Functions monotone close to boundary

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Abstract

Functions which are monotone close to boundary are defined. Some estimates are given for these functions in the class ACL^{Φ}. Criteria for monotonicity close to boundary are obtained.

1 Main Results

Let $D \subset \mathbf{R}^2$ be a domain. By $\tilde{\partial} D$ we denote the boundary of D in the extended plane $\tilde{\mathbf{R}}^2 = \mathbf{R}^2 \cup \{\infty\}$. For a subdomain $\Delta \subset D$, we set

$$\partial' \Delta = \tilde{\partial} \Delta \setminus \tilde{\partial} D \quad \text{and} \quad \partial'' \Delta = \tilde{\partial} \Delta \cap \tilde{\partial} D\,.$$

Let Γ be a subset of $\tilde{\partial} D$. A continuous function $f : D \to \mathbf{R}$ is called *monotone close to* Γ if for every subdomain $\Delta \subset D$ with $\partial'' \Delta \subset \Gamma$,

$$\operatorname{osc}(f, \Delta) \le \operatorname{osc}(f, \partial' \Delta), \tag{1.1}$$

see Martio, Miklyukov, Vuorinen [9].

If $\Gamma = \emptyset$ then we have the well-known class of functions monotone in Lebesgue's sense. If $\Gamma = \tilde{\partial} D$, then every function, monotone close to Γ, is a constant. This is evident since choosing $\Delta = D \setminus \{x_0\}$, where $x_0 \in D$ is an arbitrary point, by (1.1) we obtain

$$\operatorname{osc}(f, \Delta) \le \operatorname{osc}(f, \{x_0\}) = 0.$$

For another generalization of monotonicity in the sense of Lebesgue, see Manfredi [8].

For $a, b \in D$ we let

$$\rho_D(a, b) = \inf_{\gamma} \operatorname{diam} \gamma\,,$$

where the infimum is taken over all arcs $\gamma \subset D$ joining a and b. The quantity $\rho_D(a, b)$ is called the *inner distance* between a and b. Clearly ρ_D is a metric in D.

For arbitrary sets A, $B \subset \mathbf{R}^2$ we let

$$d(A, B) = \sup_{x \in A} \operatorname{dist}(x, B).$$

For a set $\Gamma \subset \partial D\,, D \ne \emptyset$, and points a, $b \in D$ we set

$$\delta_D(a, b; \Gamma) = \limsup\, d(\gamma_k, \Gamma),$$

where lim sup is taken over all sequences $\{\gamma_k\}$ of arcs $\gamma_k \subset D$ joining a and b such that $\operatorname{diam} \gamma_k \to \rho_D(a, b)$.

We will employ functions with Sobolev derivatives in some Orlicz classes, see [4, Chapter I]. A function $\Phi : \mathbf{R} \to \mathbf{R}_+$ is called an *N-function* if it admits the representation

$$\Phi(\tau) = \int\limits_0^{|\tau|} p(t)\,dt, \tag{1.2}$$

where $p : (0, \infty) \to (0, \infty)$ is a positive nondecreasing function, continuous to the right, such that

$$p(0+) = 0, \quad p(\infty) = \lim_{t\to\infty} p(t) = \infty. \tag{1.3}$$

Let $D \subset \mathbf{R}^2$ be an open set. Let Φ be an N-function. We say that a continuous function $f : D \to \mathbf{R}$ belongs to $ACL^\Phi(D)$ if f is ACL and

$$\int\limits_D \Phi(|\nabla f|)\,dx_1dx_2 < \infty.$$

Here ∇f means the formal gradient (f'_{x_1}, f'_{x_2}) which exists almost everywhere in D.

A continuous function $f \in ACL^p$ if $f \in ACL^\Phi$ with $\Phi = t^p$, $p \geq 1$. The class ACL^2 coincides with the class BL introduced in 1906 by B. Levi [7] (see, also, O. Nikodym [11], J. Deny and J.P. Lions [2], J. Lelong–Ferrand [6], and G.D. Suvorov [12]). For N-functions Φ of the general form, see V.M. Miklyukov [10], V.I. Kruglikov and V.M. Miklyukov [5], Astala, Iwaniec, Koskela and Martin [1].

Our main result gives the following inequality for functions monotone close to boundary:

1.4. Theorem. *Let D be a subdomain of $\mathbf{R}^2$, $\Gamma \subset \partial D$ an open Jordan arc, and $f \in ACL^\Phi(D)$. If f is monotone close to Γ, then for every pair of points a, $b \in D$ with*

$$\rho_D(a, b) < \delta_D(a, b; \Gamma)$$

the following estimate holds

$$|f(a) - f(b)| \leq \kappa_0(\rho_D(a, b); \delta_D(a, b; \Gamma), \Phi, I) \tag{1.5}$$

where

$$I = \int\limits_D \Phi(|\nabla f|)\,dx_1dx_2.$$

Here κ_0 is the function defined by (3.1).

Suppose that D is convex. Then $\rho_D(a, b) = |b - a|$ and

$$\delta_D(a, b; \Gamma) = \max\{\operatorname{dist}(a, \Gamma), \operatorname{dist}(b, \Gamma)\}.$$

If D is convex and if Φ satisfies some additional conditions, then the estimate (1.5) implies that f has a continuous extension to Γ. Indeed, if D is convex, then we suppose that for all $a \geq 1$ and all $\tau > 0$,

$$\Phi(a\,\tau) \leq c_\Phi\, a^2\, \Phi(\tau) \tag{1.6}$$

with some constant c_Φ independent of a and τ and that

$$\int\limits_1^\infty \frac{\Phi(\tau)}{\tau^3} d\tau = \infty\,. \tag{1.7}$$

1.8. Corollary. *If in the situation of Theorem 1.4 D is convex and the N-function Φ satisfies (1.6) and (1.7), then f has a continuous extension to Γ.*

Below we give two sufficient conditions for monotonicity close to boundary.

1.9. Theorem. *Let Φ and Ψ be arbitrary mutually complementary N-functions. Let $D \subset \mathbf{R}^2$ be a subdomain of $\mathbf{R}^2$, $\Gamma \subset \partial D$ and $f \in ACL^\Phi(D)$.*

Suppose that there exists a vector field $A(x) : D \to \mathbf{R}^2$ of the class $ACL^\Psi(D)$ such that for almost all $x \in D$ at which $\nabla f(x) \neq 0$,

$$\sum_{i=1}^2 f'_{x_i}(x)\, A_i(x) > 0. \tag{1.10}$$

If for every subdomain $\Delta \subset D$, such that $\partial'\Delta$ is locally rectifiable and $\partial''\Delta \subset \Gamma$, and for every $\phi \in ACL^\Phi(\Delta) \cap C(\Delta \cup \partial'\Delta)$,

$$\int\limits_{\partial'\Delta} \phi\,\langle A(x), \mathbf{n}\rangle\, |dx| = \int\limits_\Delta \sum_{i=1}^2 \phi'_{x_i}\, A_i(x)\, dx_1 dx_2, \tag{1.11}$$

then f is monotone close to Γ.

Since the assumptions in Theorem 1.9 are rather complicated, we illustrate it by an example.

1.12. Example. Let $D \subset \mathbf{R}^2$ be a bounded subdomain with smooth boundary and let $\Gamma \subset \partial D$ be an open proper subarc. Let f be a $C^1(D \cup \Gamma)$ solution of the equation

$$\operatorname{div}\left(|\nabla f|^{p-2} \nabla f\right) = 0 \quad (p \geq 1).$$

Suppose that the normal derivative $\left.\frac{\partial f}{\partial n}\right|_\Gamma = 0$. If we now choose

$$A(x) = \nabla f(x)\, |\nabla f(x)|^{p-2},$$

then

$$\sum_{i=1}^{2} f'_{x_i}(x)\, A_i(x) = |\nabla f|^p$$

and the assumption (1.10) holds.

Next for every function ϕ and every subdomain Δ as in Theorem 1.9, we have

$$\begin{aligned}
\int\limits_{\partial'\Delta} \phi\, \langle A(x), \mathbf{n}\rangle\, |dx| \; &= \int\limits_{\partial\Delta} \phi\, |\nabla f|^{p-2}\, \langle \nabla f(x), \mathbf{n}\rangle\, |dx| = \\
&= \int\limits_{\Delta} \phi\, \mathrm{div}\, |\nabla f|^{p-2} \nabla f\, dx_1 dx_2 + \\
&+ \int\limits_{\Delta} |\nabla f|^{p-2} \langle \nabla\phi, \nabla f\rangle\, dx_1 dx_2 = \\
&= \int\limits_{\Delta} \textstyle\sum_{i=1}^{2} \phi'_{x_i}\, A_i(x)\, dx_1 dx_2 .
\end{aligned}$$

This formula implies (1.11) and by Theorem 1.9, f is monotone close to Γ.

For the next result let $\Gamma \subset \tilde{D}$ and let $h : D \to \mathbf{R}$ be a locally Lipschitz function such that $\lim_{x\to\Gamma} h(x) = 0$ and

$$0 < h_0 \le \mathrm{ess\,inf}_D |\nabla h(x)| \le \mathrm{ess\,sup}_D |\nabla h(x)| \le h_1 < \infty, \tag{1.13}$$

where h_0, h_1 are some constants. We let $E_t = \{x \in D : h(x) = t\}$.

1.14. Theorem. *Let $f \in ACL^p_{\mathrm{loc}}(D)$, $p > 1$, be a bounded function, $p > 1$. Suppose that there exists a vector field*

$$A = (A_1, A_2) : D \to \mathbf{R}^2, \quad A \in ACL^q(D), \;\; q = \frac{p}{p-1},$$

such that

$$\begin{aligned}
&\sigma_1(x)\, |\nabla f(x)|^p \le \textstyle\sum_{i=1}^{2} f'_{x_i}(x)\, A_i(x), \\
&\left(\textstyle\sum_{i=1}^{2} A_i^2(x)\right)^{1/2} \le \sigma_2(x)\, |\nabla f(x)|^{p-1}
\end{aligned} \tag{1.15}$$

for some continuous functions $\sigma_j : D \to (0, \infty)$, $j = 1, 2$, and that (1.11) holds for all functions $\phi \in ACL^p(D)$, $\mathrm{supp}\,\phi \subset\subset D$, and for all subdomains $\Delta \subset\subset D$ with locally rectifiable boundaries.

If

$$\int\limits_0 dt \left(\int\limits_{D\cap E_t} \frac{\sigma_2^p}{\sigma_1^{p-1}}\, d\mathcal{H}^1(E_t) \right)^{1/(1-p)} = \infty, \tag{1.16}$$

then f is monotone close to Γ.

In particular, if $\Gamma = \tilde{\partial} D$ then $f \equiv$ const.

Let γ be a simple open Jordan arc lying in the upper half-plane with endpoints $(0,0)$ and $(a,0)$ on the x_1-axis. We set

$$\Gamma = \{(x_1, x_2) : 0 \le x_1 \le a,\ x_2 = 0\}$$

and denote by D the subdomain of $\mathbf{R}^2$ lying between γ and Γ. Choosing $h(x) = x_2$ in Theorem 1.14, we obtain:

1.17. Corollary. *Let f be as in Theorem 1.14. Suppose that there exists a vector field*

$$A = (A_1, A_2) : D \to \mathbf{R}^2, \quad A \in ACL^q(D), \;\; q = \frac{p}{p-1},$$

satisfying

$$\begin{aligned} &\sigma_1(x_2)\,|\nabla f(x)|^p \le \textstyle\sum_{i=1}^2 f'_{x_i}(x)\,A_i(x), \\ &\Big(\textstyle\sum_{i=1}^2 A_i^2(x)\Big)^{1/2} \le \sigma_2(x_2)\,|\nabla f(x)|^{p-1} \end{aligned} \tag{1.18}$$

with some continuous functions σ_1, $\sigma_2 > 0$. Suppose also that for all functions $\phi \in ACL^p(D)$, $\operatorname{supp}\phi \subset\subset D$, and all subdomains $\Delta \subset\subset D$ with locally rectifiable boundaries, the relation (1.11) holds.

If

$$\int_0 \frac{\sigma_1(t)}{\sigma_2^{1/(p-1)}(t)}\,dt = \infty, \tag{1.19}$$

then f is monotone close to Γ.

1.20. Example. Let D be as in Corollary 1.17 and let f be a C^1 solution in D of an equation

$$\operatorname{div}\Big(\sigma\,|\nabla f|^{p-2}\nabla f\Big) = 0 \quad (p > 1)$$

with a continuous weight function $\sigma(x) = \sigma(x_2)$.

Setting

$$A_i(x) = \sigma\,|\nabla f|^{p-2} f'_{x_i},$$

we find

$$\sum_{i=1}^2 f'_{x_i} A_i = \sigma\,|\nabla f|^p$$

and

$$|A| = \sigma\,|\nabla f|^{p-1}.$$

Thus, (1.18) holds with $\sigma_1 = \sigma_2 = \sigma$.

The assumption (1.19) takes the form

$$\int_0 \sigma^{(p-2)/(p-1)}(y)\,dy = \infty,$$

which guarantees that f is monotone close to Γ. □

2 Length and Area Principle

Each N-function is convex. The first of the conditions (1.3) means that

$$\lim_{\tau\to 0}\frac{\Phi(\tau)}{\tau}=0, \tag{2.1}$$

and the second that

$$\lim_{\tau\to\infty}\frac{\Phi(\tau)}{\tau}=\infty. \tag{2.2}$$

For every nondecreasing function $p:(0,\infty)\to(0,\infty)$ which is continuous to the right and satisfies (1.3), we find another function $q:(0,\infty)\to(0,\infty)$, $s\geq 0$, such that

$$q(s)=\sup_{p(t)\geq s} t.$$

It is easy to see that $q(s)$ satisfies the properties (1.3), too.

The pair of N-functions

$$\Phi(\tau)=\int\limits_0^{|\tau|} p(t)\,dt \quad \text{and} \quad \Psi(\tau)=\int\limits_0^{|\tau|} q(s)\,ds$$

is called *mutually complementary*.

We give some estimates for ACL^{Φ}-functions, which are needed for the study of monotone close to boundary functions. For this purpose we define Orlicz classes. Fix a Lebesgue measurable set $A\subset\mathbf{R}^2$ and an arbitrary N-function Φ. Consider the set of Lebesgue measurable functions u in A such that

$$I(u,A;\Phi)\equiv\int\limits_A \Phi[u(x)]\,dx_1dx_2<\infty.$$

This collection of functions is said to be the *Orlicz class* $L^{\Phi}(A)$.

In applications the functions

$$\Phi(\tau)=\tau^2\log^{\alpha}(e+\tau)$$

play an importance role and the classes $L^{\Phi}(A)$ called *Zygmund classes* [1].

Let $\Phi(\tau)$ and $\Psi(\tau)$ be mutually complementary N-functions. By $\tilde{L}^{\Phi}(A)$ we denote the set of the functions $u(x)$ satisfying

$$\left|\int\limits_A u(x)\,v(x)\,dx_1dx_2\right|<\infty \quad \text{for all} \quad v(x)\in L^{\Psi}(A).$$

Next we set

$$||u||_{\Phi,A}\equiv\sup_{I(v,A;\Psi)\leq 1}\left|\int\limits_A u(x)\,v(x)\,dx_1dx_2\right|. \tag{2.3}$$

After introduction of one of the equivalent norms, the class $\tilde{L}^{\Phi}(A)$ is converted to a linear normalized space (see [4, Chapter II]). Namely, the definition (2.3) of the norm implies that

1) $||u||_{\Phi,A} = 0$ if and only if $u(x) = 0$ almost everywhere on A;

2) $||\alpha\, u||_{\Phi,A} = |\alpha|\; ||u||_{\Phi,A}$;

3) $||u_1 + u_2||_{\Phi,A} \leq ||u_1||_{\Phi,A} + ||u_2||_{\Phi,A}$.

For the next lemma see [4],

2.4. Lemma. *(i) For each $u \in L^{\Phi}(A)$,*

$$||u||_{\Phi,A} \leq \int\limits_A \Phi(u)\, dx_1 dx_2 + 1, \tag{2.5}$$

in particular, $L^{\Phi}(A) \subset \tilde{L}^{\Phi}(A)$.

(ii) If $||u||_{\Phi,A} \leq 1$, then $u \in L^{\Phi}(A)$ and

$$\int\limits_A \Phi(u)\, dx_1 dx_2 \leq ||u||_{\Phi,A}; \tag{2.6}$$

in particular,

$$\int\limits_A \Phi\left[\frac{u(x)}{||u||_{\Phi,A}}\right] dx_1 dx_2 \leq 1.$$

(iii) For every pair of functions $u \in \tilde{L}^{\Phi}(A)$ and $v \in \tilde{L}^{\Psi}(A)$,

$$\left|\int\limits_A u(x)\, v(x)\, dx_1 dx_2\right| \leq ||u||_{\Phi,A}\; ||v||_{\Psi,A}. \tag{2.7}$$

2.8. Lemma. *If a domain $D \subset \mathbf{R}^2$ is of finite area, Φ is an N-function and $f \in ACL^{\Phi}(D)$, then $f \in ACL^1(D)$.*

Proof. Since Φ satisfies (2.2) there exists $q > 0$ such that

$$\frac{\tau}{\Phi(\tau)} \leq 1 \quad \text{for all} \quad \tau \geq q\,.$$

Write

$$D_1 = \{x \in D : |\nabla f(x)| \leq q\}, \quad D_2 = \{x \in D : |\nabla f(x)| > q\}.$$

We have

$$\begin{aligned}\int_D |\nabla f|\, dx_1dx_2 &= \int_{D_1} |\nabla f|\, dx_1dx_2 + \int_{D_2} |\nabla f|\, dx_1dx_2 \le \\ &\le q \int_{D_1} dx_1dx_2 + \int_{D_2} \tfrac{|\nabla f|}{\Phi(|\nabla f|)} \Phi(|\nabla f|)\, dx_1dx_2 \le \\ &\le q\mathcal{H}^2(D_1) + C \int_{D_2} \Phi(|\nabla f|)\, dx_1dx_2\,,\end{aligned}$$

where $\mathcal{H}^2$ is 2-dimensional Lebesgue measure and

$$C = \sup_{x\in D_2} \frac{|\nabla f(x)|}{\Phi(|\nabla f(x)|)} \le 1.$$

Hence,

$$\begin{aligned}\int_D |\nabla f|\, dx_1dx_2 &\le q\mathcal{H}^2(D_1) + \int_{D_2} \Phi(|\nabla f|)\, dx_1dx_2 \le \\ &\le q\mathcal{H}^2(D) + \int_D \Phi(|\nabla f|)\, dx_1dx_2\end{aligned}$$

and f is ACL^1 as required.

2.9. Example. The functions $\Phi(\tau) = \tau^2$ and $\Phi(\tau) = \tau^2/\ln(e+|\tau|)$ are typical examples of N-functions satisfying (1.6) and (1.7). This is evident for $\Phi(\tau) = \tau^2$. We verify the necessary conditions for $\Phi(\tau) = \tau^2/\ln(e+|\tau|)$.

At first we observe that this function satisfies (1.3). Indeed, the derivative of $\Phi(\tau)$ has the form

$$\Phi'(\tau) = \tau \frac{2\ln(e+\tau) - \tau/(e+\tau)}{\ln^2(e+\tau)} \quad (\tau > 0).$$

Clearly, $\Phi'(0+) = 0$. Next,

$$\ln(e+\tau) > 1 \quad \text{for} \quad \tau > 0.$$

Hence for all $\tau > 0$,

$$\begin{aligned}\Phi'(\tau) &= \tau \tfrac{2(e+\tau)\,\ln(e+\tau) - \tau}{(e+\tau)\,\ln^2(e+\tau)} > \\ &> \tau \tfrac{2(e+\tau) - \tau}{(e+\tau)\,\ln^2(e+\tau)} = \tau \tfrac{2e+\tau}{(e+\tau)\,\ln^2(e+\tau)} > 0\end{aligned}$$

and we find

$$\Phi'(\infty) = \lim_{\tau\to\infty} \Phi'(\tau) = \infty.$$

Secondly, we verify (1.6). For every $a \geq 1$ we have

$$\begin{aligned}\Phi(a\,\tau) &= a^2\,\tau^2\,/\ln(e + a|\tau|) \leq \\ &\leq a^2\,\tau^2\,/\ln(e + |\tau|) = a^2\,\Phi(\tau).\end{aligned}$$

The property (1.7) is evident.

We denote by $S_D(a,r)$ the intersection of the circle $S(a,r)$ and the set $D \subset \mathbf{R}^2$. Fix ϵ and ϵ_0 such that $0 < \epsilon < \epsilon_0 < \infty$. Suppose that for all $\tau \in (\epsilon, \epsilon_0)$ the intersection $S_D(a,r) \neq \emptyset$. We set

$$D_{\epsilon,\epsilon_0} = \cup_{\epsilon<\tau<\epsilon_0} S_D(a,\tau),$$

$$l(a,r) = \mathcal{H}^1(S_D(a,r)).$$

Let σ_1, $\sigma_2, \ldots$ be the componenets of $S_D(a,r)$. Then $S_D(a,r) = \cup_i\,\sigma_i$ and for a function $f : D \to \mathbf{R}$ we set

$$W_f(a,r) = \sum_i \operatorname{osc}(f,\sigma_i).$$

We shall prove the following version of the Length and Area Principle for ACL^Φ-functions. A closely related class of functions was considered in [5].

2.10. Theorem. *Let Φ be an N-function. Then for all $f \in ACL^\Phi(D)$ and all $0 < \epsilon < \epsilon_0 < \infty$,*

$$\int\limits_\epsilon^{\epsilon_0} \Phi\left(\frac{W_f(a,\tau)}{l(a,\tau)}\right) l(a,\tau)\,d\tau \leq I(D_{\epsilon,\epsilon_0}), \tag{2.11}$$

where

$$I(D_{\epsilon,\epsilon_0}) = \int\limits_{D_{\epsilon,\epsilon_0}} \Phi\left(|\nabla f|\right)\,dx_1 dx_2.$$

Proof. We may assume that $a = 0$. Lemma 2.8 shows that a function $f \in ACL^\Phi$ belongs to $ACL^1(D)$. Let (r,θ) be polar coordinates in $\mathbf{R}^2$ and set $f^* = f(r,\theta)$. Now $f^* \in ACL^1_{\text{loc}}(D)$ and, in particular, f^* is absolutely continuous on each line segment $\alpha \leq \theta \leq \beta$ on $S_D(0,\tau)$ for almost all τ. Hence for almost all $\tau \in (\epsilon, \epsilon_0)$ we have

$$\begin{aligned}W_f(a,\tau) &= \textstyle\sum_i \operatorname{osc}(f,\sigma_i) \leq \\ &\leq \textstyle\sum_i \displaystyle\int\limits_{\sigma_i} |\nabla f|\,|dx| \leq \\ &\leq \int\limits_{S_D(0,\tau)} |\nabla f|\,|dx|.\end{aligned}$$

Jensen's integral inequality yields

$$\Phi\left(\frac{\int\limits_E a(x)\,d\mathcal{H}^1}{\mathcal{H}^1(E)}\right) \le \frac{1}{\mathcal{H}^1(E)}\int\limits_E \Phi\left(a(x)\right)\,d\mathcal{H}^1 \tag{2.12}$$

for an arbitrary N-function Φ and every integrable function a [4, the inequality (8.2)]. Using (2.12) for almost all $\tau \in (\epsilon, \epsilon_0)$ we obtain

$$\begin{aligned}\Phi\left(\tfrac{W_f(0,\tau)}{l(0,\tau)}\right) &\le \Phi\left(\tfrac{1}{l(0,\tau)}\int\limits_{S_D(0,\tau)} |\nabla f|\,|dx|\right) \le \\ &\le \tfrac{1}{l(0,\tau)}\int\limits_{S_D(0,\tau)} \Phi\left(|\nabla f|\right)\,|dx|.\end{aligned}$$

The functions in the left and right sides of this inequality are measurable with respect to $\tau \in (\epsilon, \epsilon_0)$. Multiplying this inequality by $l(0,\tau)$ and integrating over (ϵ, ϵ_0), we obtain

$$\begin{aligned}\int\limits_{\epsilon}^{\epsilon_0} \Phi\left(\frac{W_f(0,\tau)}{l(0,\tau)}\right) l(0,\tau)\,d\tau &\le \int\limits_{\epsilon}^{\epsilon_0} d\tau \int\limits_{S_D(0,\tau)} \Phi\left(|\nabla f|\right)\,|dx| = \\ &= \int\limits_{D_{\epsilon,\epsilon_0}} \Phi\left(|\nabla f|\right)\,dx_1 dx_2.\end{aligned}$$

The following statement is, maybe, the most useful corollary of Theorem 2.10.

2.13. Corollary. *For every N-function Φ and $f \in ACL^{\Phi}(D)$,*

$$\int\limits_{\epsilon}^{\epsilon_0} \Phi\left(\frac{W_f(a,\tau)}{2\pi\,\tau}\right)\tau\,d\tau \le \frac{1}{2\pi}\,I(D_{\epsilon,\epsilon_0}). \tag{2.14}$$

Proof. Every N-function Φ satisfies

$$\Phi(\alpha\,\tau) \le \alpha\,\Phi(\tau) \tag{2.15}$$

for all $\tau \ge 0$ and $\alpha \in [0,1]$. Indeed, write Φ in the form $\Phi(\tau) = \int\limits_0^{\tau} p(t)\,dt$ as in (1.2). Then

$$\begin{aligned}\alpha\,\Phi(\tau) &= \int\limits_0^{\tau} p(t)\,d(\alpha\,t) \ge \\ &\ge \int\limits_0^{\tau} p(\alpha\,t)\,d(\alpha\,t) = \int\limits_0^{\alpha\,\tau} p(t)\,dt = \Phi(\alpha\,\tau)\end{aligned}$$

as required. Let $W \geq 0$. From (2.15) we conclude

$$\begin{aligned}\Phi\left(\tfrac{W}{2\pi\tau}\right) &= \Phi\left(\tfrac{W}{l(a,\tau)}\tfrac{l(a,\tau)}{2\pi\tau}\right) \leq \\ &\leq \Phi\left(\tfrac{W}{l(a,\tau)}\right)\tfrac{l(a,\tau)}{2\pi\,\tau}.\end{aligned}$$

Hence,

$$\Phi\left(\frac{W_f(a,\tau)}{2\pi\,\tau}\right) 2\pi\,\tau \leq \Phi\left(\frac{W_f(a,\tau)}{l(a,\tau)}\right) l(a,\tau),$$

and the desired conclusion follows from (2.11). □

3 The function κ_0

Here we study the inequality (2.14). For an N-function Φ and an interval (ϵ, ϵ_0) with $0 < \epsilon < \epsilon_0$, we set

$$\kappa_0(\epsilon;\epsilon_0,\Phi,I) = \sup\left\{\kappa : \int_{\epsilon}^{\epsilon_0} \Phi\left(\frac{\kappa}{2\pi\,\tau}\right) \tau\, d\tau \leq \frac{1}{2\pi} I\right\}, \tag{3.1}$$

where

$$I = \int_D \Phi(|\nabla f|)\, dx_1 dx_2.$$

From (2.1) and (2.2) it follows that the function

$$F(\kappa) = \int_{\epsilon}^{\epsilon_0} \Phi\left(\frac{\kappa}{2\pi\,\tau}\right) \tau\, d\tau$$

is continuous and strictly monotone. Moreover,

$$F(0) = 0 \quad \text{and} \quad F(\infty) = \lim_{\kappa\to\infty} F(\kappa) = \infty.$$

Now $\kappa_0(\epsilon) = \kappa_0(\epsilon;\epsilon_0,\Phi,I)$ is the unique positive root of the equation

$$F(\kappa) = \frac{1}{2\pi}\, I. \tag{3.2}$$

The function $F(\kappa)$ has the following properties:

(**i**) *For every N-function Φ the function $\kappa_0(\epsilon) = \kappa_0(\epsilon;\epsilon_0,\Phi,I)$ is continuous and strictly increasing on $(0, \epsilon_0)$.*

The proof is evident.

(ii) *If a N-function* Φ *satisfies* (1.7), *then* $\lim_{\epsilon\to 0}\kappa_0(\epsilon)=0$.

Indeed, suppose that there exists a number $m>0$ such that for all sufficiently small $\epsilon>0$,

$$\kappa_0(\epsilon)\geq m>0\,.$$

Then by the definition (3.2) of $\kappa_0(\epsilon)$ and the monotonicity of Φ, we have

$$\int\limits_{\epsilon}^{\epsilon_0}\Phi\left(\frac{m}{2\pi\,\tau}\right)\,\tau\,d\tau\leq\frac{1}{2\pi}\,I.$$

Substituting $t=m/2\pi\,\tau$ we find

$$\left(\frac{m}{2\pi}\right)^2\int\limits_{m/2\pi\,\epsilon_0}^{m/2\pi\,\epsilon}\frac{\Phi(t)}{t^3}dt\leq\frac{1}{2\pi}I\quad\text{for all}\quad 0<\epsilon<\epsilon_0.$$

Letting $\epsilon\to 0+$ we obtain a contradiction to (1.7).

(iii) *If an N-function* Φ *satisfies* (1.6), *then*

$$\liminf_{\epsilon\to 0+}\kappa_0(\epsilon)\Bigg/\left(\ln\frac{1}{\epsilon}\right)^{-\beta}=\infty\qquad\text{for every}\quad\beta>1\,.$$

Indeed, suppose that this is not true, that is, for some sequence $\epsilon_i\to 0$ of positive numbers the inequality

$$\kappa_0(\epsilon_i)\leq c\,\left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}$$

holds with some constant $c>0$. From (3.2) we obtain

$$\int\limits_{\epsilon_i}^{\epsilon_0}\Phi\left(\frac{\kappa_0(\epsilon_i)}{2\pi\,\tau}\right)\,\tau\,d\tau=\frac{1}{2\pi}I \tag{3.3}$$

and hence,

$$\int\limits_{\epsilon_i}^{\epsilon_0}\Phi\left(\frac{c}{2\pi\,\tau}\left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\right)\,\tau\,d\tau\geq\frac{1}{2\pi}I.$$

Let $\epsilon_0\leq 1/e$. We then have

$$\int\limits_{\epsilon_i}^{1/e}\Phi\left(\left(\frac{c}{2\pi\,\tau}\ln\frac{1}{\epsilon_i}\right)^{-\beta}\right)\,\tau\,d\tau+\left|\int\limits_{\epsilon_0}^{1/e}\Phi\left(\frac{c}{2\pi\,\tau}\left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\right)\,\tau\,d\tau\right|\geq\frac{1}{2\pi}I$$

and it follows from (1.6) and (2.15) that

$$\begin{aligned}
\int_{\epsilon_i}^{1/e} \Phi\left(\frac{c}{2\pi\,\tau}\left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\right)\tau\,d\tau \quad &\le c_\Phi\,\Phi\left(\tfrac{c}{2\pi}\left(\ln\tfrac{1}{\epsilon_i}\right)^{-\beta}\right)\int_{\epsilon_i}^{1/e}\frac{1}{\tau^2}\,\tau\,d\tau \le \\
&\le c_\Phi\,\Phi\left(\tfrac{c}{2\pi}\left(\ln\tfrac{1}{\epsilon_i}\right)^{-\beta}\right)(\ln\tfrac{1}{e}+\ln\tfrac{1}{\epsilon_i}) \le \\
&\le c_\Phi\,\left(\ln\tfrac{1}{\epsilon_i}\right)^{-\beta}\,\Phi\left(\tfrac{c}{2\pi}\right)\ln\tfrac{1}{\epsilon_i} \le \\
&\le c_\Phi\Phi\left(\tfrac{c}{2\pi}\right)\,\left(\ln\tfrac{1}{\epsilon_i}\right)^{1-\beta}.
\end{aligned}$$

The inequality (2.15) yields

$$\left|\int_{\epsilon_0}^{1/e}\Phi\left(\frac{c}{2\pi\,\tau}\left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\right)\tau\,d\tau\right| \le \left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\left|\int_{\epsilon_0}^{1/e}\Phi\left(\frac{c}{2\pi\,\tau}\right)\tau\,d\tau\right|$$

and combining the above estimates we arrive at the inequality

$$\frac{1}{2\pi}I \le c_\Phi\Phi\left(\frac{c}{2\pi}\right)\,\left(\ln\frac{1}{\epsilon_i}\right)^{1-\beta} + \left(\ln\frac{1}{\epsilon_i}\right)^{-\beta}\left|\int_{\epsilon_0}^{1/e}\Phi\left(\frac{c}{2\pi\,\tau}\right)\tau\,d\tau\right|.$$

For $\beta > 1$ we obtain a contradiction as $i \to \infty$.

(iv) We assume here that an N-function Φ satisfies (1.6) and (1.7). Then it is possible to find a majorant for $\kappa_0(\epsilon)$. For this fix $0 < \epsilon < \epsilon_0$. Substituting $t = \kappa_0/2\pi\,\tau$ in (3.3) we have

$$2\pi\,I = \kappa_0^2\int_{\kappa_0/2\pi\,\epsilon_0}^{\kappa_0/2\pi\,\epsilon}\frac{\Phi(t)}{t^3}\,dt.$$

By the property (ii) there exists a number $\alpha_1 = \alpha_1(\epsilon_0)$ such that

$$\kappa_0(\epsilon) \le 2\pi\epsilon_0 \quad \text{for all} \quad 0 < \epsilon < \alpha_1,$$

and hence for all sufficiently small $\epsilon > 0$,

$$2\pi\,I \ge \kappa_0^2\int_{1}^{\kappa_0/2\pi\,\epsilon}\frac{\Phi(t)}{t^3}\,dt.$$

By (iii),

$$\liminf_{\epsilon\to 0+}\frac{\kappa_0(\epsilon)}{\sqrt{\epsilon}} = \infty$$

and hence, there exists a number $\alpha_2 = \alpha_2(\epsilon_0)$ such that

$$\kappa_0(\epsilon) \geq 2\pi\sqrt{\epsilon} \quad \text{for all} \quad 0 < \epsilon < \alpha_2.$$

Therefore for all sufficiently small $\epsilon > 0$, we find

$$2\pi\, I \geq \kappa_0^2 \int\limits_1^{1/\sqrt{\epsilon}} \frac{\Phi(t)}{t^3}\, dt,$$

and we arrive at the estimate

$$\kappa_0(\epsilon,\, \epsilon_0; \Phi, I) \leq \left(2\pi\, I \Bigg/ \int\limits_1^{1/\sqrt{\epsilon}} \frac{\Phi(t)}{t^3}\, dt \right)^{1/2}, \tag{3.4}$$

which holds for all $0 < \epsilon < \min\{\alpha_1(\epsilon_0), \alpha_2(\epsilon_0)\}$.

From (3.4) it follows:

3.5. Remark. For the N-functions Φ satisfying (1.6) and (1.7) for all $\epsilon > 0$ with $0 < \epsilon < \min\{\alpha_1(\epsilon_0), \alpha_2(\epsilon_0)\}$, the inequality (2.14) can be written in the form

$$\inf_{\epsilon < \tau < \epsilon_0} W_f(a, \tau) \leq \left(2\pi\, I \Bigg/ \int\limits_1^{1/\sqrt{\epsilon}} \frac{\Phi(t)}{t^3}\, dt \right)^{1/2}. \tag{3.6}$$

Here and above

$$I = \int\limits_D \Phi(|\nabla f|)\, dx_1 dx_2.$$

4 Proof of Theorem 1.4

Fix a number $h > 0$ such that

$$\rho_D(a, b) + h < \delta_D(a, b; \Gamma) - h\,,$$

and choose an arc $\gamma \subset D$ connecting a and b with

$$\operatorname{diam} \gamma < \rho_D(a, b) + h.$$

Let $x_0 \in \gamma$ and consider a family of circles $\{S(x_0, \tau)\}$ where $\tau \in [\epsilon, \epsilon_0]$ and

$$\epsilon = \rho_D(a, b) + h, \quad \epsilon_0 = \delta_D(a, b; \Gamma) - h\,.$$

By Corollary 2.13 and the definition (3.1) we obtain

$$\inf_{\tau \in (\epsilon, \epsilon_0)} W_f(a, \tau) \leq \kappa_0(\epsilon; \epsilon_0, \Phi, I)$$

for every $\tau \in [\epsilon, \epsilon_0]$. Each circle $S(a,\tau)$ separates points a, b from $\partial D \setminus \Gamma$. Hence we can find a component σ of $S_D(x_0,\tau)$ separating a, b from $\partial D \setminus \Gamma$.

Since $\tau \in [\epsilon, \epsilon_0]$ the ends of σ lie on Γ. We denote by D_σ the set of the points which are separated by σ from $\partial D \setminus \Gamma$.

The function f is monotone close to Γ and therefore,

$$\operatorname{osc}(f, D_\sigma) \le \operatorname{osc}(f, \partial' D_\sigma).$$

Hence,

$$\begin{aligned} |f(a) - f(b)| &\le \inf_{\tau\in(\epsilon,\epsilon_0)} \operatorname{osc}(f, D_\sigma) \le \\ &\le \inf_{\tau\in(\epsilon,\epsilon_0)} \operatorname{osc}(f, \partial' D_\sigma) \le \\ &\le W_f(a,\tau) \le \kappa_0(\epsilon; \epsilon_0, \Phi, I) \le \\ &\le \kappa_0(\rho_D(a,b) + h; \delta_D(a,b;\Gamma) - h, \Phi, I) \end{aligned}$$

and letting $h \to 0$ we obtain (1.5).

By (1.7) and by the property (ii), we find $\kappa_0(\rho_D(a,b)) \to 0$ as $\rho_D(a,b) \to 0$. This means that f can be continuously extended to Γ.

5 Proof of Theorem 1.9

Fix a subdomain Δ of D with $\partial''\Delta \subset \Gamma$. First we prove that

$$\sup_{\Delta} f(x) = \sup_{\partial'\Delta} f(x). \tag{5.1}$$

Suppose that this is not the case, that is, there exists a point $x_0 \in \Delta$, such that

$$f(x_0) > \sup_{\partial'\Delta} f(x) = M.$$

Choose $\epsilon > M$ such that $f(x_0) > \epsilon$. By Lemma 2.8 the function $f \in ACL^1(D)$ and by Theorem 5.4.4 [13] for almost all $\epsilon > 0$ the sets $\{x \in \Delta : f(x) = \epsilon\}$ are locally rectifiable.

Fix a component U, $x_0 \in U$, of the set $\{x \in \Delta : f(x) > \epsilon\}$. Without loss of generality, we may assume that $\partial' U$ is locally rectifiable. Using (1.11) with $\phi = f(x) - \epsilon$, we write

$$\int\limits_U \sum_{i=1}^{2} f'_{x_i} A_i(x)\, dx_1 dx_2 = \int\limits_{\partial' U} (f - \epsilon)\langle A(x), \mathbf{n}\rangle\, |dx| = 0.$$

(Because Φ and Ψ are mutually complementary, from (2.5) and (2.7) it follows that the left-hand integral and, hence, the right-hand integral exist.)

From (1.10) we find

$$\nabla f(x) = 0 \quad \text{almost everywhere on} \quad U,$$

and $f \equiv \text{const}$ on U and this leads to a contradiction with the definition of the component U, $x_0 \in U$. Thus (5.1) follows.

Since $-f$ also satisfies (1.10), (5.1) yields

$$\inf_{\Delta} f(x) = \inf_{\partial'\Delta} f(x). \tag{5.2}$$

Finally (5.1) and (5.2) imply (1.1). $\square$

6 Proof of Theorem 1.14

Fix a subdomain Δ of D with $\partial''\Delta \subset \Gamma$. As in the proof of Theorem 1.9, it is sufficient to prove (5.1). Suppose that (5.1) is not true, that is, there is $x_0 \in \Delta$ such that

$$f(x_0) > M = \sup_{\partial'\Delta} f(x).$$

As above for some ϵ, $f(x_0) > \epsilon > M$, we choose a component U of $\{x \in \Delta : f(x) > \epsilon\}$ with a locally rectifiable boundary ∂U along which $f(x) - \epsilon = 0$.

Fix numbers $0 < \delta' < \delta'' < h(x_0)$ and a nonnegative Lipschitz function $\psi_0 : \mathbf{R}_+ \to R$. Define $\psi : \mathbf{R}_+ \to R$ by

$$\psi(\tau) = \begin{cases} 1 & \text{for} \quad \delta'' < \tau < \infty, \\ \psi_0(\tau) & \text{for} \quad \delta' \le \tau \le \delta'', \\ 0 & \text{for} \quad 0 < \tau < \delta'. \end{cases}$$

Denote $\phi = \psi^p\,(f-\epsilon)$ with $\psi = \psi(h(x))$ for $x \in U$ and $\phi \equiv 0$ for $x \in D \setminus U$. Clearly, $\phi \in ACL^p(D)$ and $\operatorname{supp}\phi \subset\subset D$. Applying (1.11) to ϕ we have

$$\begin{aligned}\int\limits_{\partial'U} \psi^p\,(f-\epsilon)\,\langle A, \mathbf{n}\rangle\,|dx| &= \int\limits_{U} \psi^p \langle \nabla f, A\rangle\, dx_1 dx_2 + \\ &+ p \int\limits_{U} \psi^{p-1}\,(f-\epsilon)\,\langle \nabla\psi, A\rangle\, dx_1 dx_2\,.\end{aligned}$$

Since the contour integral vanishes, we find

$$\begin{aligned}\int\limits_{U} \psi^p \langle \nabla f, A\rangle\, dx_1 dx_2 &\le -p \int\limits_{U} \psi^{p-1}\,(f-\epsilon)\,\langle \nabla\psi, A\rangle\, dx_1 dx_2 \le \\ &\le p \int\limits_{U} \psi^{p-1}\,|f-\epsilon|\,|\nabla\psi|\,|A|\, dx_1 dx_2\,.\end{aligned}$$

Using (1.15) we obtain

$$\int\limits_U \psi^p \,\sigma_1\, |\nabla f|^p \,dx_1dx_2 \le p \int\limits_U \psi^{p-1}\, |f-\epsilon|\, |\nabla \psi|\, \sigma_2\, |\nabla f|^{p-1}\, dx_1dx_2 \le$$

$$\le pM \int\limits_U \psi^{p-1}\, \frac{\sigma_1^{(p-1)/p}}{\sigma_1^{(p-1)/p}}\, |\nabla \psi|\, \sigma_2\, |\nabla f|^{p-1}\, dx_1dx_2 \le$$

$$\le p\, M \left(\int\limits_U \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla \psi|^p \,dx_1dx_2\right)^{1/p} \left(\int\limits_U \psi^p \,\sigma_1\, |\nabla f|^p \,dx_1dx_2\right)^{(p-1)/p},$$

where

$$M = \sup_{x\in U} |f(x)-\epsilon|$$

and hence

$$\int\limits_U \psi^p \,\sigma_1\, |\nabla f|^p \,dx_1dx_2 \le p^p\, M^p \int\limits_U \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla \psi|^p \,dx_1dx_2. \tag{6.1}$$

Let

$$U(\delta'') = \{x\in U : \delta'' < h(x)\},$$

$$U(\delta',\delta'') = \{x\in U : \delta' < h(x) < \delta''\}.$$

Since $\delta'' < h(x_0)$ and $x_0 \in U$, the set $U(\delta'') \ne \emptyset$. Bearing in mind the specific structure of ψ and using (6.1), we arrive at the inequality

$$\int\limits_{U(\delta'')} \sigma_1\, |\nabla f|^p \,dx_1dx_2 \le p^p\, M^p \int\limits_{U(\delta',\delta'')} \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla \psi_0|^p \,dx_1dx_2.$$

We have $|\nabla\psi_0| = |\psi_0'|\, |\nabla h|$. By (1.13) and by the well-known co-area formula [3, Section **3.2**], we find

$$\int\limits_{U(\delta',\delta'')} \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla \psi_0|^p \,dx_1dx_2 = \int\limits_{\delta'}^{\delta''} |\psi_0'(\tau)|^p d\tau \int\limits_{U\cap E_\tau} \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla h|^{p-1}\, d\mathcal{H}^1(E_\tau),$$

and then

$$\int\limits_{U(\delta',\delta'')} \frac{\sigma_2^p}{\sigma_1^{p-1}} |\nabla \psi_0|^p \,dx_1dx_2 \le h_1^{p-1} \int\limits_{\delta'}^{\delta''} |\psi_0'(\tau)|^p d\tau \int\limits_{U\cap E_\tau} \frac{\sigma_2^p}{\sigma_1^{p-1}}\, d\mathcal{H}^1(E_\tau).$$

Thus, we obtain

$$\int\limits_{U(\delta'')} \sigma_1\, |\nabla f|^p \,dx_1dx_2 \le p^p\, M^p\, h_1^{p-1} \int\limits_{\delta'}^{\delta''} \xi(\tau) |\psi_0'(\tau)|^p d\tau, \tag{6.2}$$

where

$$\xi(\tau) = \int\limits_{E_\tau} \frac{\sigma_2^p}{\sigma_1^{p-1}}\, d\mathcal{H}^1(E_\tau),$$

We choose

$$\psi_0(\tau) = \int\limits_{\delta'}^{\tau} \xi^{1/(1-p)}(t) \Bigg/ \int\limits_{\delta'}^{\delta''} \xi^{1/(1-p)}(t)\, dt \quad \text{for} \quad \delta' \le \tau \le \delta''.$$

Then

$$\int\limits_{\delta'}^{\delta''} \xi(\tau)|\psi_0'(\tau)|^p\, d\tau = \left(\int\limits_{\delta'}^{\delta''} \xi^{1/(1-p)}(\tau)\, d\tau \right)^{1-p}$$

and from (6.2) we have

$$\int\limits_{U(\delta'')} \sigma_1\, |\nabla f|^p\, dx_1 dx_2 \le p^p\, M^p\, h_1^{p-1} \left(\int\limits_{\delta'}^{\delta''} \xi^{1/(1-p)}(\tau)\, d\tau \right)^{1-p} .$$

This inequality holds for every $0 < \delta' < \delta''$. Let $\delta' \to 0+$ and use (1.16). Then

$$\int\limits_{U(\delta'')} \sigma_1\, |\nabla f|^p\, dx_1 dx_2 = 0$$

and, in particular, $\nabla f \equiv 0$ on $U(\delta'')$. Because $\delta'' < h(x_0)$ is arbitrary, we see that $\nabla f \equiv 0$ on U. This means that $f \equiv \epsilon$ on U. This is again a contradiction to the definition of U and the rest of the proof follows as in the end of the proof for Theorem 1.9.

7 Two Examples

Let $D \subset \mathbf{R}^2$ be a domain with a Jordan boundary ∂D. *Let $\Gamma \subset \partial D$ be an open arc. Let $f : \overline{D} \to \mathbf{R}$ be a continuous function monotone in Lebesgue's sense in D such that the restriction $f|_\Gamma$ has no points of strict local extremum. Then f is monotone close to Γ.*

Indeed, suppose that f is not monotone close to Γ, that is, there exists a bounded subdomain $\Delta \subset D$ satisfying $\partial''\Delta \subset \Gamma$ and such that

$$\operatorname{osc}(f, \Delta) > \operatorname{osc}(f, \partial'\Delta).$$

This means that at least one of following properties holds

$$\sup_{x \in \Delta} f(x) > \sup_{x \in \partial'\Delta} f(x) \quad \text{or} \quad \inf_{x \in \Delta} f(x) < \inf_{x \in \partial'\Delta} f(x).$$

We conclude that f has at least one point $a \in \overline{\Delta}$ of strict local extremum. Since f does not have inner points of local extremum in D, then $a \in \Gamma$. Hence, a is also the point of strict local extremum of $f|_\Gamma$. Contradiction.

There exist non-constant functions monotone close to boundary which do not have a continuous extension to the boundary.

Let $D = \{(x_1, x_2) : x_2 > 0\}$ be the upper half-plane. Consider the function

$$f(x_1, x_2) = \sin \frac{1}{x_2}.$$

Clearly this function is monotone close to boundary $\Gamma = \{x = (x_1, x_2) : x_2 = 0\}$, but f does not have a continuous extension to Γ.

References

[1] K. Astala, T. Iwaniec, P. Koskela and G. Martin, Mappings of BMO-bounded distortion, Math. Ann. 317, 2000, 703-726.

[2] J. Deny and J.L. Lions, Les espaces du type de Beppo Levi, Ann. Inst. Fourier, Grenoble, 5 (1953-54), 305–370.

[3] H. Federer, Geometric measure theory, Springer-Verlag, Berlin - Heidelberg - New York, 1969.

[4] M.A. Krasnoselskii and Ya.B. Rutickii, Convex Functions and Orlicz Spases, Modern Problems of Mathematics, Gosudarstvennoe Izdatelstvo Phys.-Math. Liter., Moscow, 1958 (in Russian).

[5] V.I. Kruglikov and V.M. Miklyukov, Some classes of plane topological mappings with generalized derivatives, in the book "Metric questions of functions theory and mappings", n. 4, Naukova Dumka, Kiev, (in Russian), 102–122, 1973.

[6] J. Lelong–Ferrand, Représentation conforme et transformations a intégrale de Dirichlet bornée, Gauthier–Villars, Paris, 1955.

[7] B. Levi, Sul principio di Dirichlet, Rend. Circ. Mat. di Palermo, v. 22, 293-359, 1906.

[8] J.J. Manfredi, Monotone Sobolev functions, J. Geom. Anal. 4, 393-402, 1994.

[9] O. Martio, V. Miklyukov and M. Vuorinen, Some remarks on an existence problem for degenerate elliptic systems, in prep., 2002.

[10] V.M. Miklyukov, Certain classes of mappings on the plane, Dokl. AN SSSR, 183, n. 4 (in Russian); Engl. transl. in Sov. Math. Dokl. v. 9, n. 6, p. 1455-1457, 772-774, 1968.

[11] O. Nikodym, Sur une class de fonctions considérées dans le problem de Dirichlet, Fund. Math., v. 21, 129-150, 1933.

[12] G.D. Suvorov, Generalized "Length and Area Principle" in theory of mappings, in Russian, Naukova Dumka, Kiev, 1985 (in Russian).

[13] W.P. Ziemer, Weakly Differentiable Functions, Graduate texts in Mathematics; 120, Springer-Verlag, New York, 1989.

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