\mu \le 2^{-ip}.$$

Moreover, by Lemma 4.5 we see that φ_i are p-superharmonic p-supersolutions in X.

As ϕ_i is monotonic decreasing, by construction and by Lemma 3.3 we see that $(\varphi_i)_i$ is a monotonic decreasing sequence in X. Let $\varphi := \lim_i \varphi_i$. Since

$$\int_X g_\varphi^p \, d\mu \le \int_X g_{\varphi_i}^p \, d\mu \le 2^{-ip},$$

we see that $g_\varphi = 0$ almost everywhere in X. As X supports a $(1,p)$-Poincaré inequality (and hence is an MEC_p-space; see [Sh2]), we see that $\varphi \ge 0$ is constant p-a.e. in X. To see that this constant is zero, it suffices to know that $\varphi \in N^{1,p}_{\mathrm{loc},0}(X)$.

Indeed, as ϕ_i has compact support, we see that $\varphi_i \in N^{1,p}_{\mathrm{loc},0}(X)$. As φ_i are bounded by 1, we see that the sequence $\phi_{i,j}$, obtained as the solution to the obstacle problem $\mathcal{K}_{\phi_i,\phi_i}(\Omega_j)$, is a bounded sequence in $N^{1,p}_{\mathrm{loc}}(X)$. Fix one such relatively compact domain Ω_k. Hence by Mazur's lemma we can choose a subsequence of convex combination $(\widehat{\phi}_{i,j})_j$ of the sequence $(\phi_{i,j})_j$ so that $\widehat{\phi}_{i,j} \to \varphi_i$ in $N^{1,p}(\Omega_k)$ for every relatively compact domain Ω_k in X. Fix $\epsilon > 0$. Then for every positive integer i we can find a cutoff integer j_i so that whenever $j \ge j_i$ we have $\|\varphi_i - \widehat{\phi}_{i,j}\|_{N^{1,p}(\Omega_k)} < \epsilon/2$. Similarly, as $\varphi_i \to \varphi$ in a monotonic decreasing fashion and is a bounded sequence in $N^{1,p}(\Omega_k)$, we can obtain a convex combination subsequence

$$\widetilde{\phi}_n = \sum_{i=n}^{N_n} \lambda_{i,n} \varphi_i$$

that converges in $N^{1,p}(\Omega_k)$ to φ. Thus a corresponding convex combination of the sequence $\widehat{\phi}_{i,j}$ converges in $N^{1,p}(\Omega_k)$ to φ. More specifically, with $\epsilon > 0$ fixed as above, we can find a positive integer i_0 so that $\|\widetilde{\varphi}_{i_0} - \varphi\|_{N^{1,p}(\Omega_k)} < \epsilon/2$. Choosing $j_0 = \max\{j_{i_0}, j_{i_0+1}, \dots, j_{N_{i_0}}\}$ and letting

$$\widetilde{\phi}_{i_0} := \sum_{i=i_0}^{N_{i_0}} \lambda_{i,i_0} \phi_i,$$

it is easy to see that $\|\widetilde{\phi}_{i_0} - \widetilde{\varphi}_{i_0}\|_{N^{1,p}(\Omega_k)} < \epsilon/2$. Thus we obtain an approximation $\widetilde{\phi}_{i_0}$ of φ so that this approximation is compactly supported, in $N^{1,p}(X)$, and in addition, $\|\widetilde{\phi}_{i_0} - \varphi\|_{N^{1,p}(\Omega_k)} < \epsilon$. We can now conclude that $\varphi \in N^{1,p}_{\mathrm{loc},0}(X)$, and hence cannot be a non-zero constant. Thus $\varphi = 0$, and the proof is complete. □

Remark. The above proof also shows that if $u_i \in N^{1,p}_{\mathrm{loc},0}(X)$ is a sequence that converges to a function $u \in N^{1,p}_{\mathrm{loc}}(X)$, with the convergence occuring in $N^{1,p}_{\mathrm{loc}}(X)$, and furthermore, the energies $\int_X g_{u_i}^p \, d\mu$ are bounded independent of i, then u is also in $N^{1,p}_{\mathrm{loc},0}(X)$. Hence $N^{1,p}_{\mathrm{loc},0}(X)$ is a complete space.

4.6. Proposition. *Let $\psi_i \in N^{1,p}_{\mathrm{loc},0}(X)$ be a sequence of p-q.e. monotone decreasing sequence of non-negative functions so that $\psi_i \to \psi \in N^{1,p}_{\mathrm{loc},0}(X)$, the convergence occuring also in $N^{1,p}(X)$. Let u_i be the solution to the obstacle problem $\mathcal{K}_{\psi_i}$. Then $u_i \to u$ both monotonically p-q.e. in X and weakly in $N^{1,p}_{\mathrm{loc}}(X)$, where u is the solution to the obstacle problem $\mathcal{K}_{\psi}$.*

The proof of Proposition 4.6 is along the same lines as that of Proposition 3.2.

Proof. Lemma 3.3 together with the proof of Lemma 4.5 shows that $u_i \ge u_{i+1}$ p-q.e. since $\psi_i \ge \psi_{i+1}$. Let $u = \lim_i u_i$. As $u_i \ge \psi_i \ge \psi$, we see that $u \ge \psi$. Furthermore, as $\int_X g_{\psi_i}^p \, d\mu$ is bounded independent of i, we see that so is the sequence of energies $\int_X g_{u_i}^p \, d\mu$. Hence by the remark following the proof of Lemma 4.5, $u \in N^{1,p}_{\mathrm{loc},0}(X)$ and thus we can conclude that $u \in \mathcal{K}_\psi$. Moreover, as

$$\int_X g_{u_i}^p \, d\mu \le \int_X g_{\psi_i}^p \, d\mu \to \int_X g_\psi^p \, d\mu,$$

we see that

$$\int_X g_u^p \, d\mu \le \int_X g_\psi^p \, d\mu.$$

Let v denote the solution to the obstacle problem $\mathcal{K}_\psi$ (Lemma 4.5 guarantees such a solution). From the fact that $u \in \mathcal{K}_\psi$ we see that

$$\int_X g_u^p \, d\mu \ge \int_X g_v^p \, d\mu.$$

By the uniqueness of solutions to obstacle problems (Lemma 4.5), it therefore suffices to show that

$$\int_X g_u^p \, d\mu \le \int_X g_v^p \, d\mu.$$

To this end, as in the proof of Proposition 3.2, let $\varphi_i = \max\{v, \psi_i\}$. Then $\varphi_i \in N^{1,p}_{\mathrm{loc},0}(X)$ and $\varphi_i \ge \psi_i$. Therefore, $\varphi_i \in \mathcal{K}_{\psi_i}$ and

$$\int_X g_{u_i}^p \, d\mu \le \int_X g_{\varphi_i}^p \, d\mu.$$

On the other hand, as $\psi_i \to \psi$ in $N^{1,p}(X)$ and $v \geq \psi$, we see that $\varphi_i \to v$ in $N^{1,p}(X)$; that is,

$$\int_X g_{\varphi_i}^p \, d\mu \to \int_X g_v^p \, d\mu.$$

Therefore

$$\int_X g_u^p \, d\mu \leq \liminf_i \int_X g_{u_i}^p \, d\mu \leq \int_X g_v^p \, d\mu,$$

completing the proof of the proposition. □

Proof of Theorem 4.1. The proof of Theorem 4.1 follows the exact lines as the proof of Theorem 3.4, with Proposition 4.6 playing the role of Proposition 3.2 and Lemma 3.1, Lemma 4.5, and Lemma 4.4 together playing the role of Lemma 3.1. We leave the details up to the reader. □

5. Whitney decomposition and sweeping of measures.

The aim of the next section is to generalize the result of [K2, Theorem 3.1] to the setting of metric spaces considered in this note. The proof in [K2] used the classical Whitney decomposition of the complement of a closed polar set. One of the primary characteristics of a Whitney decomposition is that the corresponding covering family of cubes is pairwise disjoint. In our setting we are unable to obtain this characteristic, but the decomposition we use in this section has a bounded overlap property that compensates for this shortcoming.

5.1. Definition. Let K be a compact subset of X. A countable cover $\mathcal{F}$ of balls in $X \setminus K$ is said to be a *Whitney decomposition* of $X \setminus K$ if there is a constant $C > 0$ so that the following three conditions hold:

(1) $\sum_{B\in\mathcal{F}} \chi_{2B} \leq C$;
(2) $\frac{1}{2}\mathrm{rad}(B) \leq \mathrm{dist}(B, K) \leq 8\,\mathrm{rad}(B)$ for every $B \in \mathcal{F}$;
(3) $2B \subset X \setminus K$ and $\bigcup_{B\in\mathcal{F}} B = X \setminus K$.

The following proposition demonstrates the existance of such a decomposition.

5.2. Proposition. *Given a compact subset K of X, there exists a Whitney decomposition $\mathcal{F}$ of $X \setminus K$.*

We refer to the proof of [CW, Theorem 1.3] in page 70 for a proof of this proposition.

Following [K2] and [Mi], for every compactly supported Radon measure ν on X we define the following Wolff potential.

5.5. Definition. Given a Radon measure ν on X, the *Wolff potential* of ν is the function

$$W_p^\nu(x, r) := \int_0^r \left(t^p \frac{\nu(B(x,t))}{\mu(B(x,t))} \right)^{1/(p-1)} \frac{dt}{t}$$

for every $x \in X$ and $r > 0$.

Following [K2, Theorem 2.15], we prove the existence of a "swept" measure of ν supported on K for every finite Radon measure ν on X and any compact set $K \subset X$.

5.6. Lemma. *Let $K \subset X$ be a compact set and ν be a finite Radon measure on X (that is, $\nu(X)$ is finite). Then there exists a finite Radon measure $\widetilde{\nu}$, supported on K, so that $\nu(X)/C \le \widetilde{\nu}(X) \le C\nu(X)$ and for every point $x \in K$ and $r > 0$,*

$$W_p^{\widetilde{\nu}}(x, 21r) \ge \frac{1}{C} W_p^{\nu}(x, r)$$

where $C > 0$ is independent of x and r.

Proof. Let $\mathcal{F}$ be the Whitney decomposition of $X \setminus K$ given by Proposition 5.2. For each ball $B \in \mathcal{F}$, choose a point $x_B \in K$ so that $d(x_B, B) = \operatorname{dist}(K, B)$ (this is possible because K is compact). By the definition of Whitney decompositions, we see that $d(x_B, B) \le 8 \operatorname{rad}(B)$. We now construct one possible measure $\widetilde{\nu}$ as follows: for sets $E \subset X$, let

$$\widetilde{\nu}(E) := \nu(E \cap K) + \sum_{B \in \mathcal{F}} \nu(B) \delta_{x_B}(E),$$

where δ_{x_B} is the Dirac mass supported at the point x_B. Clearly $\widetilde{\nu}$ is a Radon measure supported inside K, and by the bounded overlap property of $\mathcal{F}$ and by the covering property of $\mathcal{F}$, it is easily seen that $\nu(X)/C \le \widetilde{\nu}(X) \le C\nu(X)$ where C is the constant associated with the bounded overlap property of $\mathcal{F}$.

Let $x \in K$ and $r > 0$, and consider the family $\mathcal{F}(x, r)$ to be the collection of all balls $B \in \mathcal{F}$ which intersect $B(x, r)$. If $B \in \mathcal{F}(x, r)$ and $y \in B \cap B(x, r)$, then

$$\begin{aligned} d(x_B, y) &\le d(x_B, B) + 2 \operatorname{rad}(B) \le 10 \operatorname{rad}(B) \\ &\le 20 \operatorname{dist}(B, K) \le 20\, d(x, y) \le 20r. \end{aligned}$$

Therefore, $d(x_B, x) \le 21r$, and hence

$$\begin{aligned} \nu(B(x, r)) &\le \nu\left((B(x, r) \cap K) \bigcup_{B \in \mathcal{F}(x, r)} B \right) \\ &\le \nu(B(x, r) \cap K) + \sum_{B \in \mathcal{F}(x, r)} \nu(B) \\ &\le \nu(B(x, r) \cap K) + \sum_{x_B \in B(x, 21r)} \nu(B) = \widetilde{\nu}(B(x, 21r)). \end{aligned}$$

Now from the definition of Wolff potential and by the doubling property of the measure μ,

$$\begin{aligned} W_p^{\widetilde{\nu}}(x, 21r) &= \int_0^{21r} \left(t^p \frac{\widetilde{\nu}(B(x, t))}{\mu(B(x, t))} \right)^{1/(p-1)} \frac{dt}{t} \\ &\ge \int_0^{21r} \left(t^p \frac{\nu(B(x, t/21))}{\mu(B(x, t))} \right)^{1/(p-1)} \frac{dt}{t} \\ &\ge \frac{1}{C} \int_0^{r} \left(t^p \frac{\nu(B(x, t))}{\mu(B(x, t))} \right)^{1/(p-1)} \frac{dt}{t} = W_p^{\nu}(x, r), \end{aligned}$$

where the constant C depends on the doubling constant of μ but is indeed independent of x and r. $\square$

6. Finite Radon measures and p-superharmonic functions.

The p-superharmonic functions, defined using p-minimizers, are influenced by what is meant by p-minimizers. If, as in [KM2], [KSh], [Sh1], and in the first four sections of this note, we define p-minimizers v to be functions that locally minimize the integral $\int g_w^p \, d\mu$ amongst the class of all functions $w \in N_0^{1,p}(V) + v$, where g_v is the minimal p-weak upper gradient of v, then the corresponding p-minimizers may or may not satisfy an Euler-Lagrange equation. This leads to a disadvantage in that we cannot explicitly identify the space of all finite Radon measures on Ω with the corresponding space of non-negative p-superharmonic functions on Ω as in [Mi] or [BMS]. To address this shortcoming, in this section we modify the definition of p-minimizers by using the results of [C].

In [C] Cheeger gives another definition of a Sobolev type space, but for indices $p > 1$ Cheeger's construction yields the same space as $N^{1,p}(X)$; see [Sh2, Theorem 4.10]. It is a deep theorem of Cheeger that if a metric measure space supports a doubling measure and a $(1,p)$-Poincaré inequality, then the corresponding Sobolev-type space is reflexive; see [C, Theorem 4.48]. The results of [C] can easily be extended to the case of spaces supporting a locally Q-regular measure and a local $(1,p)$-Poincaré inequality.

In [C, Theorem 4.38] it was shown that if the measure on X is doubling and if X supports a $(1,p)$-Poincaré inequality for some p, then there is a "differentiable" structure on X compatible with the notion of upper gradient. More precisely, the following theorem of Cheeger holds.

6.1. Theorem. *Let X be a metric measure space equipped with a locally doubling Borel regular measure μ. Assume that X admits a weak $(1,p)$-Poincaré inequality for some $1 < p < \infty$. Then there exist measurable sets U_α with $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$ and there exist Lipschitz "coordinate" functions $\overline{X}^\alpha = (X_1^\alpha, \dots, X_{k(\alpha)}^\alpha) \colon X \to \mathbf{R}^{k(\alpha)}$ so that*

(i) *for each α, the functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ are all linearly independent on U_α and $1 \le k(\alpha) \le N$, where N is a constant depending only on the doubling constant of μ and the constants from the Poincaré inequality,*

(ii) *if $f : X \to \mathbf{R}$ is Lipschitz, then there exist unique bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbf{R}^{k(\alpha)}$ such that*

$$\lim_{r \to 0+} \sup_{x \in B(x_0, r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0) \cdot (\overline{X}^\alpha(x) - \overline{X}^\alpha(x_0))|}{r} = 0$$

for μ-a.e. $x_0 \in U_\alpha$.

In other words, there exists a finite dimensional L^∞-vector bundle of Banach spaces $T^*X = \{F_x\}_{x \in X}$, the *generalized cotangent bundle*, and a derivation operator D on the algebra of locally Lipschitz functions on X taking values in the sections of T^*X. For x in the "coordinate chart" U_α, the fiber F_x is the dual space to the space $\mathbf{R}^{k(\alpha)}$ mentioned in the theorem. Given a Newtonian function f, a norm on the space F_x is given by

$$||\lambda||_{\alpha,x} = g_{\langle \lambda, \overline{X}^\alpha \rangle}(x),$$

where, for $\lambda = (\lambda_1, \dots, \lambda_{k(\alpha)})$ in $\mathbf{R}^{k(\alpha)}$ we set $\langle \lambda, \overline{X}^\alpha \rangle$ to be the Lipschitz function given by $\sum_{i=1}^{k(\alpha)} \lambda_i X_i^\alpha$. By linear independence of $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ we mean that whenever $g_{\langle\lambda,\overline{X}^\alpha\rangle}(x) = 0$ for some $x \in U_\alpha$ we have $\lambda = 0$. Finally, in a given chart U_α, Df is given by $d^\alpha f$.

Note that the generalized cotangent bundle is only a measurable vector bundle, that is, the charts U_α are measurable subsets of X and the change of coordinate functions $\overline{X}^\beta \circ (\overline{X}^\alpha)^{-1}$ are bimeasurable invertible transformations.

By [Sh2, Theorem 4.10] and [C, Theorem 4.47], the Newtonian space $N^{1,p}(X)$ is equal to the closure in the $N^{1,p}(X)$-norm of the collection of locally Lipschitz functions on X, and by [FHK, Theorem 10], the derivation operator D can be extended to all of $N^{1,p}(X)$ so that

$$|Df(x)| \approx g_f(x)$$

for all $f \in N^{1,p}(X)$ and μ-a.e. $x \in X$. Here the norm $|\cdot|$ is obtained from a measurable inner product on T^*X, chosen to validate the above equivalence with g_f; see the discussion in [C].

One easily verifies (see also [C, p. 458]) that the derivation operator Du satisfies the product and chain rules: if v is a bounded Lipschitz function on X, $u \in N^{1,p}(X)$, and $\Phi : \mathbf{R} \to \mathbf{R}$ is continuously differentiable with bounded derivative, then uv and $\Phi(u)$ belong to $N^{1,p}(X)$ and

$$\begin{aligned} D(uv) &= u\, Dv + v\, Du, \\ D(\Phi \circ u) &= \Phi'(u)\, Du. \end{aligned}$$

We modify the notion of p-harmonicity of [KM2], [Sh2], and [KSh] by minimizing $|Du|$ rather than g_u.

6.2. Definition. Let $\Omega \subset X$ be an open set. A function $u : X \to [-\infty, \infty]$ is said to be *p-harmonic* on Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and for all relatively compact subsets U of Ω and for all functions $\varphi \in N^{1,p}_0(U)$,

$$\int_U |Du|^p \, d\mu \le \int_U |D(u + \varphi)|^p \, d\mu.$$

It must be noted that by the results in [C], the p-harmonic functions as defined above are p-quasiminimizers in the sense of [KSh]. Moreover, p-harmonic functions can be characterized in terms of a weak formulation of the appropriate Euler-Lagrange equation: f is p-harmonic if and only if

$$\int_U |Du|^{p-2} Du \cdot D\phi \, d\mu = 0$$

for all U and φ as in Definition 6.2. Here we make use of the fact that the norms $|\cdot|$ on the fibers of T^*X can be chosen to be inner product norms; see the discussion in [C, p. 460].

6.3. Definition. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a *p-supersolution* if for all relatively compact open subset V of Ω and for all functions $\varphi \in N^{1,p}_0(V)$ so that $\varphi \ge 0$,

$$\int_V |Du|^{p-2} Du \cdot D\varphi \, d\mu \ge 0,$$

or equivalently,

$$\int_V |Du|^p \, d\mu \le \int_V |D(u+\varphi)|^p \, d\mu.$$

Remarks.

(i) A function is p-harmonic if and only if it is both a p-supersolution and a p-subsolution (a function u is a p-subsolution if $-u$ is a p-supersolution). For the remainder of this section we consider the p-superharmonic functions obtained from Definition 1.12 by using p-harmonic functions instead of p-minimizers in the comparison tests. We also define solutions to obstacle problems by minimizing the integral $\int |Du|^p \, d\mu$ rather than $\int g_u^p \, d\mu$.

(ii) Since p-harmonic functions and p-supersolutions exhibit the minimizing property of p-minimizers and p-superminimizers respectively, the results of the first three sections of this note as well as the results of [Sh1], [KM2], and [KSh] apply to the modified versions of p-superharmonic functions.

(iii) The above definition of p-harmonic functions and p-supersolutions has been studied in [BMS] using the techniques developed in [Mi], and it has been shown in [BMS] that the set of all non-negative p-supersolutions $u \in N_0^{1,p}(\Omega)$ and the Radon measures ν in the positive cone in the dual space $N_0^{1,p}(\Omega)^*$ are in one-to-one correspondence given by

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu \tag{6.4}$$

whenever $\varphi \in N_0^{1,p}(\Omega)$.

We extend the results of [BMS] and construct a correspondence between compactly supported finite Radon measures on Ω and non-negative p-superharmonic functions with vanishing boundary values, thus extending the results of [Mi] to the metric space setting. We will use this correspondence together with the sweeping technique of the previous section to construct a p-superharmonic function on Ω whose infinity set is precisely the given zero capacity set.

We first show that given a p-superharmonic function there is an associated finite Radon measure ν supported on Ω, related to u via equation (6.4) above. It was shown in [KM2, Corollary 7.8] that if k is a positive integer, then $u_k = \min\{u, k\}$ is in $N^{1,p}_{\mathrm{loc}}(\Omega)$, and hence Du_k is well-defined a.e., and moreover, if $j > k$, then $Du_j = Du_k$ on the set $\{x \in \Omega \colon u(x) \le k\}$.

6.5. Definition. Let u be a non-negative p-superharmonic function on Ω. Then

$$Du := \lim_{k\to\infty} Du_k$$

is well-defined a.e. on the set $\{x \in \Omega \colon u(x) < \infty\}$.

The arguments in [HKM, Theorem 7.46] show that in the Euclidean setting with $Du = \nabla u$, $|Du| \in L^1(\Omega)$. Their argument is quite general, and is easily adapted to our situation; see below.

6.6. Lemma. *Let Ω be a bounded domain with $\operatorname{cap}_p(X \setminus \Omega) > 0$ and u be a non-negative p-superharmonic function on Ω. Then $|Du| \in L^{p-1}_{\mathrm{loc}}(\Omega)$.*

Proof. By [KM2, Corollary 7.8] the function $u_k = \min\{u, k\}$ is a p-supersolution and $u_k \in N^{1,p}_{\mathrm{loc}}(\Omega)$. By the Caccioppoli type inequality [BMS, Lemma 4.4], if B is a ball in X so that $3B \subset \Omega$ and $\eta \geq 0$ is a Lipschitz function with support in $2B$ so that $\eta = 1$ on B, then

$$\begin{aligned}\int_B |Du_k|^p u_k^{-(1+\epsilon)}\, d\mu &\leq \int_{2B} |Du_k|^p \eta^p u_k^{-(1+\epsilon)}\, d\mu \\ &\leq C \int_{2B} u_k^{p-(1+\epsilon)} |D\eta|^p\, d\mu \leq C_B \int_{2B} u_k^{p-(1+\epsilon)}\, d\mu.\end{aligned}$$

Let B be a ball in X so that $5B \subset \Omega$. By [BMS, Lemma 4.5] and by the fact that X supports a (p_0, p)-Poincaré inequality for some $p_0 > p$ (see [HaK, Section 5]), we have that

$$\left(\fint_B u^\gamma\, d\mu \right)^{1/\gamma} \leq C_\gamma \inf_{2B} u \tag{6.7}$$

whenever $\gamma < p_0(p-1)/p$. Here the constant C_γ is independent of u and B, but may depend on γ. Just as in the alternative proof of [HKM, Theorem 7.46], if $\epsilon > 0$, then

$$\begin{aligned}\int_B |Du_k|^{p-1}\, d\mu &= \int_B |Du_k|^{p-1} u_k^{-(1+\epsilon)(p-1)/p}\, u_k^{(1+\epsilon)(p-1)/p}\, d\mu \\ &\leq \left(\int_B |Du_k|^p u_k^{-(1+\epsilon)}\, d\mu \right)^{(p-1)/p} \left(\int_B u_k^{(1+\epsilon)(p-1)}\, d\mu \right)^{1/p} \\ &\leq \left(\int_B |Du_k|^p u_k^{-(1+\epsilon)}\, d\mu \right)^{(p-1)/p} \left(\int_B u^{(1+\epsilon)(p-1)}\, d\mu \right)^{1/p} \\ &\leq C_B \left(\int_{2B} u^{p-(1+\epsilon)}\, d\mu \right)^{(p-1)/p} \left(\int_B u^{(1+\epsilon)(p-1)}\, d\mu \right)^{1/p}.\end{aligned}$$

Since $p_0/p > 1$, we can find $\epsilon > 0$ so that $1 + \epsilon < p_0/p$, that is, $(1+\epsilon)(p-1) < p_0(p-1)/p$. Thus by Proposition 2.2 and inequality (6.7) above,

$$\int_B |Du_k|^{p-1}\, d\mu \leq C_{B,p,p_0} (\inf_B u)^{p-1} < \infty.$$

Therefore the sequence $|Du_k|$ is a bounded sequence in $L^{p-1}_{\mathrm{loc}}(\Omega)$, and hence by the reflexivity of $L^{p-1}_{\mathrm{loc}}(\Omega)$ we conclude that $|Du| \in L^{p-1}_{\mathrm{loc}}(\Omega)$. $\square$

The next proposition is a metric space analog of [Mi, Theorem 2.2], and associates to each non-negative p-superharmonic function a Radon measure ν.

6.8. Proposition. *If u is a non-negative p-superharmonic function on domain Ω, then there is a Radon measure ν, supported on $\overline{\Omega}$, so that for every $\varphi \in \mathrm{Lip}_c(\Omega)$,*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi\, d\mu = \int_\Omega \varphi\, d\nu.$$

Observe that the equation given in the above proposition is precisely equation (6.4).

Proof. Let $u_k = \min\{u, k\}$, and $\varphi \in \mathrm{Lip}_c(\Omega)$. Since $u_k \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and φ has compact support in Ω, by [BMS, Proposition 3.5] there is a Radon measure ν_k supported in $\overline{\Omega}$ so that

$$\int_\Omega |Du_k|^{p-2} Du_k \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu_k.$$

By Lemma 6.6 above, $|Du_k|^{p-2} Du_k \cdot D\varphi$ converges to $|Du|^{p-2} Du \cdot D\varphi$ in $L^1(\Omega)$. Therefore, $(\nu_k)_k$ is a bounded sequence of Radon measures on compact subsets of Ω, and hence weakly converge to a Radon measure ν; that is,

$$\int_\Omega \varphi \, d\nu_k \to \int_\Omega \varphi \, d\nu$$

for every $\varphi \in \mathrm{Lip}_c(\Omega)$. Thus by the convergence of $|Du_k|^{p-2}Du_k \cdot D\varphi$ to $|Du|^{p-2}Du \cdot D\varphi$ again, we see that

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu.$$

□

Remark. Unlike in [BMS, Proposition 3.5], we cannot replace the condition of $\varphi \in \mathrm{Lip}_c(\Omega)$ with $\varphi \in N^{1,p}_0(\Omega)$ because unlike the Radon measures associated with p-supersolutions, the Radon measures associated to p-superharmonic functions as in the above proposition may charge sets of zero p-capacity.

The aim of the next section is to construct a p-superharmonic function whose infinity set is precisely the closed zero p-capacity set E under study; this would be a refinement of Theorem 3.4. In order to construct such a p-superharmonic function we follow the lead of [K2]. We sweep the Radon measure associated with the p-superharmonic function constructed in Theorem 3.4 onto the set E using the technique developed in Section 5, and use the swept measure to construct the desired p-superharmonic function. The rest of this section is devoted to showing that for any finite Radon measure there is a corresponding p-harmonic function related to it as in (6.4). The argument closely follows the proof of [Mi, Theorem 2.21] and uses the machinery set up in [BMS].

6.9. Proposition. *Given a finite Radon measure ν on the domain Ω, there exists a non-negative p-superharmonic function u on Ω so that for all $\varphi \in Lip_c(\Omega)$,*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu.$$

To prove the above proposition we need the following lemma.

6.10. Lemma. *Given a sequence $(u_k)_k \in N^{1,p}_0(\Omega)$ of bounded functions that are also bounded in $N^{1,p}_0(\Omega)$ there is a subsequence $(u_{k_j})_j$ and a function $u \in N^{1,p}_0(\Omega)$ so that $u_{k_j} \to u$ in $N^{1,p}_0(\Omega)$.*

For a proof of this fact see [Mi, Theorem 2.14].

Proof of Proposition 6.9. By [BMS, Lemma 3.12] there exists a sequence $(\nu_k)_k$ of Radon measures on Ω so that $\nu_k \to \nu$ weakly and $\nu_k(\Omega) \le \nu(\Omega)$, and by [BMS, Proposition 3.9] there is an associated sequence of p-supersolutions $u_k \in N_0^{1,p}(\Omega)$ so that for every k,

$$\int_\Omega |Du_k|^{p-2} Du_k \cdot D\varphi \, d\mu = \int_\Omega \varphi \, d\nu_k.$$

Note that $\int_\Omega \varphi \, d\nu_k \to \int_\Omega \varphi \, d\nu$ as $k \to \infty$. For positive integers j set $u_{k,j} = \min\{u_k, j\}$. Then as $|Du_{k,j}| = 0$ on the set where $u_k \ge j$ and $u_{k,j} \in N_0^{1,p}(\Omega)$, we see that

$$\begin{aligned}\int_\Omega |Du_{k,j}|^p \, d\mu &= \int_\Omega |Du_{kj}|^{p-2} Du_{k,j} \cdot Du_{k,j} \, d\mu \\ &= \int_\Omega |Du_k|^{p-2} Du_k \cdot Du_{k,j} \, d\mu \\ &= \int_\Omega u_{k,j} \, d\nu_k \le j\, \nu_k(\Omega) \le j\, \nu(\Omega).\end{aligned}$$

Therefore, by the Sobolev inequality [KSh, Lemma 2.10], we have

$$\int_\Omega u_{k,j}^p \, d\mu \le C \int_\Omega |Du_{k,j}|^p \, d\mu \le C\nu(\Omega)\, j.$$

Now,

$$k^p \, \mu\left(\{x \in \Omega \,:\, j \le u_k(x) < 2j\}\right) \le \int_\Omega u_{k,2j}^p \, d\mu \le 2Cj.$$

Thus, by Hölder's inequality, when $0 < q < 1$,

$$\begin{aligned}&\int_{\{j \le u_k < 2j\}} |Du_{k,2j}|^{q(p-1)} \, d\mu \\ &\le \mu\left(\{j \le u_k < 2j\}\right)^{1-q(p-1)/p} \left(\int_\Omega |Du_{k,2j}|^p \, d\mu\right)^{q(p-1)/p} \le Cj^{-(1-q)(p-1)}.\end{aligned}$$

Note that $1 - q > 0$. Hence, when $0 < i < 2^{m+1}$,

$$\begin{aligned}\int_\Omega |Du_{k,i}|^{q(p-1)} \, d\mu &\le \sum_{j=0}^{m} \int_{\{2^j < u_k \le 2^{j+1}\}} |Du_{k,2^j}|^{q(p-1)} \, d\mu \\ &\le C \sum_{j=0}^{m} 2^{-j(1-q)(p-1)} \le C(p,q).\end{aligned}$$

Therefore, as $u_{k,i} \to u_k$ in $N_0^{1,p}(\Omega)$, we have

$$\int_\Omega |Du_k|^{q(p-1)} \, d\mu \le C(p,q),$$

the bound being independent of k. Hence without loss of generality we can assume that the sequence $|Du_k|^{p-2} Du_k$ is a weakly convergent sequence in $L^1(\Omega; \mathbf{R}^N)$.

By Lemma 6.10 and by a diagonalization argument we can assume that for every positive integer j the sequence $(u_{k,j})_k$ converges to a function $v_j \in N_0^{1,p}(\Omega)$ p-q.e. in Ω and $|Du_{k,j}|^{p-2}Du_{k,j} \to |Dv_j|^{p-2}Dv_j$ weakly in $L^1(\Omega;\mathbf{R}^N)$. Moreover, it is easy to see that $v_j \le v_{j+1}$. Since

$$v_j = \lim_{i\to\infty} \inf_{k\ge i} u_{k,j} = \lim_{i\to\infty} \lim_{l\to\infty} \inf_{i\le k\le l} u_{k,j}$$

is p-superharmonic on Ω and $v_j \le j$, by [KM2, Corollary 7.9] v_j is a p-supersolution on Ω. Hence by [KM2, Lemma 7.1] we see that the lower semicontinuous regularization of the function $v = \lim_{j\to\infty} v_j$ is either a p-superharmonic function on Ω or is identically infinite on Ω.

Suppose the lower semicontinuous regularization of v is identically infinite. Then v is infinite a.e.; that is, $v_j(x) \to \infty$ for a.e. $x \in \Omega$. Since $v_j = \liminf_{k\to\infty} u_{k,j}$, this implies that $v_j = j$ a.e. on Ω. This contradicts the fact that $v_j \in N_0^{1,p}(\Omega)$ because of the $(1,p)$-Poincaré inequality. Thus v is p-superharmonic on Ω, and hence by Proposition 2.2 v is finite p-q.e. in Ω. Consequently, $v = \liminf_{k\to\infty} u_k$ on the set where v is finite. Moreover, as $\min\{v, j\} = v_j$, we see by the definition of Dv and by the proof of Lemma 6.6 that $|Dv_j|^{p-2}Dv_j \to |Dv|^{p-2}Dv$ weakly in $L^1(\Omega;\mathbf{R}^N)$. The weak convergence of $Du_{k,j}$ to Dv_j now indicates that

$$\int_\Omega |Du_{k,j}|^{p-2}Du_{k,j}\cdot D\varphi\,d\mu \to \int_\Omega |Dv|^{p-2}Dv\cdot D\varphi\,d\mu$$

as k and then j tend to infinity. Since $Du_{k,j} = Du_k$ wherever $u_k \le j$ (almost everywhere), and $Du_{k,j} = 0$ wherever $u_k > j$, we see by the weak convergence of $|Du_k|^{p-2}Du_k$ that

$$\int_\Omega |Du_k|^{p-2}Du_k\cdot D\varphi\,d\mu \to \int_\Omega |Dv|^{p-2}Dv\cdot D\varphi\,d\mu,$$

and hence

$$\int_\Omega |Dv|^{p-2}Dv\cdot D\varphi\,d\mu = \int_\Omega \varphi\,d\nu.$$

The lower semicontinuous regularization u of v now satisfies the conditions of the proposition. □

7. Compact zero p-capacity sets and ∞-sets of modified p-superharmonic functions.

7.1. Lemma. *Let u be a non-negative p-superharmonic function on Ω and let ν be the measure associated with u as in Propositions 5.8 and 5.9. Then there is a constant $C > 0$ so that for every ball $B = B(x,r)$ with $4B \subset \Omega$,*

$$\frac{1}{C} W_p^\nu(x,r) \le u(x) \le C\left(\inf_B u + W_p^\nu(x,2r)\right).$$

The proof of this lemma follows the same lines as the proofs of [BMS, Proposition 4.9] and [BMS, Proposition 4.10] once [BMS, Lemma 4.8] is extended to p-superharmonic functions; this can be done just as in [Mi, Lemma 3.5], and is left to the reader to verify.

Now we are ready to state and prove the main result of this section.

7.2. Theorem. *Let E be a compact subset of Ω so that $\mathrm{cap}_p(E) = 0$. Then there exists a non-negative p-superharmonic function u on Ω so that*

$$E = \{x \in \Omega \colon u(x) = \infty\}.$$

Proof. By Theorem 3.4 there exists a non-negative p-superharmonic function v on Ω so that $E \subset \{x \in \Omega : v(x) = \infty\}$. Let ν be the Radon measure associated with v as in Proposition 6.8. Let $\widetilde{\nu}$ be the swept measure corresponding to ν constructed via Lemma 5.6, and let u be the non-negative p-superharmonic function on Ω associated with $\widetilde{\nu}$ as in Proposition 6.9. We show that u satisfies the above theorem.

Since $v = \infty$ on E, by Lemma 7.1 we see that $W_p^\nu(x, r) = \infty$ for all $x \in E$ and for all $r > 0$ so that $B(x, 4r) \subset \Omega$. Hence by Lemma 5.6, $W_p^{\widetilde{\nu}}(x, r) = \infty$ whenever $x \in E$ and $B(x, 84r) \subset \Omega$. Again by Lemma 7.1, we have $u(x) = \infty$ when $x \in E$.

On the other hand, since $\widetilde{\nu}$ is supported on E and E is compact, if $x \in \Omega \setminus E$ there is a radius $r > 0$ so that $W_p^{\widetilde{\nu}}(x, 2r) = 0$, and hence by Lemma 7.1 and Proposition 2.2 we see that $u(x)$ is finite. Thus $E = \{x \in \Omega \colon u(x) = \infty\}$. □

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