SOBOLEV SPACE PROPERTIES OF SUPERHARMONIC FUNCTIONS ON METRIC SPACES

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Abstract. Our objective is to study regularity of superharmonic functions related to a nonlinear potential theory on metric spaces. In particular, we are interested in which sense a superharmonic function has a derivative and what are the local integrability properties of a superharmonic function and its derivative. We show that every superharmonic function has a weak upper gradient and we provide sharp local integrability estimates.

1. Introduction

Let (X,d) be a metric space equipped with a Borel measure μ. In the metric measure space a concept of an upper gradient serves as a substitute for the Sobolev gradient. Suppose that $1 \le p < \infty$ and let u be a real-valued function on X. A nonnegative Borel measurable function g on X is said to be a p-weak upper gradient of u if for p-almost every rectifiable paths γ joining points x and y in X we have

$$|u(x)-u(y)| \le \int_\gamma g\,ds. \tag{1.1}$$

This means that (1.1) holds for all paths in X except of a family of paths which is of p-modulus zero. The Sobolev space on a metric measure space, called the Newtonian space $N^{1,p}(X)$, can be defined as a collection of equivalence classes of p-integrable functions with p-integrable upper gradients. Essentially, this is an extension of the absolute continuity property on almost all lines of Sobolev functions to metric spaces. For this approach on a metric measure space see [19]; a slightly different but equivalent approach is used in [4].

Newtonian spaces can be used to study minimizers of the p-Dirichlet integral

$$\int g_u^p\,d\mu, \tag{1.2}$$

where g_u denotes the minimal p-weak upper gradient of u. For recent results on minimizers of (1.2) we refer to [1], [2], [3], [4], [12], [13], [14], [18] and [19]. There is a nonlinear potential theory on metric measure spaces related to the minimizers of (1.2) corresponding to the theory developed in the Euclidean case for the solutions of the p-Laplace equation, see [12]. So called p-superharmonic functions play an essential role in this theory. Imitating the classical definition, the p-superharmonic

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functions on metric measure spaces are defined by a comparison principle, see [12]. Sometimes p-superharmonic functions are identified with supersolutions of the corresponding equation. This is not strictly speaking correct: Supersolutions belong to certain Sobolev space but there is no a priori Sobolev type assumption in the definition of p-superharmonic functions. The general question is: In which sense a p-superharmonic function has a derivative and what are the local integrability properties of the function and its derivative? In the Euclidean case this question has been been studied in [15] and [11]. See also [8].

Due to the notion of the upper gradient it is not clear how to employ partial differential equations and supersolutions on a metric measure space. We work with the superminimizers of (1.2) and using the Moser iteration scheme we prove a sharp version of the weak Harnack inequality for them. This implies a sharp estimate for the local integrability of a p-superharmonic function. It is known that a p-superharmonic function does not have a weak derivative if p is close to one, see [15]. We show that every p-superharmonic function has a p-weak upper gradient and we provide a sharp estimate for its local integrability. In this respect the notion of the p-weak upper gradient seems to be the correct interpretation of the derivative of a p-superharmonic function even in the Euclidean case.

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2. Preliminaries

Let X be a metric space and let μ be a Borel measure on X. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite.

Let u be a real-valued function on X. A nonnegative Borel measurable function g on X is said to be an *upper gradient* of u if (1.1) holds for all rectifiable paths γ joining points x and y in X. We recall that a path γ is rectifiable if $\text{length}(\gamma) < \infty$. See [4], [7], [9] and [19] for a discussion on upper gradients.

A property is said to hold for *p-almost all* paths, if the set of paths for which the property fails is of zero p-modulus, see for example [7] for the definition of the p-modulus. If (1.1) holds for p-almost all paths γ, then g is said to be a *p-weak upper gradient* of u. If $1 < p < \infty$, every function u that has a p-integrable p-weak upper gradient has a minimal p-integrable p-weak upper gradient in X, denoted g_u, in the sense that if g is another p-weak upper gradient of u, then $g_u \le g$ μ-almost everywhere in X.

Let $1 \le p < \infty$. Following [19] we define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p-integrable functions u on X that have a p-integrable p-weak upper gradient g on X. This space is equipped with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients of u.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of the Newtonian spaces we refer to [19]. Cheeger [4] gives an alternative definition which leads to the same space when $1 < p < \infty$, see [19].

The functions in $N^{1,p}(X)$ are *absolutely continuous on p-almost every path*, which means that $u \circ \gamma$ is absolutely continuous on $[0, \mathrm{length}(\gamma)]$ for p-almost every rectifiable arc-length parametrized path γ in X. Now (1.1) can be interpreted as follows: For p-almost every path $\gamma\colon [0, \mathrm{length}(\gamma)] \to X$ the function $u \circ \gamma$ is absolutely continuous on $[0, \mathrm{length}(\gamma)]$ and $|(u \circ \gamma)'(s)| \le g(\gamma(s))$ for almost every $s \in [0, \mathrm{length}(\gamma)]$.

The *p-capacity* of a set $E \subset X$ is defined by

$$\mathrm{C}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, whose restriction to E is bounded below by 1. The discussion in [12] can easily be adapted to show that the capacity is an outer measure, see also [19].

Let E be an arbitrary subset of X. Following the method of [10], we define $\widetilde{N}^{1,p}_0(E)$ to be the set of functions $u\colon E \to [-\infty, \infty]$ for which there exists a function $\widetilde{u} \in \widetilde{N}^{1,p}(X)$ such that $\widetilde{u} = u$ μ-almost everywhere in E and

$$\mathrm{C}_p(\{x \in X \setminus E\colon \widetilde{u}(x) \neq 0\}) = 0.$$

Next we define an equivalence relation on $\widetilde{N}^{1,p}_0(E)$ by requiring that $u \sim v$ if $u = v$ μ-almost everywhere on E. Finally we let $N^{1,p}_0(E) = \widetilde{N}^{1,p}_0(E)/\sim$, equipped with the norm

$$\|u\|_{N^{1,p}_0(E)} = \|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)},$$

be the *Newtonian space with zero boundary values.* The norm is unambiguously defined by [19] and the space $N^{1,p}_0(E)$ is a Banach space.

A subset A of Ω is *compactly contained* in Ω, abbreviated $A \Subset \Omega$, if the closure of A is a compact subset of Ω. We say that u belongs the the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(A)$ for every measurable set $A \Subset \Omega$.

A metric measure space X is said to be *doubling* if there is a constant $c_\mu \ge 1$ so that

$$\mu(B(z, 2r)) \le c_\mu \mu(B(z, r))$$

for every open ball $B(z, r)$ in X. The constant c_μ is called the doubling constant of μ. Note that an iteration of the doubling property implies, that if $B(y, R)$ is a ball in X, $z \in B(y, R)$ and $0 < r \le R < \infty$, then

$$\frac{\mu(B(z, r))}{\mu(B(y, R))} \ge c\Big(\frac{r}{R}\Big)^Q$$

for some $c = c(c_\mu)$ and $Q = \log c_\mu / \log 2$. The exponent Q serves as a counterpart of dimension related to the measure and, for example, in $\mathbf{R}^n$ with the Lebesgue measure Q is equal to the dimension n.

The space X is said to *support a weak* $(1, p)$*-Poincaré inequality* if there are constants $c > 0$ and $\tau \ge 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \le cr\Big(\fint_{B(z,\tau r)} g^p \, d\mu\Big)^{1/p}$$

for all balls $B(z,r)$ in X, for all integrable functions u in $B(z,\tau r)$ and for all p-weak upper gradients g of u.

A result of [5] (see also Theorem 5.1 in [6]) shows that in a doubling measure space a Poincaré inequality implies a *Sobolev-Poincaré inequality*. Indeed, suppose that μ is doubling and X supports a weak $(1,p)$-Poincaré inequality. If $1<p<Q$ and $1\le\kappa\le Q/(Q-p)$, there is $c=c(p,\kappa,c_\mu)>0$ such that

$$\Big(\fint_{B(z,r)}|u-u_{B(z,r)}|^{\kappa p}\,d\mu\Big)^{1/\kappa p}\le cr\Big(\fint_{B(z,5\tau r)}g^p\,d\mu\Big)^{1/p}$$

for every p-weak upper gradient g of u. If $p=Q$, then there are $c_1=c_1(c_\mu)>0$ and $c_2=c_2(\mu)>0$ such that

$$\fint_{B(z,r)}\exp\Big(\frac{c_1\mu(B(z,r))^{1/p}|u-u_{B(z,r)}|}{r\|g\|_{L^p(B(z,5\tau r)}}\Big)\,d\mu\le c_2.$$

If $p>Q$, then there is $c=c(p,c_\mu)>0$ such that

$$|u(x)-u(y)|\le cr^{Q/p}d(x,y)^{1-Q/p}\Big(\fint_{B(z,5\tau r)}g^p\,d\mu\Big)^{1/p}$$

for every $x,y\in B(z,r)\setminus N$ with $\mu(N)=0$ and every p-weak upper gradient g of u. In particular, this implies that for every $1<p<\infty$ we have there is $c=c(p,\kappa,c_\mu)>0$ such that

$$\Big(\fint_{B(z,r)}|u-u_{B(z,r)}|^{\kappa p}\,d\mu\Big)^{1/\kappa p}\le cr\Big(\fint_{B(z,5\tau r)}g^p\,d\mu\Big)^{1/p}\tag{2.1}$$

for every p-weak upper gradient g of u, where $1\le\kappa\le Q/(Q-p)$ if $1<p<Q$ and $\kappa=2$ if $p\ge Q$.

We also need an inequality for Sobolev functions with zero boundary values. If $u\in N_0^{1,p}(B(z,r))$, then there exists $c=c(p,c_\mu)>0$ such that

$$\Big(\fint_{B(z,r)}|u|^{\kappa p}\,d\mu\Big)^{1/\kappa p}\le cr\Big(\fint_{B(z,r)}g^p\,d\mu\Big)^{1/p}\tag{2.2}$$

provided $0<2r<\operatorname{diam}(X)$. For this result we refer to Lemma 2.1 in [14].

From now on we assume, without further notice, that the complete metric measure space X is equipped with a doubling Borel measure for which the measure of every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1,q)$-Poincaré inequality for some q with $1<q<p$.

The main reason for this somewhat tehcnical assumption is that we apply results from papers [12] and [14] and those results are known to be valid with the assumption above. This assumption is not needed in the proof of the Caccioppoli type estimate (3.2).

Let $1 < p < \infty$. Suppose that $\Omega \subset X$ is an open set, let $\vartheta \in N^{1,p}(\Omega)$. A function $u \in N^{1,p}(\Omega)$ such that $u - \vartheta \in N_0^{1,p}(\Omega)$ is a *p-minimizer with boundary values ϑ in Ω*, if

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu \tag{2.3}$$

for every $v \in N^{1,p}(\Omega)$ such that $v - \vartheta \in N_0^{1,p}(\Omega)$. Here g_u and g_v are the minimal p-weak upper gradients of u and v in Ω, respectively. Let $\Omega \subset X$ be an open set. A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is called a *p-minimizer* in Ω if (2.3) holds in every open set $\Omega' \Subset \Omega$ for all v such that $v - u \in N_0^{1,p}(\Omega')$. A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *p-superminimizer* in Ω if (2.3) holds in every open set $\Omega' \Subset \Omega$ for all v such that $v - u \in N_0^{1,p}(\Omega')$, $v \ge u$ μ-almost everywhere in Ω'. We observe that u is a p-minimizer in Ω if and only if both u and $-u$ are superminimizers in Ω.

Let Ω be an open subset of X. A function $v\colon \Omega \to \mathbf{R}$ is *p-harmonic*, or simply *harmonic*, in Ω if it is a continuous p-minimizer in Ω. Observe that every p-minimizer is locally Hölder continuous after redefinition on a set of measure zero by [14].

Suppose that $\Omega' \Subset \Omega$ is open and $v \in N^{1,p}(\Omega')$. Let $h_{\Omega'}(v) = h(v)$ be the unique harmonic function in Ω' with $v - h(v) \in N_0^{1,p}(\Omega')$.

Following [12] we say that a function $u\colon \Omega \to (-\infty, \infty]$ is *p-superharmonic*, or simply *superharmonic* in Ω, if

(i) u is lower semicontinuous in Ω,
(ii) u is not identically ∞ in any component of Ω,
(iii) for every open $\Omega' \Subset \Omega$ the comparison principle holds: if $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \le u$ on $\partial\Omega'$, then $h(v) \le u$ in Ω'.

In the Euclidean case with the Lebesgue measure the definition coincides with the standard definition of superharmonicity when $p = 2$. If $1 < p < \infty$ and $\mu = w\,dx$, where w is an admissible weight function in the sense of [8], then the definition gives the p-superharmonic functions studied in the weighted nonlinear potential theory, see Chapter 7 of [8].

We collect some results which clarify the interplay between p-superharmonic functions and p-superminimizers in the following lemma.

2.4. Theorem.

(i) *If u is a p-superminimizer in Ω and*

$$v(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y)$$

for every $x \in \Omega$, then v is p-superharmonic in Ω and $u = v$ μ-almost everywhere in Ω.

(ii) *If u is p-superharmonic in Ω and, in addition, u is locally bounded above, then $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ and u is a p-superminimizer.*

(iii) *If u is p-superharmonic and, in addition, $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$, then u is a p-superminimizer.*

(iv) *If u is p-superharmonic in Ω, then*

$$u(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y)$$

for every $x \in \Omega$.

For the proofs we refer to [12]. For the Euclidean case, see [8].

3. A Caccioppoli estimate

The following Caccioppoli type estimate is crucial for us.

3.1. Lemma. *Suppose that $u > 0$ is a p-superminimizer in Ω and let $\varepsilon > 0$. Let η be a compactly supported Lipschitz continuous function in Ω such that $0 \le \eta \le 1$. Then*

$$\int_\Omega g_u^p u^{-1-\varepsilon} \eta^p \, d\mu \le c \int_\Omega u^{p-1-\varepsilon} g_\eta^p \, d\mu, \tag{3.2}$$

where $c = (p/\varepsilon)^p$.

Proof. We may assume that $u \ge \varepsilon^{1/(\varepsilon+1)}$, since otherwise we study αu for $\alpha > 0$ large enough. Let $w = u + \eta^p u^{-\varepsilon}$. Then $w \ge u$ and $w \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with

$$g_w \le \big(1 - \varepsilon \eta^p u^{-\varepsilon-1}\big) g_u + p\eta^{p-1} u^{-\varepsilon} g_\eta$$

μ-almost everywhere in Ω.

To see this, let γ be the arc-length parametrization of a rectifiable path on which the function u is absolutely continuous. Since η is Lipschitz continuous, it is absolutely continuous on γ. Define $h\colon [0, \mathrm{length}(\gamma)] \to [0, \infty)$,

$$h(s) = (u \circ \gamma)(s) + (\eta \circ \gamma)(s)^p (u \circ \gamma)(s)^{-\varepsilon}.$$

Then h is absolutely continuous and for almost every $s \in [0, \mathrm{length}(\gamma)]$, with respect to the one-dimensional Lebesgue measure, we have

$$\begin{aligned} h'(s) =& (u \circ \gamma)'(s) + p(\eta \circ \gamma)(s)^{p-1} (\eta \circ \gamma)'(s) (u \circ \gamma)(s)^{-\varepsilon} \\ &- \varepsilon (u \circ \gamma)(s)^{-\varepsilon-1} (u \circ \gamma)'(s) (\eta \circ \gamma)(s)^p \\ =& \big(1 - \varepsilon (\eta \circ \gamma)(s)^p (u \circ \gamma)(s)^{-\varepsilon-1}\big) (u \circ \gamma)'(s) \\ &+ p(\eta \circ \gamma)(s)^{p-1} (u \circ \gamma)(s)^{-\varepsilon} (\eta \circ \gamma)'(s). \end{aligned}$$

Since $|(u \circ \gamma)'(s)| \le g_u(\gamma(s))$ and $|(\eta \circ \gamma)'(s)| \le g_\eta(\gamma(s))$ for almost every $s \in [0, \mathrm{length}(\gamma)]$, we obtain

$$\begin{aligned} |(w \circ \gamma)'(s)| = |h'(s)| \le& \big(1 - \varepsilon \eta(\gamma(s))^p u(\gamma(s))^{-\varepsilon-1}\big) g_u(\gamma(s)) \\ &+ p\eta(\gamma(s))^{p-1} u(\gamma(s))^{-\varepsilon} g_\eta(\gamma(s)) \end{aligned}$$

for almost every $s \in [0, \mathrm{length}(\gamma)]$. This proves the claim.

Since $0 \le \varepsilon \eta^p u^{-\varepsilon-1} \le 1$, by convexity we obtain

$$\begin{aligned} g_w^p &\le \big(1 - \varepsilon \eta^p u^{-\varepsilon-1}\big) g_u^p + \varepsilon \eta^p u^{-\varepsilon-1} \Big(\frac{p}{\varepsilon}\frac{u}{\eta} g_\eta\Big)^p \\ &\le \big(1 - \varepsilon \eta^p u^{-\varepsilon-1}\big) g_u^p + \varepsilon^{1-p} p^p u^{p-1-\varepsilon} g_\eta^p. \end{aligned}$$

Since u is a p-superminimizer, we have

$$\begin{aligned} \int_{\Omega'} g_u^p \, d\mu &\le \int_{\Omega'} g_w^p \, d\mu \\ &\le \int_{\Omega'} g_u^p \, d\mu - \varepsilon \int_{\Omega'} \eta^p u^{-\varepsilon-1} g_u^p \, d\mu + \varepsilon^{1-p} p^p \int_{\Omega'} u^{p-1-\varepsilon} g_\eta^p \, d\mu, \end{aligned}$$

where $\Omega' \Subset \Omega$ is an open set containing the support of η. From this we conclude that

$$\int_\Omega \eta^p u^{-\varepsilon-1} g_u^p \, d\mu \le c \int_\Omega u^{p-1-\varepsilon} g_\eta^p \, d\mu,$$

where $c = (p/\varepsilon)^p$. $\square$

4. Weak Harnack inequalities

For the next result, called *the weak Harnack inequality* on metric spaces, see Theorem 7.1 in [14] and Lemma 4.7 in [12]. Originally it was proven in the Euclidean case by J. Moser in [16]. See also [17] and [20].

4.1. Lemma. *Suppose that $u \geq 0$ is a p-superminimizer in an open set $\Omega \subset X$. Then for every ball $B(z,r)$ with $B(z,5r) \subset \Omega$ we have*

$$\Big(\fint_{B(z,r)} u^\sigma \, d\mu \Big)^{1/\sigma} \leq c \operatorname*{ess\,inf}_{B(z,3r)} u, \tag{4.2}$$

where $c < \infty$ and $\sigma > 0$ depend only on p, the doubling constant and the constants in the Poincaré inequality.

Now we are ready to prove a sharp version of the weak Harnack inequality. In the Euclidean case this result is due to N. S. Trundinger, see [20].

4.3. Theorem. *Suppose that $u \geq 0$ is a p-superminimizer in an open set $\Omega \subset X$. If $0 < s < \kappa(p-1)$, where $\kappa > 1$ is as in (2.1), then for every ball $B(z,r)$ with $B(z,10r) \subset \Omega$ we have*

$$\Big(\fint_{B(z,r)} u^s \, d\mu \Big)^{1/s} \leq c \operatorname*{ess\,inf}_{B(z,6r)} u, \tag{4.4}$$

where $c < \infty$ and $\sigma > 0$ depend on the data in the same way as in Lemma 4.1.

Proof. Without loss of generality we may assume that $u \geq \delta$ in $B(z,4r)$ for some $\delta > 0$ and that $s > \sigma$, where σ is as in (4.2). Let k be an integer such that

$$\kappa^{-k} s \leq \sigma < \kappa^{-k+1} s$$

and write $B_j = B(z,(1+j/k)r)$ for $j = 0,1,\dots,k+1$. Let η_j be a Lipschitz cut-off function such that $0 \leq \eta_j \leq 1$, $\eta_j = 0$ in $X \setminus B_j$, $\eta_j = 1$ on $\overline{B}_{j-1}$, and $g_{\eta_j} \leq 4k/r$ for $j = 1,\dots,k+1$. Let $\varepsilon_j = p - 1 - \kappa^{-j} s$ and

$$v_j = \eta_j u^{1-(1+\varepsilon_j)/p}, \quad j = 1,2,\dots,k+1.$$

Observe that $\varepsilon_j > 0$ for every $j = 1,2,\dots$. Then as in the proof of Lemma 3.1 we have

$$g_{v_j} \leq g_{\eta_j} u^{1-(1+\varepsilon_j)/p} + \Big(1 - \frac{1+\varepsilon_j}{p}\Big) u^{-(1+\varepsilon_j)/p} g_u \eta_j$$

and consequently

$$g_{v_j}^p \leq c g_{\eta_j}^p u^{p-1-\varepsilon_j} + c\Big(1 - \frac{1+\varepsilon_j}{p}\Big)^p u^{-1-\varepsilon_j} g_u^p \eta_j^p$$

μ-almost everywhere with $c = 2^{p-1}$. The Sobolev inequality (2.2) and Lemma 3.1 imply for $j = 1,\dots,k$, that

$$\begin{aligned} \Big(\fint_{B_j} v_j^{\kappa p} \, d\mu \Big)^{1/\kappa p} &\leq cr \Big(\fint_{B_j} g_{v_j}^p \, d\mu \Big)^{1/p} \leq cr \Big(\fint_{B_j} u^{p-1-\varepsilon_j} g_{\eta_j}^p \, d\mu \Big)^{1/p} \\ &\leq c \Big(\fint_{B_j} u^{\kappa^{-j} s} \, d\mu \Big)^{1/p} = c \Big(\fint_{B_j} v_{j+1}^{\kappa p} \, d\mu \Big)^{1/p}. \end{aligned} \tag{4.5}$$

This implies

$$\fint_{B_j} v_j^{\kappa p}\,d\mu \le c\Big(\fint_{B_{j+1}} v_{j+1}^{\kappa p}\,d\mu\Big)^{\kappa}, \quad j=1,2,\dots,k-1.$$

We iterate this estimate $k-1$ times and we obtain

$$\fint_{B_1} v_1^{\kappa p}\,d\mu \le c\Big(\fint_{B_k} v_k^{\kappa p}\,d\mu\Big)^{\kappa^{k-1}}.$$

Since $B_k = B(z,2r)$ and $\kappa^{-k}s \le \sigma$, using (4.5) once more we arrive at

$$\begin{aligned}\Big(\fint_{B(z,r)} u^s\,d\mu\Big)^{1/s} &\le c\Big(\fint_{B_1} v_1^{\kappa p}\,d\mu\Big)^{1/s} \le c\Big(\fint_{B_k} v_k^{\kappa p}\,d\mu\Big)^{\kappa^{k-1}/s}\\ &\le c\Big(\fint_{B_k} v_{k+1}^{\kappa p}\,d\mu\Big)^{\kappa^k/s} \le c\Big(\fint_{B_k} u^{\kappa^{-k}s}\,d\mu\Big)^{\kappa^k/s}\\ &\le c\Big(\fint_{B(z,2r)} u^{\sigma}\,d\mu\Big)^{1/\sigma} \le c\operatorname*{ess\,inf}_{B(z,6r)} u.\end{aligned}$$

In the last inequality we used (4.2). □

4.6. Remark. The upper bound for the exponent s in Theorem 4.3 is the largest possible, see p. 74 of [8] and Remark 5.3 (i) below.

5. Local integrability

It follows from Lemma 4.1 and the fact that every p-superharmonic function can be locally approximated by an increasing sequence of p-superminimizers, that every p-superharmonic function is locally integrable to the power σ in (4.2), see Lemma 7.11 in [12]. The power σ given by Lemma 4.1 is usually small and the next result gives a sharp estimate.

5.1. Theorem. *If u is p-superharmonic in Ω, then $u \in L^s_{\mathrm{loc}}(\Omega)$ whenever $0 < s < \kappa(p-1)$, where $\kappa > 1$ is as in* (2.1)*.*

Proof. Let $B(z,11r) \subset \Omega$. Since u is lower semicontinuous, we may assume that $u(x) > 0$ for every $x \in B(z,10r)$. By Corollary 7.8 in [12] we see that $u_k = \min(u,k)$, $k=1,2,\dots$, are p-superminimizers in Ω and hence Theorem 4.3 implies that

$$\Big(\fint_{B(z,r)} u_k^s\,d\mu\Big)^{1/s} \le c\operatorname*{ess\,inf}_{B(z,6r)} u_k,$$

where $0 < s < \kappa(p-1)$. Since $(u_k)_{k=1}^{\infty}$ is an increasing sequence which converges to u pointwise, the monotone convergence yields

$$\Big(\fint_{B(z,r)} u^s\,d\mu\Big)^{1/s} \le c\operatorname*{ess\,inf}_{B(z,6r)} u. \tag{5.2}$$

□

5.3. Remark. (i) The upper bound $\kappa(p-1)$ for the exponent s in Theorem 5.1 is the largest possible when $p \le Q$. To see this, let $X = \mathbf{R}^n$ and μ be the Lebesgue measure. Then $Q = n$, $\kappa = n/(n-p)$ and

$$u(x) = \begin{cases} |x|^{(p-n)/(p-1)}, & \text{if} \quad p < n, \\ -\log|x|, & \text{if} \quad p = n, \end{cases}$$

is a p-superharmonic function which shows that the upper bound is sharp.

(ii) If $p > Q$, then all p-superharmonic functions are continuous and hence locally bounded. In this case the upper bound is not sharp.

(iii) It follows from (5.2) that a positive p-superharmonic function on X is a Muckenhoupt A_1 weight if $p > 2Q/(Q+1)$.

It was shown in Corollary 7.8 of [12] that if u is p-superharmonic in Ω, then $u_k = \min(u,k) \in N^{1,p}_{\mathrm{loc}}(\Omega)$ for every $k = 1, 2, \ldots$. This implies that g_{u_k} is well defined μ-almost everywhere and if $j > k$, then $g_{u_j} = g_{u_k}$ μ-almost everywhere on the set $\{x \in \Omega \colon u(x) \le k\}$. It follows that $(g_{u_k})_{k=1}^{\infty}$ is an increasing sequence and

$$G_u = \lim_{k\to\infty} g_{u_k} \tag{5.4}$$

is well defined μ-almost everywhere on the set $\{x \in \Omega \colon u(x) < \infty\}$. By Theorem 5.1 the measure of $\{x \in \Omega \colon u(x) < \infty\}$ is zero so that G_u is finite μ-almost everywhere on Ω.

Let u be p-superharmonic in Ω. In the next theorem we need the fact that the polar set $E = \{x \in \Omega \colon u(x) = \infty\}$ is of zero p-capacity, see [13]. In a forthcoming paper the authors will return to this question in a more general setup.

5.5. Theorem. *If u is p-superharmonic in Ω, then G_u is a p-weak upper gradient of u.*

Proof. Let $E = \{x \in \Omega \colon u(x) = \infty\}$. As mentioned above, we have $\mathrm{C}_p(E) = 0$. If Γ_E denotes the family of all rectifiable paths that intersect the set E, it is possible to show that $\mathrm{C}_p(E) = 0$ implies that the p-modulus of Γ_E is zero, see [19]. Let Γ_k, $k = 1, 2, \ldots$, be the family of rectifiable curves for which

$$|u_k(x) - u_k(y)| \le \int_\gamma g_{u_k}\, ds$$

fails to hold. Then $\Gamma = \Gamma_E \cup \bigcup_{k=1}^{\infty} \Gamma_k$ is of zero p-modulus and

$$|u(x) - u(y)| \le \int_\gamma G_u\, ds$$

whenever γ is a rectifiable path so that $\gamma \notin \Gamma$. It follows that G_u is a p-weak upper gradient of u. □

5.6. Theorem. *If u is p-superharmonic in Ω, then $G_u \in L^{q(p-1)}_{\mathrm{loc}}(\Omega)$ whenever $0 < q < \kappa p/(\kappa(p-1)+1)$, where $\kappa > 1$ is as in* (2.1).

Proof. Let $B(z, 22r) \subset \Omega$ and $\varepsilon > 0$. Then by Lemma 3.1 we have

$$
\begin{aligned}
&\int_{B(z,r)} g_{u_k}^{q(p-1)}\,d\mu \\
&= \int_{B(z,r)} g_{u_k}^{q(p-1)} u_k^{-(1+\varepsilon)(p-1)q/p} u_k^{(1+\varepsilon)(p-1)q/p}\,d\mu \\
&\le \Big(\int_{B(z,r)} g_{u_k}^p u_k^{-1-\varepsilon}\,d\mu\Big)^{q(p-1)/p}\Big(\int_{B(z,r)} u_k^{(1+\varepsilon)q(p-1)/(p-q(p-1))}\,d\mu\Big)^{(p-q(p-1))/p} \\
&\le c\Big(\int_{B(z,2r)} u^{p-1-\varepsilon}\,d\mu\Big)^{q(p-1)/p}\Big(\int_{B(z,r)} u^{(1+\varepsilon)q(p-1)/(p-q(p-1))}\,d\mu\Big)^{(p-q(p-1))/p}.
\end{aligned}
$$

The integrals are finite by Theorem 5.1 if we choose $\varepsilon > 0$ so that $(1+\varepsilon)q(p-1)/(p-q(p-1)) < \kappa(p-1)$. □

5.7. Remark. If we insert $\kappa = Q/(Q-p)$ into the upper bound for the exponent q in Theorem 5.6 we obtain $0 < q < Q/(Q-1)$. For this we conclude that $G_u \in L^1_{\mathrm{loc}}(\Omega)$ if $p > 2 - 1/Q$. If $1 < p \le 2 - 1/Q$, then G_u need not be locally integrable as the example in Remark 5.3 (i) shows. This implies that, even in the Euclidean case, a p-superharmonic function does not have weak derivatives when $1 < p \le 2 - 1/Q$. P. Lindqvist proved in [15] that every p-superharmonic function is absolutely continuous on almost every line parallel to the coordinate axes if $p > 2 - 1/n$. It can be shown that u is absolutely continuous on p-almost every rectifiable path in an open set $\Omega \subset \mathbf{R}^n$ if and only if u is absolutely continuous on almost every line parallel to the coordinate axes. Theorems 5.5 and 5.6 show that every p-superharmonic function on metric spaces is absolutely continuous on p-almost every rectifiable path if $p > 2 - 1/Q$.

6. Logarithmic estimates

Although a p-superharmonic function need not belong to the natural Sobolev space $N^{1,p}_{\mathrm{loc}}(\Omega)$, it is known that a locally bounded p-superharmonic function belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$, see Theorem 2.4. We close this work by showing that a logarithm of a nonnegative p-superharmonic function belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$. Furthermore, it can be also shown that it is a p-superminimizer, but we leave this to the interested reader. In the Euclidean case these results have been proven in [15]. See also [8].

6.1. Theorem. *Suppose that $u > 0$ is a p-superminimizer in Ω and let $v = \log u$. Then $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $g_v \le g_u/u$ μ-almost everywhere in Ω.*

Proof. Let $B(z,r)$ be a ball such that $B(z,5r) \subset \Omega$. Using (4.2) we see that $u(x) \ge c > 0$ for μ-almost all $x \in B(z,r)$. Hence v is bounded below in $B(z,r)$ and $v \in L^p(B(z,r))$. As in the proof of Lemma 3.1 we see that $g_v \le g_u/u$. Let η be a Lipschitz function such that $0 \le \eta \le 1$, $\eta = 1$ on $B(z,r)$, $\eta = 0$ on $X \setminus B(z,2r)$ and $g_\eta \le c/r$. If we choose $\varepsilon = p-1$ in (3.2) we have

$$\int_\Omega g_v^p \eta^p\,d\mu \le \int_\Omega g_u^p u^{-p}\eta^p\,d\mu \le c\int_\Omega g_\eta^p\,d\mu,$$

where $c = (p/(p-1))^p$. From this we obtain

$$\int_{B(z,r)} g_v^p \, d\mu \le c r^{-p} \mu(B(z,r)). \tag{6.2}$$

It follows that $g_v \in L^p_{\mathrm{loc}}(\Omega)$ and consequently that $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$. $\square$

6.3. Remark. Poincaré inequality and (6.2) imply that

$$\fint_{B(z,r)} |v - v_{B(z,r)}|^p \, d\mu \le c r^p \fint_{B(z,r)} g_v^p \, d\mu \le c < \infty,$$

where the constant c is independent of the ball $B(z,r)$ for which $B(z,5r) \subset \Omega$. This means that $v = \log u$ is of bounded mean oscillation over such balls.

6.4. Corollary. *Suppose that u is a positive p-superharmonic function in Ω. Then $\log u \in N^{1,p}_{\mathrm{loc}}(\Omega)$.*

Proof. Let $B(z,5r) \subset \Omega$ and $u_k = \min(u,k)$, $k = 1,2,\dots$. Then

$$\int_{B(z,r)} |\log u_k|^p \, d\mu \le \int_{B(z,r)} |\log u|^p \, d\mu \le c \int_{B(z,r)} u^s \, d\mu,$$

where $0 < s < \kappa(p-1)$ as in Theorem 4.3. By Theorem 6.1 we conclude that $\log u_k \in N^{1,p}(B(z,r))$ and

$$\|\log u_k\|_{N^{1,p}(B(z,r))} \le c < \infty,$$

where the constant c is independent of k, see (6.2). This implies that the sequence $(\log u_k)_{k=1}^\infty$ is bounded in $N^{1,p}(B(z,r))$ and a weak compactness argument implies that $\log u \in N^{1,p}(B(z,r))$. $\square$

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