Some remarks on an existence problem for degenerate elliptic systems

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`FILE: degen19.tex` 2002-10-09

Abstract

The system $au_x + bu_y = v_y, \quad cu_x + du_y = -v_x$, which yields Beltrami's system if $b = c$, is considered. Under a condition for the coefficients a, b, c, d a non-existence theorem is proved.

1 Main Results

Let D, $\mathcal{D} \subset \mathbf{R}^2$ be simply connected domains of $z = (x, y)$ and $w = (u, v)$ – planes, respectively. Below we consider the problem of existence of homeomorphic $ACL^2_{\rm loc}$ mappings $w = w(z)$ from D onto $\mathcal{D}$ satisfying the system

$$a\, u_x + b\, u_y = v_y, \quad c\, u_x + d\, u_y = -v_x. \tag{1.1}$$

Here coefficients a, b, c, d are measurable functions in D and such that $a > 0$ and

$$\operatorname{ess\,inf}_{D'} \Big(ad - (b + c)^2/4\Big) > 0 \quad \text{for any} \quad D' \subset\subset D. \tag{1.2}$$

In the particular case $a = d = 1$ and $b = c = 0$, (1.1) is the classic Cauchy-Riemann system and the solution of the existence problem is given by the Riemann mapping theorem. In the case $b = c$, we have the well-known Beltrami system. About existence theorems here see, for example, G. David [4], U. Srebro and E. Yakubov [13], V. Gutlyanskii, O. Martio, T. Sugawa and M. Vuorinen [7], O. Martio, V. Miklyukov [9] etc.

In the general case, the global mapping problem is essentially more complicated than in the aforementioned particular cases. Under the assumption

$$\operatorname{ess\,inf}_{D} \Big(ad - (b + c)^2/4\Big) > 0 \tag{1.3}$$

the existence problem was solved by B. Bojarski [1]. If this condition is not fullfilled, then there are isolated results only. The case in which (1.3) is violated at a finite number of boundary points was considered by A. Dzuraev [5] and E.A. Chtcherbakov [3]. On the other hand, I.S. Ovchinnikov [12] and A.P. Mihailov [10] proved some results about the solvability problem of the system (1.1) with (1.2) under some special conditions, which allow degeneration close to a boundary arc.

Here we give a condition for coefficients of (1.1) under which the coordinate function $u(z)$ of $w(z)$ is monotone close to boundary [11]. We use this condition for a non-existence theorem.

Let $D \subset \mathbf{R}^2$ be a domain. By $\tilde{\partial} D$ we denote the boundary of D in the extended plane $\tilde{\mathbf{R}}^2 = \mathbf{R}^2 \cup \{\infty\}$. For an arbitrary subdomain $\Delta \subset D$, we set

$$\partial' \Delta = \tilde{\partial}\Delta \setminus \tilde{\partial} D \quad \text{and} \quad \partial''\Delta = \tilde{\partial}\Delta \cap \tilde{\partial} D\,.$$

Let Γ be a subset of $\tilde{\partial} D$. A continuous function$f : D \to \mathbf{R}$ is called *monotone close to* Γ if for any subdomain $\Delta \subset D$ with $\partial''\Delta \subset \Gamma$,

$$\operatorname{osc}(f, \Delta) \le \operatorname{osc}(f, \partial'\Delta). \tag{1.4}$$

Here the symbol $\operatorname{osc}(f, E)$ means the oscillation of f along the set E.

Every function f monotone close to Γ is monotone in the classical sense of Lebesgue since for domains $\Delta \subset\subset D$, (1.4) reduces to $\operatorname{osc}(f, \Delta) \le \operatorname{osc}(f, \partial\Delta)$. On the other hand, if $\Gamma = \tilde{\partial} D$ then every function monotone close to Γ is a constant function. This is obvious since choosing $\Delta = D \setminus \{z_0\}$, where $z_0 \in D$ is an arbitrary point, by (1.4) we obtain

$$\operatorname{osc}(f, \Delta) \le \operatorname{osc}(f, \{z_0\}) = 0.$$

We shall describe the behaviour of coefficients of the system close to a set of degeneracy by means of a special exhaustion function h. Fix a set $\Gamma \subset \tilde{\partial} D$ and a positive locally Lipschitz function $h : D \to \mathbf{R}$ such that $\lim_{z\to\Gamma} h(z) = 0$ and

$$0 < h_0 \le \operatorname{ess\,inf}_D |\nabla h(z)| \le \operatorname{ess\,sup}_D |\nabla h(z)| \le h_1 < \infty, \tag{1.5}$$

where h_0, h_1 are some constants.

Below we let $E_t = \{z \in D : h(z) = t\}$ denote the level line of h.

1.6. Theorem. *Let $\Gamma \subset \tilde{\partial} D$ be an arbitrary set and let $w = (u, v)$ be an ACL^2_{loc} homeomorphic solution of (1.1) with (1.2) from D onto $\mathcal{D}$ such that $|u| < M$ in D. If*

$$\int\limits_0 dt \left(\int\limits_{D\cap E_t} (a + d)\frac{ad - bc}{\Delta}\, d\mathcal{H}^1(E_t) \right)^{-1} = \infty, \quad \Delta = ad - \left(\frac{b + c}{2}\right)^2, \tag{1.7}$$

then u is monotone close to Γ.

For an arbitrary $\Gamma \subset \tilde{\partial} D$ we set

$$w(\Gamma) = \{w \in \tilde{\partial}\mathcal{D} : \exists \text{ a sequence } z_n \in D,\ z_n \to \Gamma, \text{ such that } w(z_n) \to w\}.$$

Clearly, if $\Gamma \subset \partial D$ is connected and ∂D is a simple Jordan curve, then $w(\Gamma)$ also is connected.

A set $L \subset \tilde{\partial}\mathcal{D}$ is called *u-forked* if there are at least two points $w' = (u', v')$, $w'' = (u', v'') \in L$ such that the segment

$$l = \{(u, v) \in \mathbf{R}^2 : u = u',\ v' < v < v''\} \quad \text{lies in} \quad \mathcal{D}$$

and separates from $\tilde{\partial} D \setminus L$ some subdomain $U \subset \mathcal{D}$ with $\tilde{\partial} U \subset \bar{l} \cup L$.

For example, let $\mathcal{D}$ be a disk and let $\partial\mathcal{D}$ be its boundary circle. Then the right and left semicircles are u-forked, however the upper and lower semicircles are not u-forked.

1.8. Theorem. *Suppose that $\mathcal{D}$ is a subdomain of $\{(u,v) : |u| < M\}$, coefficients of (1.1) satisfy (1.7) and $L \subset \tilde{\partial}\mathcal{D}$ is u-forked. Then there is no ACL^2_{loc} homeomorphic solution $w = w(z)$ of (1.1) from D onto $\mathcal{D}$ such that $w(\Gamma) \supset L$.*

2 Proof of Theorem 1.6

Let Δ be a subdomain of D with $\partial''\Delta \subset \Gamma$. We prove that

$$\sup_{\Delta} u(z) = \sup_{\partial'\Delta} u(z). \tag{2.9}$$

Assume the contrary, that is, there exists a point $z_0 \in \Delta$ such that

$$u(z_0) > \sup_{\partial'\Delta} u(z) = A.$$

Choose $\epsilon > A$ such that $u(z_0) > \epsilon$. Fix the connected component U of the set $\{z \in \Delta : u(z) > \epsilon\}$ containing z_0. By [14, Theorem 5.4.4] for almost all $\epsilon > A$, sets $\{z \in \Delta : u(z) = \epsilon\}$ are locally rectifiable. Therefore, without loss of generality, we may assume that $\partial' U$ is locally rectifiable.

Fix numbers $0 < \delta' < \delta'' < h(z_0)$ and an arbitrary nonnegative Lipschitz function $\psi_0 : \mathbf{R}_+ \to \mathbf{R}$ such that $\psi : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$\psi(\tau) = \begin{cases} 1 & \text{for} \quad \delta'' < \tau < \infty, \\ \psi_0(\tau) & \text{for} \quad \delta' \le \tau \le \delta'', \\ 0 & \text{for} \quad 0 < \tau < \delta' \end{cases}$$

is continuous. (We shall specify ψ_0 later.) Denote $\phi = \psi^2\,(u - \epsilon)$ (with $\psi = \psi(h(z))$) for $z \in U$ and $\phi \equiv 0$ for $z \in D \setminus U$. Using [8, Theorem 1.20] we conclude that $\phi \in ACL^2(D)$. Because $\operatorname{supp}\phi \subset\subset D$ by the Green formula, we have

$$\int_U d(\phi\, dv) = 0$$

and hence

$$\int_U d\phi \wedge dv = 0.$$

Because

$$d\phi \wedge dv = \psi^2 \, du \wedge dv + 2\psi \, (u-\epsilon) \, d\psi \wedge dv,$$

then we obtain

$$\int\limits_U \psi^2 (u_x v_y - u_y v_x) \, dxdy = -2 \int\limits_U \psi \, (u-\epsilon) \, (\psi_x v_y - \psi_y v_x) \, dxdy.$$

This is clear for $u, v \in C^2(D)$. In the general case we can easily prove it if we use a standard approximation procedure (see, for example, [8, Lemma 14.15]).

From here

$$\int\limits_U \psi^2 (u_x v_y - u_y v_x) \, dxdy \leq 2 \int\limits_U \psi |u-\epsilon| |\nabla\psi| \, |\nabla v| \, dxdy.$$

Using (1.1) we find

$$\int\limits_U \psi^2 \left(a u_x^2 + (b+c) u_x u_y + d u_y^2 \right) \, dxdy \leq$$

$$\leq 4M \int\limits_U \psi |\nabla\psi| \sqrt{(a u_x + b u_y)^2 + (c u_x + d u_y)^2} dxdy =$$

$$= 4M \int\limits_U \psi |\nabla\psi| \sqrt{(a^2+c^2) u_x^2 + 2(ab+cd) u_x u_y + (b^2+d^2) u_y^2} dxdy \leq$$

$$\leq 4M \left(\int\limits_U |\nabla\psi|^2 \frac{(a^2+c^2) u_x^2 + 2(ab+cd) u_x u_y + (b^2+d^2) u_y^2}{a u_x^2 + (b+c) u_x u_y + d u_y^2} dxdy \right)^{1/2} \times$$

$$\times \left(\int\limits_U \psi^2 \left(a u_x^2 + (b+c) u_x u_y + d u_y^2 \right) dxdy \right)^{1/2} .$$

Thus we obtain

$$\int\limits_U \psi^2 \left(a u_x^2 + (b+c) u_x u_y + d u_y^2 \right) \, dxdy \leq$$

$$\leq 16M^2 \int\limits_U |\nabla\psi|^2 \frac{(a^2+c^2) u_x^2 + 2(ab+cd) u_x u_y + (b^2+d^2) u_y^2}{a u_x^2 + (b+c) u_x u_y + d u_y^2} \, dxdy. \tag{2.10}$$

We write

$$I(\xi, \eta) = (a^2+c^2)\xi^2 + 2(ab+cd)\xi\eta + (b^2+d^2)\eta^2$$

and

$$II(\xi,\eta) = (a\xi^2 + (b+c)\xi\eta + d\eta^2).$$

¿From (1.2) it follows that the quadratic form $II(\xi,\eta)$ is positive definite and hence a bundle of quadratic forms

$$H(\xi,\eta;\lambda) = I(\xi,\eta) - \lambda\, II(\xi,\eta)$$

is regular.

Let $\lambda_{\max}$ be a maximal eigenvalue of $H(\xi,\eta;\lambda)$. Then using well-known properties of quadratic forms (see [6, Theorem **13** Chapter X]) for any $\xi,\eta \neq 0$, we have

$$\frac{I(\xi,\eta)}{II(\xi,\eta)} \leq \lambda_{\max}. \tag{2.11}$$

The characteristic equation of $H(\xi,\eta;\lambda)$ has the form

$$\begin{vmatrix} a^2+c^2-\lambda a & ab+cd-\lambda(b+c)/2 \\ ab+cd-\lambda(b+c)/2 & b^2+d^2-\lambda d \end{vmatrix} = 0$$

i.e.,

$$\Delta\,\lambda^2 - (a+d)(ad-bc)\,\lambda + (ad-bc)^2 = 0.$$

Solving this equation for λ and using (1.2) we see that it is easy to find that

$$\lambda_{\max} \leq \mu \equiv \frac{(ad-bc)(a+d)}{\Delta}.$$

Clearly, $ad - bc > 0$ since from (1.2) it follows that

$$0 < 4ad - (b+c)^2 \leq 4ad - 4bc.$$

By (2.11) for any $(x,y) \in U$, we obtain

$$\frac{(a^2+c^2)u_x^2 + 2(ab+cd)u_x u_y + (b^2+d^2)u_y^2}{au_x^2 + (b+c)u_x u_y + du_y^2} \leq \mu\,. \tag{2.12}$$

Thus from (2.10) we arrive at the estimate

$$\int\limits_U \psi^2 \left(au_x^2 + (b+c)u_x u_y + du_y^2\right) dxdy \leq 16M^2 \int\limits_U \mu\,|\nabla\phi|^2\,dxdy\,. \tag{2.13}$$

Now we shall estimate the integral in the right side of (2.13). We have

$$I \equiv \int\limits_U \mu\,|\nabla\phi|^2\,dxdy = \int\limits_{\substack{\delta'<h(z)<\delta'' \\ z\in U}} \mu\,\psi_0'^2(h(z))\,|\nabla h|^2\,dxdy$$

and next, by the well-known Kronrod-Federer formula [2, Chapter 3, §**2.4**],

$$I = \int\limits_{\delta'}^{\delta''} \psi_0'^2(t)\,dt \int\limits_{E_t\cap U} \mu\,|\nabla h|\,d\mathcal{H}^1(E_t).$$

Using (1.5) we find

$$I \le h_1 \int\limits_{\delta'}^{\delta''} \psi_0'^2(t)\,dt \int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t). \tag{2.14}$$

Observe that

$$1 \le \left(\int\limits_{\delta'}^{\delta''} |\psi_0'|(t)\,dt\right)^2 \le \int\limits_{\delta'}^{\delta''} |\psi_0'|^2 dt \int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t) \times$$

$$\times \int\limits_{\delta'}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1}.$$

Therefore, for any ψ_0 of the described form,

$$\left(\int\limits_{\delta'}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1}\right)^{-1} \le \int\limits_{\delta'}^{\delta''} |\psi_0'|^2 dt \int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t).$$

Choosing for $\delta' < \tau < \delta''$,

$$\psi_0(\tau) = \int\limits_{\delta'}^{\tau} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1} \Bigg/ \int\limits_{\delta'}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1},$$

we have

$$\min_{\psi_0} \int\limits_{\delta'}^{\delta''} |\psi_0'|^2 dt \int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t) = \left(\int\limits_{\delta'}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1}\right)^{-1}.$$

Thus from (2.13) and (2.14) we see that,

$$\int\limits_{\substack{\delta''<h(z)\\ z\in U}} \left(au_x^2 + (b+c)u_xu_y + du_y^2\right)\,dxdy \le 16M^2h_1 \left(\int\limits_{\delta'}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1}\right)^{-1}.$$

Setting here $\delta' \to 0$ we obtain

$$\int\limits_{\substack{\delta''<h(z)\\ z\in U}} \left(au_x^2 + (b+c)u_xu_y + du_y^2\right)\,dxdy \le 16M^2h_1 \left(\int\limits_{0}^{\delta''} dt \left(\int\limits_{E_t\cap U} \mu\, d\mathcal{H}^1(E_t)\right)^{-1}\right)^{-1}.$$

Previously we had been chosen $\delta'' < h(z_0)$ where $z_0 \in U$. Thus the open set $U_1 = \{z \in U : \delta'' < h(z_0)\}$ is not empthy. Now from (1.7) we conclude that

$$\int\limits_{U_1} \left(au_x^2 + (b+c)u_xu_y + du_y^2\right)\,dxdy = 0,$$

that is,

$$au_x^2 + (b+c)u_x u_y + du_y^2 = 0 \quad \text{a.e. in} \quad U_1.$$

The assumption (1.2) implies that $\nabla u = 0$ a.e. in U_1 and hence $u \equiv \text{const}$ in U_1. But $\delta'' > \delta' > 0$ can be arbitrarily small and hence $\nabla u = 0$ a.e. in U. There is a contradiction with the definition of the connected component U and (2.9) is true.

Analogously, we have

$$\inf_{\Delta} u(z) = \inf_{\partial'\Delta} u(z)$$

and this together with (2.9) yields (1.4). $\Box$

3 Proof of Theorem 1.8

Suppose that there exists a homeomorphic solution $w(z)$ of (1.1), (1.2) such that $w(D) = \mathcal{D}$ and $L \subset w(\Gamma)$. Since L is u-forked then there is a vertical segment $l \subset \mathcal{D}$ with the end points in L and moreover, there exists a subdomain U of $\mathcal{D}$ with the boundary $\tilde{\partial} U \subset \bar{l} \cup L$. Consider its preimage $\Delta = w^{-1}(U)$. By Theorem 1.6 the function u is monotone close to Γ. Hence,

$$\begin{aligned} \operatorname{osc}(u, \Delta) &\leq \operatorname{osc}(u, \tilde{\partial}'\Delta) = \\ &= \operatorname{osc}(u, w^{-1}(l)) = 0. \end{aligned}$$

Thus, $u \equiv \text{const}$ in Δ what is impossible.$\Box$

4 An example

Let γ be a simple open Jordan arc lying in the upper half-plane with end points $(0,0)$ and $(1,0)$ on the x-axis. We set

$$\Gamma = \{(x,y) : 0 \leq x \leq 1,\ y = 0\}$$

and denote by D the subdomain of $\mathbf{R}^2$ lying between γ and Γ.

Choose in Theorem 1.8 $h(z) = y$. Evidently, (1.5) is fulfilled.

Consider the case of (1.1) in which $a = a(y)$, $d = d(y)$ are positive and $b = c = 0$. Then (1.1) admits the form

$$a(y)\, u_x = v_y, \quad d(y)\, u_y = -v_x. \tag{4.1}$$

The assumption (1.2) is fullfilled too.

The assumption (1.7) here has the form

$$\int\limits_0 \frac{dy}{a(y) + d(y)} = \infty. \tag{4.2}$$

We obtain:

4.3. Corollary. *Let $\mathcal{D}$ be a simply connected subdomain of $\{(u,v) : |u| < M\}$, where $0 < M < \infty$ is a constant. Assume that (4.2) is fullfilled and $L \subset \widetilde{\partial}\mathcal{D}$ is u-forked. Then there are no $ACL^2_{\rm loc}$ solutions $w = w(z)$ of (4.1) realizing a homeomorphic mapping from D onto Δ such that $L \subset w(\Gamma)$.*

Acknowledgements. The authors are indebted to Prof. V.A. Klyachin. His remarks were very useful.

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