FINITE MEAN OSCILLATION IN THE MAPPING THEORY

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Abstract

It is studied a class of the so–called Q–homeomorphisms. It is proved that an isolated singularity is removable for Q–homeomorphisms if $Q(x)$ has finite mean oscillation at the point. It is also shown that super Q–homeomorphisms admit a homeomorphic extension to an exceptional set of the length zero if $Q(x)$ has finite mean oscillation at every point of the set. The results are applicable to many significant classes of mappings with finite distortion which are intensively investigated recently.

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1 Introduction

We investigate Q–homeomorphisms which have been recently introduced by Professor Olli Martio, see e.g. [MRSY$_1$]–[MRSY$_3$]. Note that the corresponding modulus inequality in the below definition of Q–homeomorphisms has been first appeared in [RSY$_1$]–[RSY$_4$] for $n = 2$ under the study of the so–called BMO–quasiconformal mappings. In practice, the theory of Q–homeomorphisms and Q–mapping, see for the latter [MRSY$_4$], is applicable wherever the modulus techniques are met and, in particular, in the investigations of various classes of mappings with finite distortion which are intensively studied during the last decade in many works, see e.g. [AIKM], [GI], [HK], [IM], [IS], [KKM$_1$], [KKM$_2$], [KO], [MV$_1$] and [MV$_2$].

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \mathbb{R}^n$ is called a **Q–homeomorphism** if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x)\, dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

Recall that, given a family of paths Γ, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int_\gamma \rho(x)\, |dx| \geq 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\rho \in adm\, \Gamma} \int\limits_D \rho^n(x)\, dm(x). \tag{1.3}$$

Let a mapping $f : D \to \mathbb{R}^n$ be differentiable at a point $x \in D$. Then the **outer dilatation** of f at x is defined by

$$K_O(x,f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.4}$$

and otherwise we set $K_O(x,f) = \infty$. Similarly, the **inner dilatation** of f at x is defined as

$$K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{1.5}$$

and $K_O(x,f) = \infty$ otherwise. The **maximal dilatation**, or in short the **dilatation**, of f at x is

$$K(x,f) = \max\left(K_O(x), K_I(x)\right). \tag{1.6}$$

Here $f'(x)$ denotes the Jacobian matrix of f and $J(x,f)$ is its determinant and $|f'(x)|$ is the operator norm of $f'(x)$, i.e., $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$, and $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$.

An example of $Q(x)$–homeomorphisms is provided by a class of homeomorphisms $f \in \mathrm{W}^{1,n}_{loc}(D)$ whose maximal dilatation majorant Q is in $L^{n-1}_{loc}(D)$, see [MRSY$_1$] and [MRSY$_2$]. In particular, that holds for $Q \in BMO$. Given a function $Q : D \to [1,\infty]$, they say that a sense-preserving homeomorphism $f : D \to \mathbb{R}^n$ is **Q(x)–quasiconformal**, abbr. **Q(x)–qc**, if $f \in \mathrm{W}^{1,n}_{loc}(D)$ and

$$K(x,f) \leq Q(x) \qquad \text{a.e.} \tag{1.7}$$

They also say that $f : D \to \mathbb{R}^n$ is **BMO–quasiconformal**, abbr. **BMO–qc**, if f is $Q(x)$-qc for some BMO function $Q : D \to [1,\infty]$. In particular, for $Q \in L^\infty(D)$, we obtain usual quasiconformal mappings and, omitting here the condition that f is a sense-preserving homeomorphism, we obtain quasiregular mappings.

Recall that a real valued function $\varphi \in \mathrm{L}^1_{loc}(D)$ is said to be of **bounded mean oscillation** in D, abbr. $\varphi \in$ BMO(D) or simply $\varphi \in$ **BMO**, if

$$\|\varphi\|_* = \sup_{B \subset D} \frac{1}{|B|} \int\limits_B |\varphi(x) - \varphi_B|\, dm(x) \quad < \quad \infty \tag{1.8}$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \frac{1}{|B|} \int\limits_B \varphi(x)\, dm(x) \tag{1.9}$$

is the mean value of the function φ over B. It is well–known that $\mathrm{L}^\infty(\mathrm{D}) \subset \mathrm{BMO(D)} \subset \mathrm{L}^p_{loc}(\mathrm{D})$ for all $1 \leq p < \infty$, see e.g. [JN] and [RR]. BMO functions are related in many ways to quasiconformal and quasiregular mappings, see e.g. [As], [AG], [Ge$_1$], [Jo], [MRV] and [Rei].

2 On finite mean oscillation

Let D be a domain in $\mathbb{R}^n$, $n \geq 1$. We say that a function $\varphi : D \to \mathbb{R}$ has a **finite mean oscillation** at a point $x_0 \in \overline{D}$ if

$$\overline{\lim_{\varepsilon\to 0}} \ \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi_\varepsilon}| \, dm(x) < \infty \tag{2.1}$$

where

$$\overline{\varphi_\varepsilon} = \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} \varphi(x) \, dm(x) \tag{2.2}$$

is the mean value of the function $\varphi(x)$ over $D(x_0,\varepsilon) = D \bigcap B(x_0,\varepsilon)$. Here

$$B(x_0,\varepsilon) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}\,. \tag{2.3}$$

Note, under (2.1) it is possible that $\overline{\varphi_\varepsilon} \to \infty$ as $\varepsilon \to 0$.

We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the domain D, abbr. $\varphi \in$ FMO(D) or simply $\varphi \in$ **FMO**, if φ has finite mean oscillation at every point $x \in D$. We also say that a function $\varphi : D \to \mathbb{R}$ is of finite mean oscillation in the closure of D, abbr. $\varphi \in \mathrm{FMO}(\overline{D})$, if φ has finite mean oscillation at every point $x \in \overline{D}$.

2.4. Proposition. *If, for some numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x) < \infty\,, \tag{2.5}$$

then (2.1) holds, too.

Proof. Indeed, by the triangle inequality

$$\frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi_\varepsilon}| \, dm(x) \leq$$

$$\leq \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x) + |\varphi_\varepsilon - \overline{\varphi}_\varepsilon| \leq$$

$$\leq \frac{2}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x)\,.$$

2.6. Corollary. *If, for a point $x_0 \in \overline{D}$,*

$$\overline{\lim_{\varepsilon\to 0}} \ \frac{1}{|D(x_0,\varepsilon)|} \int\limits_{D(x_0,\varepsilon)} |\varphi(x)| \, dm(x) < \infty\,, \tag{2.7}$$

then φ has finite mean oscillation at x_0.

2.8. Lemma. *Let $\varphi : D \to \mathbb{R}, n \geq 2$, be a nonnegative function with finite mean oscillation at $0 \in \overline{D}$ and let the set $\mathbb{R}^n \backslash D$ has a measure zero in a neighborhood of 0. Then, for some $\varepsilon_0 > 0$,*

$$\int\limits_{D \bigcap A(\varepsilon,\varepsilon_0)} \frac{\varphi(x)\, dm(x)}{\left(|x| log \frac{1}{|x|}\right)^n} = O\left(loglog \frac{1}{\varepsilon}\right) \tag{2.9}$$

as $\varepsilon \to 0$ where

$$A(\varepsilon,\varepsilon_0) = \{x \in \mathbb{R}^n : \varepsilon < |x| < \varepsilon_0 < 1\}\,. \tag{2.10}$$

Proof. Take $\varepsilon_0 \in (0, e^{-1})$ such that φ is integrable over $D_1 = D \bigcap B$ where $B = B(0,\varepsilon_0)$, and

$$\delta = \sup_{r \in (0,\varepsilon_0)} \frac{1}{|D(r)|} \int\limits_{D(r)} |\varphi(x) - \overline{\varphi_r}|\, dm(x) < \infty\,,$$

where $D(r) = D \bigcap B(r)$, $B(r) = B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$. Further, let $\varepsilon < e^{-1}\varepsilon_0$, $\varepsilon_k = e^{-k} e \varepsilon_0$, $A_k = \{x \in \mathbb{R}^n : \varepsilon_{k+1} \leq |x| < \varepsilon_k\}$, $B_k = B(\varepsilon_k)$ and let φ_k be the mean value of $\varphi(x)$ over $D_k = D \bigcap B_k, k = 1, 2 \ldots$. Take a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$ and denote $\alpha(t) = (t \log 1/t)^{-n}$. Then $D \bigcap A(\varepsilon,\varepsilon_0) \subset \Delta(\varepsilon) = \bigcup\limits_{k=1}^{N} \Delta_k$ where $\Delta_k = D \bigcap A_k$ and

$$\eta(\varepsilon) = \int\limits_{\Delta(\varepsilon)} \varphi(x)\, \alpha(|x|)\, dm(x) \leq |S_1| + S_2$$

where

$$S_1(\varepsilon) = \sum_{k=1}^{N} \int\limits_{\Delta_k} (\varphi(x) - \varphi_k)\, \alpha(|x|)\, dm(x)\,,$$

$$S_2(\varepsilon) = \sum_{k=1}^{N} \varphi_k \int\limits_{\Delta_k} \alpha(|x|)\, dm(x)\,.$$

Since $\Delta_k \subset D_k \subset B_k$, $|x|^{-n} \leq \Omega_n e^n / |D_k|$ for $x \in \Delta_k$ where Ω_n is the volume of the unit ball in $\mathbb{R}^n$, and $\log \frac{1}{|x|} > k$ in Δ_k, then

$$|S_1| \leq \delta \Omega_n e^n \sum_{k=1}^{N} \frac{1}{k^n} < 2\delta \Omega_n e^n$$

because

$$\sum_{k=2}^{\infty} \frac{1}{k^n} < \int\limits_{1}^{\infty} \frac{dt}{t^n} = \frac{1}{n-1} \leq 1\,.$$

Now,

$$\int\limits_{\Delta_k} \alpha(|x|)\, dm(x) \leq \frac{1}{k^n} \int\limits_{A_k} \frac{dm(x)}{|x|^n} = \frac{\omega_n}{k^n}$$

where ω_n is the ($(n-1)$–dimensional) volume of the unit sphere in $\mathbb{R}^n$. Moreover,

$$|\varphi_k - \varphi_{k-1}| = \frac{1}{|D_k|}\left|\int\limits_{D_k} (\varphi(x) - \varphi_{k-1})\, dm(x)\right| \leq$$

$$\leq \frac{e^n}{|D_{k-1}|}\int\limits_{D_{k-1}} |(\varphi(x) - \varphi_{k-1})|\, dm(x) \leq \delta e^n$$

and by the triangle inequality

$$\varphi_k = |\varphi_k| \leq \varphi_1 + \sum_{l=1}^{k} |\varphi_l - \varphi_{l-1}| \leq \varphi_1 + k\delta e^n\,.$$

Hence

$$S_2 = |S_2| \leq \omega_n \sum_{k=1}^{N} \frac{\varphi_k}{k^n} \leq 2\varphi_1\omega_n + \delta\omega_n e^n \sum_{k=1}^{N} \frac{1}{k^{(n-1)}}\,.$$

Finally, since

$$\sum_{k=2}^{N} \frac{1}{k} < \int\limits_1^N \frac{dt}{t} = \log N$$

and, for $\varepsilon_0 \in (0, e^{-1})$ and $\varepsilon < \varepsilon_N$,

$$N < N + \log\left(\frac{1}{e\varepsilon_0}\right) = \log\frac{1}{\varepsilon_N} < \log\frac{1}{\varepsilon}$$

then

$$\sum_{k=1}^{N} \frac{1}{k^{(n-1)}} \leq \sum_{k=1}^{N} \frac{1}{k} < 1 + \log\log\frac{1}{\varepsilon}\,,$$

and we come to the conclusion of the lemma.

As it follows from the proof, the integral given in (2.9) is even bounded as $\varepsilon \to 0$ if $n \geq 3$ because

$$\sum_{k=2}^{\infty} \frac{1}{k^{(n-1)}} < \int\limits_1^{\infty} \frac{dt}{t^{n-1}} = \frac{1}{n-2} \leq 1\,.$$

2.11. Corollary. *Under the conditions of Lemma 2.8, the singular integral*

$$\int\limits_{|x|<\varepsilon_0} \frac{\varphi(x)\, dm(x)}{\left(|x| log \frac{1}{|x|}\right)^n} < \infty\,, \tag{2.12}$$

i.e., it is convergent for $n \geq 3$.

3 On removability of isolated singularities

The following statement shows that any degree of integrability of $Q(x)$ cannot guarantee the removability of isolated singularities of Q–homeomorphisms. Here $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ denotes the unit ball in $\mathbb{R}^n$.

3.1. Proposition. *For any $p \in [1, \infty)$, there is a Q–homeomorphism $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n, n \geq 2$, with $Q \in L^p(\mathbb{B}^n)$ which has no continuous extension to $\mathbb{B}^n$.*

Proof. The desired homeomorphism f can be given in the sphereical coordinates:

$$R = 1 + r^\alpha, \quad \Theta = \vartheta\,,$$

where $\alpha \in (0, n/p)$. f maps the punctured unit ball $\mathbb{B}^n \backslash \{0\}$ onto the spherical ring $1 < |y| < 2$ in $\mathbb{R}^n$. f has no continuous extension onto $\mathbb{B}^n$.

Further, note that f has the maximal dilatation

$$Q(x) = max\left(\frac{\alpha r^\alpha}{1+r^\alpha}, \frac{1+r^\alpha}{\alpha r^\alpha}\right), \quad r = |x|\,.$$

$f \in C^1 \subset W^{1,n}_{loc}$ in $\mathbb{B}^n \backslash \{0\}$. Hence f is a Q–homeomorphism, see e.g. [MRSY$_1$] and [MRSY$_2$], cf. Theorem 4.2 below. For small $r = |x|$,

$$Q(x) = \frac{1+r^\alpha}{\alpha r^\alpha} = \frac{1}{\alpha}\left(1 + \frac{1}{r^\alpha}\right) \leq \frac{2}{\alpha} \cdot \frac{1}{r^\alpha}\,,$$

and for the rest r, Q(x) is bounded. Consequently, $Q \in L^p(\mathbb{B}^n \backslash \{0\})$ because $\alpha p < n$.

However, as the next lemma shows, it is sufficient for this goal to require the integrability of $Q(x)$ with suitable weights.

3.2. Lemma. *Let $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n, n \geq 2$, be a Q–homeomorphism. If*

$$\int\limits_{\varepsilon<|x|<1} Q(x) \cdot \psi^n(|x|) \;\; dm(x) = o(I(\varepsilon)^n) \tag{3.3}$$

as $\varepsilon \to 0$ where $\psi(t)$ is a Borel function on $(0, 1)$ such that

$$I(\varepsilon) = \int\limits_{\varepsilon}^{1} \psi(t) \;\; dt < \infty, \quad \varepsilon \in (0, 1)\,. \tag{3.4}$$

Then f has a continuous extension to $\mathbb{B}^n$ which is a Q–homeomorphism.

3.5. Remark. In particular, that holds if

$$\int\limits_{\mathbb{B}^n} Q(x) \cdot \psi^n(|x|) \;\; dm(x) < \infty \tag{3.6}$$

and $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof. Since the modulus of a family of paths passing through a fixed point $x_0 = 0$ is equal to 0 it is sufficient to show that $f(x)$ has a limit as $x \to 0$.

Let Γ_ε be a family of all paths joining the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x| = \varepsilon_0\}$ in the ring $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x| < \varepsilon_0\}$. Then the function

$$\rho_\varepsilon(x) = \begin{cases} \psi(|x|)/I(\varepsilon), & \text{if } x \in A_\varepsilon, \\ 0, & \text{if } x \in \mathbb{R}^n \backslash A_\varepsilon, \end{cases}$$

is admissible for Γ_ε and hence

$$M(f\Gamma_\varepsilon) \le \int\limits_{0<|x|<1} Q(x) \cdot {\rho_\varepsilon}^n(|x|) \;\; dm(x)\,,$$

i.e., $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (3.3).

By the generalized Schoenflies theorem the images of the above spheres fS_t, $t \in (0,1)$, split the space $\mathbb{R}^n$ into two components and, thus, $\mathbb{R}^n \backslash fA_\varepsilon$ consists exactly of two components, see e.g. [Br], [Maz] and [Mo]. Denote by Γ_ε^* the family of all paths in $\mathbb{R}^n$ joining the images of the spheres fS_ε and fS_0. Then

$$M(\Gamma_\varepsilon^*) = M(f\Gamma_\varepsilon)$$

because on the one hand $f\Gamma_\varepsilon \subset \Gamma_\varepsilon^*$ and hence $M(f\Gamma_\varepsilon) \le M(\Gamma_\varepsilon^*)$, and on the other hand $f\Gamma_\varepsilon < \Gamma_\varepsilon^*$ (i.e., every path in Γ_ε^* contains a subpath in $f\Gamma_\varepsilon$ just because fA_ε separates the two components of $\mathbb{R}^n$) and, consequently, $M(f\Gamma_\varepsilon) \ge M(\Gamma_\varepsilon^*)$, see e.g. either Theorem 1(c) in [Fu] or 6.4 in [Vä].

By the Gehring lemma, see [Ge$_2$], cf. 7.37 in [Vu],

$$M(\Gamma_\varepsilon^*) \ge a_n \Bigg/ \left(\log \frac{b_n}{\delta_0 \delta \varepsilon}\right)^{n-1}$$

where constants a_n and b_n depend only on n, δ_0 and δ_ε denote the spherical (chordal) diameters of fS_0 and fS_ε. Thus, $\delta_\varepsilon \to 0$ and fS_ε are contracted to a point as $\varepsilon \to 0$.

Combining Lemmas 2.8 and 3.2 we have the following theorem.

3.7. Theorem. *Let $f : D\backslash\{x_0\} \to \mathbb{R}^n, n \ge 2$, be a Q–homeomorphism where $Q(x)$ has finite mean oscillation at a point $x_0 \in D$. Then $f = \overline{f}|_{D\backslash\{x_0\}}$ where $\overline{f}$ is a Q–homeomorphism in D.*

In other words, an isolated singularity of a Q–homeomorphism is removable if $Q(x)$ has a finite mean oscillation at the point, say if $Q(x)$ is continuous at x_0. By Corollary 2.6 we obtain also from Theorem 3.7 that **Lebesgue points** of $Q \in L^1$ are removable singularities for Q–homeomorphisms and, more generally:

3.8. Corollary. *If $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{R}^n, n \ge 2$, is a Q–homeomorphism with*

$$\overline{\lim_{\varepsilon\to 0}}\; \frac{1}{\varepsilon^n} \int\limits_{0<|x|<\varepsilon} Q(x)\, dm(x) < \infty\,, \tag{3.9}$$

then f has a Q–homeomorphic extension to $\mathbb{B}^n$.

Similarly, choosing in Lemma 3.2 as a weight the function $\psi(t) = 1/t$ we come to the following more general statement.

3.10. Theorem. *Let $f : \mathbb{B}^n \backslash \{0\} \to \mathbb{R}^n, n \geq 2$, be a Q–homeomorphism. If*

$$\int\limits_{\varepsilon<|x|<1} Q(x)\ \frac{dm(x)}{|x|^n} = o\left(\left[log\frac{1}{\varepsilon}\right]^n\right) \tag{3.11}$$

as $\varepsilon \to 0$, then f has a Q–homeomorphic extension to $\mathbb{B}^n$.

3.12. Corollary. *The condition (3.11) and the conclusion of Theorem 3.10 hold if*

$$Q(x) = o\left(\left[\log\frac{1}{|x|}\right]^{n-1}\right) \tag{3.13}$$

as $x \to 0$. The same holds if

$$q(r) = o\left(\left[\log\frac{1}{r}\right]^{n-1}\right) \tag{3.14}$$

as $r \to 0$ where $q(r)$ is the mean value of the function $Q(x)$ over the sphere $|x| = r$.

3.15. Remark. Choosing in the Lemma 3.2 instead of $\psi(t) = 1/t$ the function $\psi(t) = 1/(t \log 1/t)$, we obtain instead of (3.11) the condition

$$\int\limits_{\varepsilon<|x|<1} \frac{Q(x)\, dm(x)}{\left(|x| \log \frac{1}{|x|}\right)^n} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^n\right)\ , \tag{3.16}$$

and instead of (3.14) the condition

$$q(r) = o\left(\left[\log\frac{1}{r}\log\log\frac{1}{r}\right]^{n-1}\right) \tag{3.17}$$

and, thus, it is sufficient to inquire instead of (3.14) the condition

$$q(r) = O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)\ . \tag{3.18}$$

In general, we could give here the whole scale of the corresponding conditions in log.

4 On super Q–homeomorphisms

From this section we start to study **super Q–homeomorphisms**, i.e., such Q–homeomorphisms $f : D \to \mathbb{R}^n, n \geq 2$, that the inequality (1.1) holds not only for all families Γ of continuous paths $\gamma : (0, 1) \to D$ but also dashed lines $\gamma : \Delta \to D$

where the mapping γ is continuous on an arbitrary open subset Δ of the real axis $\mathbb{R}$ consisting of a sequence of intervals $\Delta_i \subset \mathbb{R}$, $i = 1, 2, \ldots$. We say that a family Γ of dashed lines is minorized by another such family Γ^*, abbr. $\Gamma \geq \Gamma^*$, if for every line $\gamma \in \Gamma, \gamma : \Delta \to \mathbb{R}^n$, there is a line $\gamma^* \in \Gamma^*, \gamma^* : \Delta^* \to \mathbb{R}^n$, which is a restriction of γ, i.e., $\Delta^* \subset \Delta$ and $\gamma^* = \gamma|_{\Delta^*}$. Later on, the following property is useful, see Theorem 1(c) in [Fu], c. 178.

4.1. Proposition. *Let Γ and Γ^* be families of dashed lines. If $\Gamma \geq \Gamma^*$, then $M(\Gamma) \leq M(\Gamma^*)$.*

We say that a property P holds for **almost every (a.e.)** dashed lines γ in a family Γ if the subfamily of all lines in Γ for which P fails has modulus zero. In particular, almost all dashed lines in $\mathbb{R}^n$ are rectifiable. All definitions of the modulus, rectifiability and so on are perfectly similar and hence omitted. For more general considerations of systems of measures, see [Fu].

4.2. Theorem. *Let a function $Q : D \to [1, \infty]$ belongs to L^{n-1}_{loc} in a domain $D \subset \mathbb{R}^n, n \geq 2$. Then every $Q(x)$-quasiconformal mapping $f : D \to \mathbb{R}^n$ is a super Q–homeomorphism.*

Proof. By Theorem 6.1 in [HK] $f^{-1} \in W^{1,n}_{loc}$ and, consequently, $f^{-1} \in ACL^n_{loc}$, see e.g. [Ma], p. 8. By the Fuglede theorem, see [Fu] and [Vä], the modulus of all locally rectifiable paths in $f(D)$ with at least one closed subpath where f^{-1} is not absolutely continuous has the modulus zero. This family of paths minorizes the corresponding family of dashed lines and, thus, by Proposition 4.1 the latter family is also has the modulus zero.

Let Γ be an arbitrary family of dashed lines in D. Let us denote by Γ^* the family of all dashed lines $\gamma^* \in f\Gamma$ for which f^{-1} is absolutely continuous on every closed subpath of γ^*. Then $M(f\Gamma) = M(\Gamma^*)$.

For $\rho \in adm\,\Gamma$, set $\rho^*(y) = \rho(f^{-1}(y)) \cdot |(f^{-1})'(y)|$ if $y \in f(D)$ and $\rho^*(y) = 0$ outside of $f(D)$. Then

$$\int\limits_{\gamma^*} \rho^* ds^* = \int\limits_{f^{-1}\circ\gamma^*} \rho\, ds \geq 1 \tag{4.3}$$

for all $\gamma^* \in \Gamma^*$, i.e., $\rho^* \in adm\,\Gamma^*$.

By [HK] f^{-1} has the property (N) and is differentiable a.e. Hence

$$\begin{aligned} M(f\Gamma) = M(\Gamma^*) &\leq \int\limits_{f(D)} \rho^* dm(y) \leq \\ \leq \int\limits_{f(D)} \rho(f^{-1}(y))^n\, K_O(y, f^{-1})\, J(y, f^{-1})\, dm(y) &= \\ = \int\limits_{D} \rho(x)^n\, K_I(x, f)\, dm(x) &\leq \int\limits_{D} Q(x)\rho(x)^n\, dm(x), \end{aligned}$$

i.e., f is really a super Q–homeomorphism.

Theorem 4.2 shows that super Q–homeomorphisms form a very wide important subclass of Q–homeomorphisms including many mappings with finite distortion.

5 Topological lemmas

Here we prove a pair of lemmas that will replace the Schoenflies theorem in the cases when we will consider, instead of an isolated singular point, a singular set which is infinite and even uncountable.

Recall that by the well–known Borsuk theorem, see e.g. [Du], p. 357, a compact set $K \subset \mathbb{R}^n$ separates $\mathbb{R}^n$ if and only if there is a continuous mapping $f : K \to S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ which is not homotopic to a constant mapping. Conversely, the statement that a compactum K does not separate $\mathbb{R}^n$ is equivalent to the statement that each continuous mapping $f : K \to S^{n-1}$ is homotopic to a constant mapping. Thus, we obtain the following simple corollary of the Borsuk theorem.

5.1. Proposition. *Let K_1 and K_2 be compact sets in $\overline{\mathbb{R}^n}$ every of which does not separate $\overline{\mathbb{R}^n}$. Then the compactum $K = K_1 \bigcup K_2$ does not separate $\overline{\mathbb{R}^n}$, too.*

On this base we prove the following statement.

5.2. Lemma. *Let D be a domain in $\overline{\mathbb{R}^n}, n \geq 2$. Then the boundary of every component of its complement $\overline{\mathbb{R}^n} \backslash D$ is a component of ∂D.*

Proof. Let assume that the conclusion of Lemma 5.2 is not true, i.e., there is a domain D in $\overline{\mathbb{R}^n}, n \geq 2$, such that the boundary of a component C_0 of $\overline{\mathbb{R}^n} \backslash D$ does not coincide with a component of ∂D.

As known $\partial C_0 \subset \partial D$, see e.g., [Du], p. 356. Moreover, if a component of K of the boundary ∂D has a nonempty intersection with ∂C_0, then $K \subset \partial C_0$. Thus, the negation of the statement of Lemma 5.2 is equivalent to the property of ∂C_0 to consist of more than one component of ∂D, i.e., to the statement that ∂C_0 is not a connected set.

Every component of $\overline{\mathbb{R}^n} \backslash D$ is a closed set. Hence joining to the domain D, if it is necessary, all components of its complement $\overline{\mathbb{R}^n} \backslash D$ with the exception of C_0, we may consider without loss of generality that $\overline{\mathbb{R}^n} \backslash D$ has only one component C_0 and, consequently, $\overline{\mathbb{R}^n} \backslash C_0 = D$.

By the above assumption ∂C_0 can be split into two disjoint closed sets S_1 and S_2 which are compact in $\overline{\mathbb{R}^n}$. Let Ω_1 be a component of the complement $\overline{\mathbb{R}^n} \backslash S_1$ including the domain D. Then the compactum $K_1 = \overline{\mathbb{R}^n} \backslash \Omega_1$ includes S_1 and does not separate $\overline{\mathbb{R}^n}$. Moreover, by the construction $K_1 \subset C_0$ and $S_2 \subset \Omega_1$ (the latter because of every neighborhood of each point in $S_2 \subset \partial C_0$ must involve points of $D \subset \Omega_1$ but the spherical distance $h(S_2, \partial\Omega_1) \geq h(S_2, S_1) > 0$ since $\partial\Omega_1 \subset S_1$, see [Du], p. 356). Ω_2 and K_2 are defined in a similar way through S_2.

Note that $K_1 \bigcap K_2 = \emptyset$. Indeed,

$$\partial K_1 = \partial\Omega_1 \subset S_1 \subset \Omega_2 = \overline{\mathbb{R}^n} \backslash K_2$$

and

$$\partial K_2 = \partial\Omega_2 \subset S_2 \subset \Omega_1 = \overline{\mathbb{R}^n} \backslash K_1 \,.$$

Hence $\partial K_1 \bigcap K_2 = \emptyset$ and $\partial K_2 \bigcap K_1 = \emptyset$. Thus, if

$$K_1 \bigcap K_2 \neq \emptyset\,, \tag{5.3}$$

then

$$Int\, K_1 \bigcap Int\, K_2 \neq \emptyset\,,$$

i.e., there exist components $\Omega^{(1)} \neq \Omega_1$, $\Omega^{(1)} \subset Int\, K_1$ and $\Omega^{(2)} \neq \Omega_2$, $\Omega^{(2)} \subset Int\, K_2$ of the complements $\overline{\mathbb{R}^n}\backslash S_1$ and $\overline{\mathbb{R}^n}\backslash S_2$, correspondingly, such that $\Omega^{(1)} \bigcap \Omega^{(2)} \neq \emptyset$. If $\Omega^{(1)} = \Omega^{(2)}$, then $\partial\Omega^{(1)} = \partial\Omega^{(2)} \subset S_1 \bigcap S_2 = \emptyset$, i.e., $\partial\Omega^{(1)} = \partial\Omega^{(2)} = \emptyset$ and $\Omega^{(1)} = \Omega^{(2)} = \overline{\mathbb{R}^n}$ (because $\overline{\mathbb{R}^n}$ is a connected space) that is impossible by the construction. Let for the definiteness $\Omega^{(1)}\backslash\Omega^{(2)} \neq \emptyset$. Then $\partial\Omega^{(2)} \bigcap \Omega^{(1)} \neq \emptyset$ because $\emptyset \neq \Omega^{(1)} \bigcap \Omega^{(2)} \subset \Omega^{(1)}$ and $\Omega^{(1)}$ is connected. However, $\partial\Omega^{(2)} \subset S_2$ and $\Omega^{(1)} \subset K_1$. Consequently, the assumption (5.3) contradicts to the inclusion $S_2 \subset \Omega_1 = \overline{\mathbb{R}^n}\backslash K_1$.

Then by Proposition 5.1 the compactum $K = K_1 \bigcup K_2 \subset C_0$ does not separate $\overline{\mathbb{R}^n}$, i.e., $\Omega = \overline{\mathbb{R}^n}\backslash K = \Omega_1 \bigcap \Omega_2$ is a domain in $\overline{\mathbb{R}^n}$.

Further, since $\partial C_0 = S_1 \bigcup S_2 \subset K_1 \bigcup K_2 = K$ and $K_1 \bigcap K_2 = \emptyset$ and C_0 is connected, then $\Omega_0 = \Omega \bigcap\, Int\, C_0 \neq \emptyset$ (if $Int\, C_0 \subset K$, then $C_0 = K = K_1 \bigcup K_2$). However, by the construction $D \subset \Omega$, $D \bigcup C_0 = \overline{\mathbb{R}^n}$, $D \bigcap C_0 = \emptyset$ and $\Omega = D \bigcup \Omega_0$. The latter contradicts to the connectivity of Ω. Thus, the assumption that ∂C_0 is not connected is really not true and we come to the conclusion of the lemma.

5.4. Corollary. *The interiority $Int\, K$ of every component K of the complement $\overline{\mathbb{R}^n}\backslash D$ is separated from $\overline{\mathbb{R}^n}\backslash K$ by a component of ∂D.*

5.5. Lemma. *Let D be domain in $\overline{\mathbb{R}^n}$, $n \geq 2$, and let $f : D \to \overline{\mathbb{R}^n}$ be a homeomorphism. Then $D' = f(D)$ is a domain and there is a natural one–to–one correspondence between components K and K' of the boundaries ∂D and $\partial D'$ such that $C(f, K) = K'$ and $C(f^{-1}, K') = K$.*

Here we use the notation

$$C(f, E) = \{y \in \overline{\mathbb{R}^n} : y = \lim_{l\to\infty} f(x_l), x_l \to x \in E\} \tag{5.6}$$

for sets $E \subset \partial D$.

Proof. $D' = f(D)$ is a domain by the well–known Brouwer theorem, see e.g. [Du], p. 358.

Further, for every set $E \subset \partial D$, $C(f, E) \subset \partial D'$, and, similarly, for every set $E' \subset \partial D'$, $C(f^{-1}, E') \subset \partial D$.

Indeed, by the definition $C(f, E) \subset \overline{D'}$. Let us assume that $y_0 \in C(f, E) \bigcap D'$. Let $x_0 = f^{-1}(y_0)$. $x_0 \in D$ and hence $\delta_0 = dist\,(x_0, \partial D)/2 > 0$. Let $x_k \in D$ such that $f(x_k) \to y_0$ and $dist\,(x_k, E) \to 0$ as $k \to \infty$. Then $x_k \overline{\in} B(x_0, \delta_0)$ for great enough k and, simultaneously, $x_k = f^{-1}(f(x_k)) \to f^{-1}(y_0) = x_0$ as $k \to \infty$ by the continuity of f^{-1}. The contradiction disproves the assumption.

Let K be a component of the boundary of D. It is clear that K is a closed subset of ∂D which is a compact set in $\overline{\mathbb{R}^n}$ and hence K is a continuum. Further, let

$$\delta_\varepsilon = \{x \in D : h(x, K) < \varepsilon\}$$

where h is a spherical (chordal) distance in $\overline{\mathbb{R}^n}$. Then

$$C(f,K)=\bigcap_{\varepsilon>0}\overline{f(\delta_\varepsilon)}\,.$$

Denote by D_ε a component of $\overline{f(\delta_\varepsilon)}$ including $C(f,K)$. The sets D_ε form decreasing family of continua and

$$C(f,K)=\bigcap_{\varepsilon>0}D_\varepsilon\,.$$

Thus, $C(f,K)$ is a continuum, see e.g. [Wh], p. 15.

Denote by K' the component of $\partial D'$ including $C(f,K)$. Then arguing as above we obtain that $C(f^{-1},K')$ is a continuum which by the construction includes K and hence $K=C(f^{-1},K')$. In view of symmetry of the conditions of the lemma with respect to f and f^{-1}, it is also $K'=C(f,K)$.

5.7. Corollary. *Let $f: D\setminus X\to D'$ be a homeomorphism where X is a closed totally disconnected subset of D. If f has a continuous extension $\overline{f}$ to D, then $\overline{f}$ is a homeomorphism of D onto $\overline{f}(D)$.*

Here the set $X\subset\mathbb{R}^n$ is called **totally disconnected** if every (connected) component of X consists of one point.

6 On singular sets of the length zero

In this section we consider the problem of removability of singularities for super Q–homeomorphisms. A set X in $\mathbb{R}^n$ is called **a set of the length zero** if X can be covered by a sequence of balls in $\mathbb{R}^n$ with an arbitrary small sum of diameters. As known, such sets have the (Lebesgue) measure zero and

$$dim\,X=0 \tag{6.1}$$

and hence they are totally disconnected, see e.g. [HW]. A classical example of such sets is the Cantor discontinuum every point of which is its point of the condensation, i.e., each neighborhood of the point includes an uncountable number of other points of the set.

By the well–known theorem of Menger and Urysohn (6.1) guarantees that X does not separate a domain D in $\mathbb{R}^n$ and, thus, if X is closed in D, then $D^*=D\setminus X$ is also a domain.

6.2. Lemma. *Let D be a domain in $\mathbb{R}^n, n\geq 2$, X be a closed subset of D of the length zero and let $f: D\setminus X\to\mathbb{R}^n$ be a super Q–homeomorphism. If, for every $x_0\in X$,*

$$\int\limits_{\varepsilon<|x|<\delta(x_0)} Q(x_0+x)\cdot\psi^n_{x_0,\varepsilon}(|x|)\;\;dm(x)=o(I(x_0,\varepsilon)^n) \tag{6.3}$$

as $\varepsilon \to 0$ *where* $0 < \delta(x_0) < dist\,(x_0, \partial D)$ *and* $\psi_{x_0,\varepsilon}(t)$, $\varepsilon \in (0, \delta(x_0))$, *are nonnegative Borel functions on* $(0,1)$ *such that*

$$I(x_0, \varepsilon) = \int\limits_{\varepsilon}^{\delta(x_0)} \psi_{x_0,\varepsilon}(t) \ \ dt < \infty\,, \tag{6.4}$$

then f *has a continuous extension to* D *which is a homeomorphism.*

Proof. By Corollary 5.7 it is sufficient to prove that the mapping f is extended by continuity to D, i.e., there is a limit $f(x)$ as $x \to x_0$, $x \in D\backslash X$, for every $x_0 \in X$.

Let Γ_ε be the family of all open arcs (injective paths) joining the spheres $|x| = \varepsilon$ and $|x| = \delta(x_0)$ in the ring $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x| < \delta(x_0)\}$, and Γ_ε° be the family of the corresponding dashed lines in $A_\varepsilon \backslash X$ obtained from the arcs of Γ_ε by the rejection of all the points in X. Then the function

$$\rho_\varepsilon(x) = \begin{cases} \psi_{x_0,\varepsilon}(|x|)/I(x_0,\varepsilon), & \text{if } x \in A_\varepsilon \setminus X, \\ 0, & \text{if } x \in \mathbb{R}^n \backslash A_\varepsilon \setminus X, \end{cases} \tag{6.5}$$

is admissible for Γ_ε° because X is of the length zero, see e.g. Remark 30.11 in [Vä], and hence

$$M(f\Gamma_\varepsilon^\circ) \le \int\limits_{D\backslash X} Q(x) \cdot \rho_\varepsilon^n(|x|) \ \ dm(x)\,. \tag{6.6}$$

Consequently,

$$\lim_{\varepsilon \to 0} M(f\Gamma_\varepsilon^\circ) = 0 \tag{6.7}$$

by the condition (6.3).

Denote by Γ_ε^* the family of all open arcs in $\mathbb{R}^n$ joining the images of the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x| = \delta(x_0)\}$ under the mapping f. Then, as it is clear from Lemmas 5.2 and 5.5,

$$f\Gamma_\varepsilon^\circ \le \Gamma_\varepsilon^* \tag{6.8}$$

and, consequently,

$$M(f\Gamma_\varepsilon^\circ) \ge M(\Gamma_\varepsilon^*)\,, \tag{6.9}$$

see Theorem 1(c) in [Fu], p. 178.

Recall also that

$$M(\Gamma_\varepsilon^*) = M(\Gamma_\varepsilon') \tag{6.10}$$

where Γ_ε' the family of all paths joining fS_ε and fS_0 in $\mathbb{R}^n$, see e.g. Remark 7.11 in [Vä]. On the other hand, by the Gehring lemma, see [Ge$_2$], cf. 7.37 in [Vu],

$$M(\Gamma_\varepsilon') \ge a_n \Bigg/ \left(\log \frac{b_n}{\delta_0 \delta_\varepsilon}\right)^{n-1} \tag{6.11}$$

where a_n and b_n are constants depending only on n, and δ_ε and δ_0 are the diameters of fS_ε and fS_0 in the spherical (chordal) metric in $\overline{\mathbb{R}^n}$.

Finally, the relations (6.7)–(6.11) imply that $\delta_\varepsilon \to 0$, i.e., fS_ε is contracted to a point and, thus, the conclusion of the lemma follows.

Combining Lemmas 2.8 and 6.2, we obtain the following result.

6.12. Theorem. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, X be a closed subset of D of the length zero and let $f : D \backslash X \to \mathbb{R}^n$ be a super Q–homeomorphism. If the function $Q(x)$ has finite mean oscillation at every point $x_0 \in X$, then f has a homeomorphic extension to D.*

6.13. Corollary. *In particular, if the function $Q(x)$ is integrable in a neighborhood of every point $x_0 \in X$ and has finite limit as $x \to x_0$, then there is a limit $f(x)$ as $x \to x_0$ for every $x_0 \in X$.*

6.14. Corollary. *Let X be a closed subset of the length zero in D and let*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(x_0, \varepsilon)|} \int\limits_{B(x_0,\varepsilon)} Q(x) \ \ dm(x) < \infty \tag{6.15}$$

for every $x_0 \in X$. Then every super Q–homeomorphism $f : D \backslash X \to \mathbb{R}^n$ has homeomorphic extension to D.

6.16. Corollary. *Let X be a closed subset of the length zero in D and let $Q(x)$ be integrable in a neighborhood of X and every point of X is a Lebesgue point of $Q(x)$. Then every super Q–homeomorphism $f : D \backslash X \to \mathbb{R}^n$ has homeomorphic extension to D.*

Choosing in Lemma 6.2 $\psi(t) = 1/t$, we obtain the following theorem.

6.17. Theorem. *Let D be a domain in $\mathbb{R}^n, n \geq 2$, X be a closed subset of D of the length zero and let, for every $x_0 \in X$,*

$$\int\limits_{\varepsilon < |x| < \delta(x_0)} Q(x_0 + x) \ \ \frac{dm(x)}{|x|^n} = o\left(\left[\log \frac{1}{\varepsilon}\right]^n\right) \tag{6.18}$$

as $\varepsilon \to 0$ where $0 < \delta(x_0) < dist(x_0, \partial D)$. Then every super Q–homeomorphism $f : D \backslash X \to \mathbb{R}^n$ has a homeomorphic extension to D.

6.19. Corollary. *In particular, if the singular integral*

$$\int\limits_U \frac{Q(y) - Q(x)}{|y - x|^n} \, dm(y) \tag{6.20}$$

is convergent for every $x \in X$ over a neighborhood U of the set X, then f has a homeomorphic extension to D.

6.21. Remark. As it is clear from the well–known example of (conformal) Zhukovskii mapping f of the complement of a segment onto the complement of the unit disk in $\mathbb{C}$, the condition of the length zero for singular sets is essential and the results cannot be extended (without additional geometric conditions) to singular sets of a finite positive length even under the best possible maximal dilatation $K(x, f) \equiv 1$, cf. in the context the interesting work [Mi].

The results of the paper are applicable, in particular, to BMO–quasiconformal mappings and homeomorphisms of finite length distortion, see e.g. [MRSY_1]–[MRSY_4].

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References

[As] Astala K., *A remark on quasiconformal mappings and BMO–functions*, Michigan Math. J. 30 (1983), 209–212.

[AG] Astala K. and Gehring F.W., *Injectivity, the BMO norm and the universal Teichmuller space*, J. Anal. Math. 46 (1986), 16–57.

[AIKM] Astala K., Iwaniec T., Koskela P.and Martin G., *Mappings of BMO-bounded distortion*, Math. Annalen 317 (2000), 703–726.

[Br] Brown M., *A proof of generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), 74-76.

[Du] Dugundji J., *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[Fu] Fuglede B., *Extremal length and functional completion*, Acta Math. 98 (1957), 171–219.

[Ge_1] Gehring F.W., *Characteristic Properties of Quasidisk*, Les presses de l'Universite de Montreal, Montreal, 1982.

[Ge_2] Gehring F.W., *Quasiconformal mappings, in Complex Analysis and its Applications*, V. 2., International Atomic Energy Agency, Vienna, 1976.

[GI] Gehring F.W. and Iwaniec T., *The limit of mappings with finite distortion*, Ann. Acad. Sci. Fenn. Math. 24 (1999), 253–264.

[HK] Heinonen J. and Koskela P., *Sobolev mappings with integrable dilatations*, Arch. Rational Mech. Anal. 125 (1993), 81–97.

[HW] Hurewicz W. and Wallman H., *Dimension Theory,* Princeton Univ. Press, Princeton, 1948.

[IM] Iwaniec T. and Martin G., *Geometrical Function Theory and Non–linear Analysis*, Clarendon Press, Oxford, 2001.

[IS] Iwaniec T. and Šverák V., *On mappings with integrable dilatation*, Proc. Amer. Math. Soc. 118 (1993), 181–188.

[Jo] P.M. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. 29 (1980), 41–66.

[JN] John F., and Nirenberg L., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.

[KKM$_1$] Kauhanen J., Koskela P. and Maly J., *Mappings of finite distortion: discreteness and openness*, Arch. Rational Mech. Anal. 160 (2001), 135–151.

[KKM$_2$] Kauhanen J., Koskela P. and Maly J., *Mappings of finite distortion: condition N*, Michigan Math. J. 49 (2001), 169–181.

[KO] Koskela P. and Onninen J., *Mappings of finite distortion: capacity and modulus inequalities*, Dept. Math. Stat., University of Jyväskylä, Preprint 257 (2002), 1–32.

[Ma] Maz'ya V., *Sobolev classes*, Springer, Berlin–New York, 1985.

[Maz] Mazur B., *On embeddings of spheres*, Bull. Amer. Math. Soc. 65 (1959), 59-65.

[Mi] Mikljukov V.M., *Removable singularities of quasiconformal mappings in space*, Dokl. Akad. Nauk SSSR 188 (1969), 525-527.

[Mo] Morse M., *A reduction of the Schoenflies extension theorem*, Bull. Amer. Math. Soc. 66 (1960), 113-115.

[MRSY$_1$] Martio O., Ryazanov V., Srebro U. and Yakubov E., *BMO–quasiconformal mappings and Q-homeomorphisms in space*, 2001, 24 pp. Preprint of Department of Mathematics, University of Helsinki, 2001.288.

[MRSY$_2$] Martio O., Ryazanov V., Srebro U. and Yakubov E., *To the theory of Q-homeomorphisms*, Dokl. Akad. Nauk Rossii 381, no. 1 (2001), 20–22.

[MRSY$_3$] Martio O., Ryazanov V., Srebro U. and Yakubov E., *On the boundary behavior of Q-homeomorphisms*, 2002, 12 pp. Preprint of Department of Mathematics, University of Helsinki, 2002.318.

[MRSY$_4$] Martio O., Ryazanov V., Srebro U. and Yakubov E., *Mappings of finite length distortion*, 2002, 20 pp. Preprint of Department of Mathematics, University of Helsinki, 2002.322.

[MRV] Martio O., Ryazanov V. and Vuorinen M., *BMO and Injectivity of Space Quasiregular Mappings*, Math. Nachr. 205 (1999), 149–161.

[MV$_1$] Manfredi J.J. and Villamor E., *Mappings with integrable dilatation in higher dimensions*, Bull. Amer. Math. Soc. 32, no. 2 (1995), 235-240.

[MV$_2$] Manfredi J.J. and Villamor E., *An extension of Reshetnyak's theorem*, Indiana Univ. Math. J. 47, no. 3 (1998), 1131-1145.

[Rei] Reimann H.M., *Functions of bounded mean oscillation and quasiconformal mappings*, Comment. Math. Helv. 49 (1974), 260–276.

[RR] Reimann H.M. and Rychener T., *Funktionen Beschrnkter Mittlerer Oscillation*, Springer, Berlin etc., 1975.

[RSY$_1$] V. Ryazanov, U. Srebro and E. Yakubov, *BMO–quasiconformal mappings*, 1997, 22 pp. Preprint of Department of Mathematics, University of Helsinki, 97.155.

[RSY$_2$] V. Ryazanov, U. Srebro and E. Yakubov, *To the theory of BMO-quasiregular mappings*, Dokl. Akad. Nauk Rossii 369, no 1 (1999), 13–15.

[RSY$_3$] V. Ryazanov, U. Srebro and E. Yakubov, *BMO-quasiconformal mappings*, J. d'Analyse Math. 83 (2001), 1–20.

[RSY$_4$] V. Ryazanov, U. Srebro and E. Yakubov, *Plane mappings with dilatation dominated by functions of bounded mean oscillation*, Sib. Adv. in Math. 11, No. 2 (2001), 94–130.

[Vä] Väisälä J., *Lectures on n–Dimensional Quasiconformal Mappings,* Lecture Notes in Math. 229, Berlin etc., Springer–Verlag, 1971.

[Vu] Vuorinen M., *Conformal Geometry and Quasiregular Mappings,* Lecture Notes in Math. 1319, Berlin etc., Springer–Verlag, 1988.

[Wh] Whyburn G.Th., *Analytic Topology*, Providence, Amer. Math. Soc., 1942.

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