ON GAUSSIAN PROCESSES EQUIVALENT IN LAW TO FRACTIONAL BROWNIAN MOTION

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ABSTRACT. We consider Gaussian processes that are equivalent in law to the fractional Brownian motion and their canonical representations. We prove a Hitsuda type representation theorem for the fractional Brownian motion with Hurst index $H \leq \frac{1}{2}$. For the case $H > \frac{1}{2}$ we show that such a representation cannot hold. We also consider briefly the connection between Hitsuda and Girsanov representations. Using the Hitsuda representation we consider a certain special kind Gaussian stochastic equation with fractional Brownian motion as noise.

1. INTRODUCTION

The fractional Brownian motion $Z = (Z_t^H)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ is a mean square continuous Gaussian process. Thus, we know that every Gaussian process that is equivalent in law to it has a canonical nonanticipative representation in the following sense (cf. Kallianpur [10], Theorem 9.2.1 or Kallianpur and Oodaira [11]): Any Gaussian process $\tilde{Z} = (\tilde{Z}_t)_{t\in[0,1]}$ that is equivalent in law to the fractional Brownian motion Z can be represented in terms of Z in the sense that

$$\tilde{Z}_t \;\in\; \mathscr{H}_1(Z,t)$$

where $\mathscr{H}_1(Z,t)$ is the linear space generated by Z on the interval $[0,t]$. We shall call this the Kallianpur representation of $\tilde{Z}$.

In the case of ordinary Brownian motion W the elements in the linear space $\mathscr{H}_1(W,t)$ can be considered as Wiener integrals

$$\int_0^t f(s)\,\mathrm{d}W_s$$

for some $f \in L^2([0,1])$ and the Kallianpur representation takes a less abstract form known as the Hitsuda representation (cf. Hida and Hitsuda [8], Theorem 6.3' or Hitsuda [9]): A Gaussian process $\tilde{W}$ is equivalent in law to the Brownian

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motion W if and only if there exists a Volterra kernel $k \in L^2([0,1]^2)$ and a function $a \in L^2([0,1])$ such that $\tilde{W}$ can be represented as

$$\tilde{W}_t = W_t - \int_0^t \int_0^s k(s,u)\,\mathrm{d}W_u\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s. \tag{1.1}$$

The Volterrra kernel k and the function a are unique and the representation (1.1) is proper canonical in the sense that the processes $\tilde{W}$ and W generate the same filtration: $\mathbf{F}^{\tilde{W}} = \mathbf{F}^W$. Moreover, the Brownian motion W is constructed from $\tilde{W}$ by

$$\begin{aligned} W_t &= \tilde{W}_t - \int_0^t \int_0^s r(s,u)\,\mathrm{d}\tilde{W}_u\,\mathrm{d}s - \int_0^t \left\{ a(s) - \int_0^s r(s,u)a(u)\,\mathrm{d}u \right\} \mathrm{d}s \\ &= \tilde{W}_t - \int_0^t \int_0^s r(s,u)(\mathrm{d}\tilde{W}_u - a(u)\,\mathrm{d}u)\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s \end{aligned} \tag{1.2}$$

where $r \in L^2([0,1]^2)$ is the resolvent of k. By the resolvent of k we mean the Volterra kernel r satisfying

$$\begin{aligned} r(t,s) + k(t,s) &= \int_s^t k(t,u)r(u,s)\,\mathrm{d}u, \\ r(t,s) + k(t,s) &= \int_s^t r(t,u)k(u,s)\,\mathrm{d}u. \end{aligned}$$

The existence and uniqueness of such a resolvent kernel follows from the theory of integral equations (cf. Smithies [22], Chapter 2). In fact, r can be represented as the Neumann series

$$r(t,s) = -\sum_{n=1}^{\infty} k^{(n)}(t,s)$$

where

$$\begin{aligned} k^{(1)}(t,s) &= k(t,s), \\ k^{(n)}(t,s) &= \int_s^t k(t,u)k^{(n-1)}(u,s)\,\mathrm{d}u. \end{aligned}$$

The Hitsuda representation (1.1) is connected to the Girsanov representation of the density between $\mathbf{P}_{\tilde{W}}$ and $\mathbf{P}_W$ in the following way (cf. Hida and Hitsuda [8], Theorem 6.3'):

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_{\tilde{W}}}{\mathrm{d}\mathbf{P}_W}\Big|\mathscr{F}_t &= \exp\biggl(\int_0^t \left(\int_0^s k(s,u)\,\mathrm{d}W_u + a(s) \right) \mathrm{d}W_s \\ &\qquad - \frac{1}{2}\int_0^t \left(\int_0^s k(s,u)\,\mathrm{d}W_u + a(s) \right)^2 \mathrm{d}s \biggr). \end{aligned} \tag{1.3}$$

We study to what extend one can have a fractional version of the Hitsuda representation for the fractional Brownian motion, i.e. is it possible to characterise *all* Gaussian processes $\tilde{Z}$ that are equivalent in law to the fractional Brownian motion Z with certain deterministic kernels f and functions A such that

$$\tilde{Z}_t \;=\; Z_t - \int_0^t f(t,s)\,\mathrm{d}Z_t - A(t). \tag{1.4}$$

(The terms in (1.4) is of different form than in (1.2), i.e. they are not differentiable. The reason for this lies in the structure of the reproducing kernel Hilbert space of fractional Brownian motion, cf. Proposition 1) It turns out that this is possible only in the case when the Hurst index H of the fractional Brownian motion satisfies $H \leq \frac{1}{2}$. For the case $H > \frac{1}{2}$ we have a partial result stating when a process $\tilde{Z}$ given by (1.4) is equivalent in law to the fractional Brownian motion. However, in this case there exist Gaussian processes that are equivalent in law to the fractional Brownian motion that do not admit the representation (1.4). The reason for the lack of the representation theorem for $H > \frac{1}{2}$ is that there are random variables linear spaces $\mathscr{H}_1(Z,t),\ t \in (0,1]$, that cannot be represented as Wiener integrals with respect to the fractional Brownian motion Z (cf. Pipiras and Taqqu [20], Theorem 5.1).

For $H \neq \frac{1}{2}$ a Girsanov representation like (1.3) can be constructed in terms of a Brownian motion constructed from the fractional one. To construct representation in terms of the fractional Brownian motion itself is difficult due to the problems indicated in the paragraph above. Furthermore, the situation is now even more complicated than the in the representation (1.1) due to the double integral in (1.3). Thus, we have a proper stochastic integral and there are at least two sensible (and different) ways to understand it.

A Kallianpur representation for the fractional Brownian motion is constructed with the help of an ordinary Brownian motion constructed from the fractional one.

Remark 1. We study Gaussian processes admitting continuous versions. Therefore, for the sake of definiteness one can consider the canonical space $\Omega = C([0,1])$ of real valued continuous functions over the interval $[0,1]$ equipped with the supremum norm and the corresponding Borel σ-algebra $\mathscr{F}$. The probability measure $\mathbf{P} = \mathbf{P}_W$ on Ω will make the coordinate process $(t,\omega) \mapsto \omega(t) =: W_t(\omega)$ a Brownian motion. The filtration $\mathbf{F} = (\mathscr{F}_t)_{t\in[0,1]}$ is generated in the natural way by the coordinate mappings $\omega \mapsto \omega(s),\ s \leq t$. The σ-algebras $\mathscr{F},\ \mathscr{F}_t,\ t \in [0,1]$ are augmented with the $\mathbf{P}$-null sets. In what follows all the σ-algebras generated by the different processes we consider turn out to be equal.

The work is organised as follows. In Section 2 we recall some preliminaries of deterministic fractional calculus and how one constructs Wiener integrals with respect to the fractional Brownian motion. Representations, especially the Hitsuda representation, with respect to the fractional Brownian motion are considered in Section 3. Finally, in Section 4 we study a certain kind of Gaussian stochastic equation by using the Hitsuda representation theorem.

2. Preliminaries

2.1. **Fractional integrals and derivatives.** We recall some preliminaries of fractional calculus. For details we refer to Samko et al. [21].

Definition 1. Let f be a function over the interval $[0,1]$ and $\alpha > 0$. Then

$$I_+^\alpha f\,(t) \quad := \quad \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s,$$

$$I_-^\alpha f\,(t) \quad := \quad \frac{1}{\Gamma(\alpha)} \int_t^1 \frac{f(s)}{(s-t)^{1-\alpha}}\,\mathrm{d}s$$

are called the left-sided and right-sided *Riemann–Liouville fractional integrals* of order α; I_+^0 and I_-^0 are identity operators.

Definition 2. Let f be a function over the interval $[0,1]$ and $\alpha \in (0,1)$. Then

$$D_+^\alpha f\,(t) \quad := \quad \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}\,\mathrm{d}s,$$

$$D_-^\alpha f\,(t) \quad := \quad -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^1 \frac{f(s)}{(s-t)^{\alpha}}\,\mathrm{d}s$$

are called the left-sided and right-sided *Riemann–Liouville fractional derivatives* of order α; D_+^0 and D_-^0 are identity operators.

If one ignores the troubles concerning divergent integrals one obtains and formally changes the order of differentiation and integration one obtains

$$I_\pm^{-\alpha} \quad = \quad D_\pm^\alpha.$$

We shall take the above as the definition for fractional integral of negative order.

The following is proved in Samko et al. [21], Theorem 2.5.

Lemma 1. *The composition formula*

$$I_\pm^\alpha I_\pm^\beta f \quad = \quad I_\pm^{\alpha+\beta} f$$

is valid in any of the following cases.

(i) $\beta \geq 0,\ \alpha+\beta \geq 0$ *and* $f \in L^1([0,1])$,
(ii) $\beta \leq 0,\ \alpha \geq 0$ *and* $f \in I_\pm^{-\beta} L^1([0,1])$,
(iii) $\alpha \leq 0,\ \alpha+\beta \leq 0$ *and* $f \in I_\pm^{-\alpha-\beta} L^1([0,1])$.

2.2. **Fractional Brownian motion and Wiener integrals.** The fractional Brownian motion $Z = (Z_t^H)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ is the (upto a multiplicative constant) unique centred H-self-similar Gaussian process with stationary increments. By H-self-similarity we mean that

$$(Z_t)_{t\in[0,1]} \quad \stackrel{d}{=} \quad (a^{-H} Z_{at})_{t\in[0,1]}$$

for all $a > 0$ where d means equality in distributions. It follows that

$$\mathbf{E}Z_tZ_s = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

and by Kolmogorov criterion Z admits a version with continuous sample paths.

When $H = \frac{1}{2}$ we have the ordinary Brownian motion: $W = Z^{1/2}$. If $H > \frac{1}{2}$ then the increments of the fractional Brownian motion exhibit the so-called long-range dependency property. The case $H < \frac{1}{2}$ corresponds to short-range dependence. For details of short and long-range dependence see Beran [3].

The fractional Brownian motion was originally defined and studied by Kolmogorov [12] within a Hilbert space framework where it was called a "Wiener helix". It was further studied by the Russian school (see e.g. [14, 23]). The name "fractional Brownian motion" comes from Mandelbrot and Van Ness [13].

By Norros et al. [15] there exist Volterra kernels z and z^* such that Z and W can be constructed from each others as

$$Z_t = \int_0^t z(t,s)\,\mathrm{d}W_s, \tag{2.1}$$

$$W_t = \int_0^t z^*(t,s)\,\mathrm{d}Z_s. \tag{2.2}$$

Here

$$z(t,s) = c_1\left(\left(\frac{t}{s}\right)^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}\,\mathrm{d}u\right),$$

$$z^*(t,s) = c_2\left(\left(\frac{t}{s}\right)^{\alpha}(t-s)^{-\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{-\alpha}\,\mathrm{d}u\right)$$

where we have denoted

$$\alpha := H - \frac{1}{2}$$

for notational convenience. The normalising constants are

$$c_1 = \sqrt{\frac{(2\alpha+1)\Gamma(1-\alpha)}{\Gamma(\alpha+1)\Gamma(1-2\alpha)}},$$

$$c_2 = \frac{\Gamma(\alpha+1)\Gamma(1-2\alpha)}{\mathrm{B}(1-\alpha,\alpha+1)\sqrt{(2\alpha+1)\Gamma(1-\alpha)}}$$

where Γ and B denote the gamma and beta functions, respectively.

Remark 2. The integrals (2.1) and (2.2) are defined in the mean square sense as well as improper ω-by-ω Riemann–Stieltjes integrals. Let us also note that the representation (2.1) first appeared in Molchan and Golosov [14] already in the late sixties (see also [2, 7, 17]).

Remark 3. Actually, the kernel z^* does not appear in Norros et al. [15]. However, it is easy to derive it by the same methods that yield the kernel z.

Using the notions of fractional calculus we can write z and z^* neatly as

$$z(t,s) = c_3 s^{-\alpha}\left(I_-^{\alpha} u^{\alpha}\mathbf{1}_{[0,t]}(u)\right)(s), \tag{2.3}$$

$$z^*(t,s) = c_3 s^{-\alpha}\left(I_-^{-\alpha} u^{\alpha}\mathbf{1}_{[0,t]}(u)\right)(s). \tag{2.4}$$

The normalising constant c_3 is

$$c_3 = \sqrt{\frac{\alpha(2\alpha+1)\Gamma(\alpha)^2}{\mathrm{B}(\alpha,1-2\alpha)}}.$$

For details concerning (2.3) and (2.4) we refer to Pipiras and Taqqu [19, 20].

Let us recall the construction of Wiener integral with respect to fractional Brownian motion. For details we refer to Pipiras and Taqqu [20]. Let $f\in\mathscr{E}$, i.e.

$$f = \sum_{k=1}^{n} a_k \mathbf{1}_{(t_{k-1},t_k]},$$

$0\le t_0 < t_1 < \cdots < t_n \le 1$ and $a_1,\ldots,a_n\in\mathbb{R}$. In this case it is natural to set

$$\int_0^1 f(t)\,\mathrm{d}Z_t := \sum_{k=1}^{n} a_k (Z_{t_k} - Z_{t_{k-1}}).$$

Representations (2.3) and (2.4) suggest to consider the operators

$$\mathrm{K}f(t) := c_3 t^{-\alpha}\left(I_-^{\alpha} s^{\alpha} f(s)\right)(t),$$

$$\mathrm{K}^*f(t) := c_3 t^{-\alpha}\left(I_-^{-\alpha} s^{\alpha} f(s)\right)(t)$$

So, K and K^* are defined by extending linearly the relations

$$z(t,s) = \mathrm{K}\mathbf{1}_{[0,t]}(s),$$

$$z^*(t,s) = \mathrm{K}^*\mathbf{1}_{[0,t]}(s).$$

In particular, for any $f\in\mathscr{E}$ and almost all ω we have

$$\int_0^1 f(t)\,\mathrm{d}Z_t = \int_0^1 \mathrm{K}f\,(t)\,\mathrm{d}W_t.$$

Since the classical Wiener integral is defined for any $f\in L^2([0,1])$ the discussion above leads to the following definition.

Definition 3. Set

$$\Lambda := \left\{ f : \mathrm{K}f \in L^2([0,1]) \right\}.$$

The *Wiener integral of $f\in\Lambda$ with respect to fractional Brownian motion* is

$$\int_0^1 f(t)\,\mathrm{d}Z_t := \int_0^1 \mathrm{K}f\,(t)\,\mathrm{d}W_t.$$

The integral in Definition 3 can be considered as a limit of elementary functions: by theorems 4.1 and 4.2 of [20] the set Λ is a Pre-Hilbert space with

$$\langle f, g\rangle_\Lambda \;:=\; \langle \mathrm{K}f, \mathrm{K}g\rangle_{L^2([0,1])}.$$

Moreover, the set of elementary functions $\mathscr{E}$ is dense in Λ.

Lemma 2. *The equalities*

$$\mathrm{K}\mathrm{K}^* f \;=\; f \;=\; \mathrm{K}^*\mathrm{K} f$$

hold for any $f \in \mathscr{E}$*. If* $H > \frac{1}{2}$ *then the equality*

$$\mathrm{K}^*\mathrm{K} f \;=\; f$$

holds for any $f \in L^2([0,1])$*. If* $H < \frac{1}{2}$ *then, for any* $f \in L^2([0,1])$*, we have*

$$\mathrm{K}\mathrm{K}^* f \;=\; f.$$

Proof. The first claim follows from the integral representations (2.1) and (2.2). The latter claims follow from Lemma 1. □

Lemma 3. *Let* $H > \frac{1}{2}$*. Then there exist functions* $f \in L^2([0,1])$ *such that the equation*

$$\mathrm{K}g \;=\; f \tag{2.5}$$

has no solution in g*.*

The reason for the lack of solutions in (2.5) is that for $H > \frac{1}{2}$ the operator K is a weighted fractional *integral* operator. So, one can *differentiate* f fractionally (K^* is a weighted fractional differential operator). However, there are "arbitrarily non-smooth" functions in $L^2([0,1])$. An example of such a function is

$$f(t) \;=\; t^{\frac{1}{2}-H}\psi(t) \tag{2.6}$$

where ψ is the real part of the Weierstrass function

$$\psi^*(t) \;=\; \sum_{n=1}^{\infty} b^{-pn} e^{ib^n t},$$

$b > 1$ and $p \in (0, \alpha)$. For the rigorous proof of Lemma 3 see [20], Lemma 5.3.

Recall that the *reproducing kernel Hilbert space* $\mathscr{H}(Z)$ of Z with covariance function R is the closure of $\{R(t,\cdot) : t \in [0,1]\}$ with respect to the inner product

$$\langle R(t,\cdot), R(s,\cdot)\rangle_{\mathscr{H}(Z)} \;:=\; R(t,s).$$

The *linear space* $\mathscr{H}_1(Z,t)$ is the set of random variables that can be approximated in $L^2(\Omega, \mathscr{F}^Z, \mathbf{P})$ by Wiener integrals of elementary functions over the interval $[0,t]$ with respect to Z . We shall denote briefly $\mathscr{H}_1 := \mathscr{H}_1(Z,1)$ and $\mathscr{H} := \mathscr{H}(Z)$. The mapping $R(t,\cdot) \mapsto Z_t$ extends to an isometry between $\mathscr{H}$ and $\mathscr{H}_1$.

The space $\mathscr{H}$ can be described in the following way (cf. Decreusefond and Üstünel [7], Theorem 3.3 and Remark 3.1).

Proposition 1. *A function $f \in \mathscr{H}$ if and only if it can be represented as*

$$f(t) \;=\; \int_0^t z(t,s)\tilde{f}(s)\,\mathrm{d}s$$

for some $\tilde{f} \in L^2([0,1])$. The scalar product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ on $\mathscr{H}$ is given by

$$\langle f,g\rangle_{\mathscr{H}} \;=\; \langle \tilde{f},\tilde{g}\rangle_{L^2([0,1])}.$$

Moreover, as a vector space

$$\mathscr{H} \;=\; I_+^{H+\frac{1}{2}}(L^2([0,1])).$$

It is desirable that also the space Λ is isometric to $\mathscr{H}_1$, i.e. one wants to identify any $F \in \mathscr{H}_1$ with a single function $f \in \Lambda$ so that

$$F \;=\; \int_0^1 f(t)\,\mathrm{d}Z_t$$

for some $f = f_F \in \Lambda$. Obviously, this is possible if and only if Λ is complete. Otherwise Λ is isometric to a proper subspace of $\mathscr{H}_1$. Indeed, by lemmas 2 and 3 we have the following.

Proposition 2. *If $H \le \frac{1}{2}$ then*

$$\Lambda \;=\; \left\{\mathrm{K}^* f \,:\, f \in L^2([0,1])\right\}.$$

In particular, the inner product space Λ is complete and hence isometric to $\mathscr{H}_1$. For $H > \frac{1}{2}$ the inner product space Λ is not complete and hence isometric to a proper subspace of $\mathscr{H}_1$.

Let us summarise the discussion above as a lemma.

Lemma 4. *For $H \le \frac{1}{2}$ the equality*

$$\int_0^1 g(t)\,\mathrm{d}W_t \;=\; \int_0^1 \mathrm{K}^* g\,(t)\,\mathrm{d}Z_t \tag{2.7}$$

holds for any $g \in L^2([0,1])$. For $H > \frac{1}{2}$ the equality (2.7) holds only for $g \in \mathrm{K}(\Lambda)$ and the inclusion $\mathrm{K}(\Lambda) \subset L^2([0,1])$ is strict.

Remark 4. It is tempting to extend Definition 3 to random f. This turns out to be rather problematic, however. The problem is that while the isometry

$$\mathbf{E}\Big(\int_0^1 \mathrm{K}f\,(t)\,\mathrm{d}W_t\Big)^2 \;=\; \mathbf{E}\int_0^1 (\mathrm{K}f\,(t))^2\,\mathrm{d}t$$

holds for deterministic f it fails for random f. This is due to the fact that even though f were adapted $\mathrm{K}f$ may not be, i.e. K "looks into the future". So, we are not in the domain of the classical Ito isometry and one has to consider "anticipative" stochastic integrals (which has been done e.g. in [1, 7]).

3. Representations

Let us denote by $L^2_V([0,1]^2)$ the set of square integrable Volterra kernels, i.e. the set of functions $k:[0,1]^2\to\mathbb{R}$ satisfying

$$\begin{aligned}\int_0^1\int_0^1 k(t,s)^2\,\mathrm{d}s\mathrm{d}t &< \infty,\\ k(t,s) &= 0, \qquad \text{whenever } s>t.\end{aligned}$$

Denote by $\mathbf{P}_X$ the law of X. We say that a process $\tilde{X}$ is equivalent in law to a process X if their corresponding distributions are mutually absolutely continuous, i.e. $\mathbf{P}_{\tilde{X}}\sim\mathbf{P}_X$.

Let us introduce operators acting on Volterra kernels.

$$\begin{aligned}\mathrm{V}^*k(t,s) &:= \int_0^1 \mathrm{K}\mathbf{1}_{[0,t]}(u)\mathrm{K}^*k(u,\cdot)(s)\,\mathrm{d}u,\\ \mathrm{V}k(t,s) &:= \int_0^1 \mathrm{K}^*\mathbf{1}_{[0,t]}(s)\mathrm{K}k(u,\cdot)(s)\,\mathrm{d}u.\end{aligned}$$

Note that V^* is defined for kernels $k\in L^2_V([0,1])^2$ with the property that $k(t,\cdot)\in\Lambda$ for all $t\in[0,1]$. If $k\in\mathrm{V}^*\left(L^2_V([0,1]^2)\right)$ then $\mathrm{V}k$ exists. In particular, $k=\mathrm{V}\mathrm{V}^*k$.

Our main theorem is the following.

Theorem 1. *Let Z be a fractional Brownian motion with index $H\in(0,1)$.*

(i) *A Gaussian process $\tilde{Z}$ given by*

$$\tilde{Z}_t = Z_t - \int_0^t f(t,s)\,\mathrm{d}Z_s - A(t) \tag{3.1}$$

is equivalent in law Z if and only if

$$\begin{aligned}f &\in \mathrm{V}^*\left(L^2([0,1]^2)\right),\\ A &\in I_+^{H+\frac12}\left(L^2([0,1])\right).\end{aligned}$$

Moreover, the fractional Brownian motion Z is constructed from $\tilde{Z}$ by

$$Z_t = \tilde{Z}_t - \int_0^t \mathrm{V}^*r(t,s)\,\mathrm{d}\tilde{Z}_s - A(t) + \int_0^t \mathrm{V}^*r(t,s)\,\mathrm{d}A(s) \tag{3.2}$$

where r is the resolvent of $\mathrm{V}f$.

(ii) *If $H\le\frac12$ then a Gaussian process is equivalent to fractional Brownian motion if and only if it is given by (3.1). For $H>\frac12$ there are Gaussian processes equivalent in law to the fractional Brownian motion that do not admit the representation (3.1).*

Before going into the proof of Theorem 1 note that a straightforward application of the Hitsuda representation theorem to the representations (2.1) and (2.2) yields the following (cf. [8], Theorem 6.4).

Proposition 3. *A Gaussian process $\tilde{Z}$ is equivalent in law to a fractional Brownian motion Z if and only if it can be represented as*

$$\tilde{Z}_t = Z_t - \int_0^t z(t,s)\int_0^s k(s,u)\,\mathrm{d}W_u\,\mathrm{d}s - \int_0^t z(t,s)a(s)\,\mathrm{d}s \qquad (3.3)$$

where W is a Brownian motion constructed from Z by (2.2), $a \in L^2([0,1])$ and $k \in L^2_V([0,1]^2)$. Moreover, the fractional Brownian motion Z is constructed from $\tilde{Z}$ by

$$\begin{aligned} Z_t &= \tilde{Z}_t - \int_0^t z(t,s)\int_0^s r(s,u)\,\mathrm{d}\tilde{W}_u\,\mathrm{d}s \\ &\quad + \int_0^t z(t,s)\left\{a(s) - \int_0^s r(s,u)a(u)\,\mathrm{d}u\right\}\,\mathrm{d}s \\ &= \tilde{Z}_t - \int_0^t z(t,s)\int_0^s r(s,u)(\mathrm{d}\tilde{W}_u - a(u)\,\mathrm{d}u)\,\mathrm{d}s - \int_0^t z(t,s)a(s)\,\mathrm{d}s \end{aligned}$$

where $\tilde{W}$ is constructed from $\tilde{Z}$ by

$$\tilde{W}_t = \int_0^t z^*(t,s)\,\mathrm{d}\tilde{Z}_s$$

and $r \in L^2_V([0,1]^2)$ is the resolvent of k.

Using Proposition 1 we can rephrase (3.3) in Proposition 3.

Proposition 4. *A Gaussian process $\tilde{Z}$ is equivalent in law to a fractional Brownian motion Z with index H if and only if it can be represented as*

$$\tilde{Z}_t = Z_t - \int_0^t f(t,s)\,\mathrm{d}W_s - A(t) \qquad (3.4)$$

where W is a Brownian motion constructed from Z by (2.2),

$$A = I_+^{H+\frac{1}{2}}a$$

for some $a \in L^2([0,1])$ and for all $s \in [0,1]$

$$f(\cdot,s) = I_+^{H+\frac{1}{2}}k(\cdot,s)$$

for some $k \in L^2_V([0,1]^2)$.

Remark 5. The equation (3.4) is a generalisation of Hitsuda representation (1.1) in the sense that for the ordinary Brownian motion, i.e. in the case $H = \frac{1}{2}$, the operator $I_+^{H+1/2} = I_+^1$ is an ordinary Lebesgue integral and the equation (1.1) can be obtained from (3.4) by changing the order of the Wiener integral and the Lebesgue integral.

The obvious difference between the representations (1.1) and (3.3) (or (3.4)) is that in the latter one one needs to construct a Brownian motion from Z in order to represent $\tilde{Z}$. Nevertheless, (3.3) and (3.4) are Kallianpur representations of $\tilde{Z}$ with respect to Z, since obviously $\tilde{Z}_t \in \mathscr{H}_1(Z,t)$.

In the case $H \leq \frac{1}{2}$ (and in this case only) we can use Lemma 4 to obtain a representation without the auxiliary Brownian motion. This is the essence of Theorem 1.

Proof of Theorem 1. (i) The form of the drift A is obvious by Proposition 4. Consequently, in proving (3.1) we may well assume that the process $\tilde{Z}$ has mean zero. By Proposition 3 the Gaussian process $\tilde{Z}$ is equivalent in law to Z if and only if

$$\tilde{Z}_t = Z_t - \int_0^t \int_s^t z(t,u)k(u,s)\,\mathrm{d}u\,\mathrm{d}W_s$$

where W is a Brownian motion constructed from the fractional one Z by (2.2), and $k \in L^2_V([0,1]^2)$. On the other hand, for the Wiener integral in (3.1), we have by Definition 3

$$\int_0^t f(t,s)\,\mathrm{d}Z_s = \int_0^t \mathrm{K}f(t,\cdot)\,(s)\,\mathrm{d}W_s.$$

Thus,

$$\mathrm{K}f(t,\cdot)\,(s) = \int_s^t z(t,u)k(u,s)\,\mathrm{d}u.$$

Operating with K^* on the both sides of the equation above and using Fubini theorem we obtain

$$\begin{aligned} f(t,s) &= \left(\mathrm{K}^* \int_s^t z(t,u)k(u,\cdot)\,\mathrm{d}u\right)(s) \\ &= \int_s^t z(t,u)\mathrm{K}^*k(u,\cdot)\,(s)\,\mathrm{d}u \\ &= \int_0^t z(t,u)\mathrm{K}^*k(u,\cdot)\,(s)\,\mathrm{d}u \\ &= \mathrm{V}^*k\,(t,s). \end{aligned}$$

Here we have used the fact that both z and k are Volterra kernels. We have shown (3.1).

Let us show the inverse relation (3.2). Assume for the while that $A \equiv 0$. By Proposition 3 we have

$$Z_t = \tilde{Z}_t - \int_0^t z(t,s) \int_0^s r(s,u)\,\mathrm{d}\tilde{W}_u\,\mathrm{d}s \tag{3.5}$$

where r is the resolvent of k and

$$\tilde{W}_t = \int_0^t z^*(t,s)\,\mathrm{d}\tilde{Z}_s$$

Thus, it remains to show that $k(u,\cdot) \in \mathrm{K}(\Lambda)$ implies $r(u,\cdot) \in \mathrm{K}(\Lambda)$. If so, we can invert (3.5) to get (3.2). But this follows from the resolvent relation (defining the resolvent)

$$r(t,s) = -k(t,s) + \int_s^t r(t,u)k(u,s)\,\mathrm{d}u.$$

Indeed, by Fubini theorem we see that

$$\mathrm{K}^* r(t,\cdot) = -\mathrm{K}^* k(t,\cdot) + \int_s^t r(t,u)\mathrm{K}^* k(u,\cdot)\,\mathrm{d}u.$$

The integral in the right hand side of the equation above exists since k and r are in $L_V^2([0,1]^2)$. The representation (3.2) follows now by adding in the drift A of $\tilde{Z}$.

(ii) If $H \le \frac{1}{2}$ then (3.4) can always be written as (3.1) by Lemma 4. So, there are no Gaussian processes equivalent in law to the fractional Brownian motion with $H \le \frac{1}{2}$ without the representation (3.1).

Finally, let us construct a Gaussian process $\tilde{Z}$ that is equivalent in law to a fractional Brownian motion with index $H > \frac{1}{2}$ that does not admit the representation (3.1). Let $g \in L^2([0,1])$ be given by (2.6). Now, by Proposition 3 the process

$$\tilde{Z}_t := Z_t - \int_0^t z(t,s) \int_0^s g(u)\,\mathrm{d}W_u\,\mathrm{d}s \tag{3.6}$$

is equivalent in law to the fractional Brownian motion Z. However, the Wiener integral in (3.6) cannot be represented in terms of Z since the function $\mathrm{K}^* g$ does not exist by Lemma 3. □

Remark 6. Note that since each of the processes W, $\tilde{W}$, Z, and $\tilde{Z}$ can be constructed from each others the corresponding filtrations coincide: $\mathbf{F}^W = \mathbf{F}^{\tilde{W}} = \mathbf{F}^Z = \mathbf{F}^{\tilde{Z}}$.

Let us end this section by saying a few words of the connection to the Girsanov representation.

Suppose that a Gaussian process $\tilde{Z}$ is equivalent in law to a fractional Brownian motion Z. If $\tilde{Z}$ is given by (3.3) then the density is of course given by (1.3):

$$\frac{\mathrm{d}\mathbf{P}_{\tilde{Z}}}{\mathrm{d}\mathbf{P}_Z}\Big|\mathscr{F}_t = \exp\Bigg(\int_0^t \left(\int_0^s k(s,u)\,\mathrm{d}W_u + a(s) \right) \mathrm{d}W_s - \frac{1}{2}\int_0^t \left(\int_0^s k(s,u)\,\mathrm{d}W_u + a(s)\right)^2 \mathrm{d}s \Bigg)$$

where the Brownian motion W is constructed from from the fractional one. If $\tilde{Z}$ is given by (3.1) then we know that the inner integral in (1.3) can be represented in terms of the fractional Brownian motion Z. However, to consider the outer integral we have to define what we mean by a stochastic or multiple Wiener integral with respect to fractional Brownian motion. An approach taken by Pérez-Abreu and Tudor [18] is to define

$$\int_0^1 \int_0^1 k(t,s)\,\mathrm{d}Z_s \mathrm{d}Z_t \;:=\; \int_0^1 \int_0^1 \mathrm{K}_2 k\,(t,s)\,\mathrm{d}W_s\,\mathrm{d}W_t \tag{3.7}$$

where

$$\mathrm{K}_2 \;:=\; \mathrm{K} \otimes \mathrm{K}$$

and the right hand side of (3.7) is a double Wiener integral in the sense of Ito. So, one can define the fractional double Wiener integral for integrands from the set

$$\Lambda_2 \;:=\; \left\{ k \,:\, \mathrm{K}_2 f \in L^2([0,1]^2) \right\}.$$

The situation is here similar to the single Wiener integral case: If $H \le \frac{1}{2}$ then Λ_2 is complete and otherwise not (cf. [18]). Thus, if $H \le \frac{1}{2}$ we may assume that $\tilde{Z}$ is given by (3.1) and we have

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_{\tilde{Z}}}{\mathrm{d}\mathbf{P}_Z}\Big|\mathscr{F}_t \;=\; \exp\Bigg(& \int_0^t \int_0^t \mathrm{K}_2^* k\,(s,u)\,\mathrm{d}Z_u\,\mathrm{d}Z_s + \int_0^t \mathrm{K}^* a\,(s)\,\mathrm{d}Z_s \\ & -\frac{1}{2}\int_0^t \left(\int_0^t \mathrm{K}^* k(s,\cdot)\,(u)\,\mathrm{d}Z_u + \mathrm{K}^* a\,(s) \right)^2 \mathrm{d}s \Bigg) \end{aligned} \tag{3.8}$$

where

$$\mathrm{K}_2^* \;:=\; \mathrm{K}^* \otimes \mathrm{K}^*,$$

$k = \mathrm{V}f$ and $A = \int_0^1 z(\cdot,s)a(s)\,\mathrm{d}s$. If $H > \frac{1}{2}$ then there are processes $\tilde{Z}$ that are equivalent in law to the fractional Brownian motion Z that that do not admit the representation (3.8) even if $\tilde{Z}$ admits the representation (3.1).

Finally, let us note that the operators K_2 and K_2^* do not preserve Volterra kernels. So, it might be more natural to define the double Wiener integral as a double Stratonovich–Wiener integral. This has been done for $H > \frac{1}{2}$ by Dasgupta and Kallianpur [5, 6]. This definition leads into a Shepp-type representation of the density.

4. Application to stochastic equations

We apply the Hitsuda representation theorem to a special kind of Gaussian stochastic equation where the fractional Brownian motion Z is the noise term. More generally stochastic differential equations with respect to the fractional Brownian motion have been studied e.g. in [4, 16] (with different notions of stochastic integrals).

Henceforth, we assume to be in the centred Gaussian realm and in the canonical space $\Omega = C([0,1])$ considered in the Remark 1. The measure $\mathbf{P}_Z$ is such that the coordinate process is Z. Similarly, $\mathbf{P}_W$ is the Wiener measure. The filtration generated by the coordinate measure is denoted by $\mathbf{F}$. By Remark 6 the filtrations under $\mathbf{P}_Z$ and $\mathbf{P}_W$ coincide although the measures themselves are singular.

A functional $\gamma : [0,1]\times\Omega \to \mathbb{R}$ satisfying

(i) γ is $\mathscr{B}([0,1])\times\mathscr{F}$ measurable,
(ii) $\gamma(t,\cdot)$ is measurable with respect to $\mathscr{F}_t$

is called *nonanticipative* or *causal.*

Theorem 2. *Let $\beta : [0,1]\times\Omega \to \mathbb{R}$ be a nonanticipative functional satisfying*

$$\mathbf{P}_Z\Big(\omega\in\Omega \ : \ \int_0^1 \beta(t,\omega)^2\,\mathrm{d}t < \infty\Big) \ = \ 1. \tag{4.1}$$

(i) *Then the stochastic equation*

$$\zeta_t \ = \ \int_0^t z(t,s)\beta(s,\zeta)\,\mathrm{d}s + Z_t. \tag{4.2}$$

has a Gaussian weak solution if and only if β is of the form

$$\beta(t,\omega) \ = \ \int_0^t k(t,u)\,\mathrm{d}\omega^*(u) \qquad \mathrm{Leb}\times\mathbf{P}_Z - \text{a.e.}$$

where

$$\omega^*(t) \ = \ \int_0^t z^*(t,s)\,\mathrm{d}\omega(s) \tag{4.3}$$

and $k\in L^2_V([0,1]^2)$.

(ii) *If the index H of the fractional Brownian motion Z satisfies $H\le\frac12$ then β can be written as*

$$\beta(t,\omega) \ = \ \int_0^t f(t,s)\,\mathrm{d}\omega(s)$$

where $f(t,\cdot) = \mathrm{K}^*k(t,\cdot)$. If $H>\frac12$ then there are functionals β that cannot be represented as an integral with respect to ω.

(iii) *If a Gaussian weak solution to (4.2) exists then so does a Gaussian strong solution, and the latter is unique.*

To prove Theorem 2 let us recall the framework of Section 9.4 of Kallianpur [10].

Consider a stochastic differential equation of the form

$$\mathrm{d}\xi_t \;=\; \gamma(t,\xi)\,\mathrm{d}t + \mathrm{d}W_t \tag{4.4}$$

where γ is a nonanticipative functional satisfying

$$\mathbf{P}_W\Big(\omega\in\Omega\;:\;\int_0^1 \gamma^2(t,\omega)\,\mathrm{d}t<\infty\Big) \;=\; 1. \tag{4.5}$$

Theorem 9.4.2 of [10] states that the equation (4.4) has a Gaussian weak solution if and only if γ is of the form

$$\gamma(t,\omega) \;=\; \int_0^t r(t,u)\,\mathrm{d}\omega(u) \qquad \mathrm{Leb}\times\mathbf{P}_W - \text{a.e.}$$

for some $r\in L^2_V([0,1]^2)$. Moreover, if a Gaussian weak solution exists then so does a Gaussian strong solution. The latter is unique.

Remark 7. The condition (4.5) has two roles. Firstly, it implies that $\mathbf{P}_\xi \ll \mathbf{P}_W$ (cf. Kallianpur [10], Theorem 7.3.1). Now, since we assumed ξ to be Gaussian we have $\mathbf{P}_\xi\sim\mathbf{P}_W$. Therefore, if a weak solution $\mathbf{P}_\xi$ exists we can construct a strong solution ξ' by using the Hitsuda representation theorem as

$$\begin{aligned}\xi'_t \;&=\; -\int_0^t\int_0^s k(s,u)\,\mathrm{d}W_u\,\mathrm{d}s + W_t\\ &=\; \int_0^t\int_0^s r(s,u)\,\mathrm{d}\xi'_u\,\mathrm{d}s + W_t\end{aligned}$$

where r is a suitable kernel depending on the functional γ (or equivalently, of the law $\mathbf{P}_\xi$). Secondly, using the condition (4.5) again one can show that if ξ'' is any strong solution to (4.4) then $\xi'=\xi''$ $\mathbf{P}_W$- (and hence $\mathbf{P}_\xi$) almost surely. So, the uniqueness of ξ follows from the uniqueness of the Hitsuda representation.

Remark 8. In the fractional case $H\neq\frac12$ it is natural to replace the "differential drift" $\gamma(t,\xi)\,\mathrm{d}t$ in (4.4) by a fractionally differentiable one of (4.2) (cf. propositions 1 and 4). Note that for $H>\frac12$ the drift in (4.2) is differentiable. Indeed, by Lemma 1 and the representation (2.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t z(t,s)\beta(s,\zeta)\,\mathrm{d}s \;=\; \int_0^t \frac{\partial z}{\partial t}(t,s)\beta(s,\zeta)\,\mathrm{d}s.$$

Proof of Theorem 2. (i) and (iii) Define a nonanticipative functional γ by setting

$$\gamma(t,\omega^*) \;:=\; \beta(t,\omega)$$

where ω^* is defined by (4.3). The transformation $\omega\mapsto\omega^*$ is well-defined for $\mathbf{P}_Z$-almost all ω by Remark 2. Consider then the equation

$$\mathrm{d}\xi_t \;=\; \gamma(t,\xi)\,\mathrm{d}t + \mathrm{d}W_t. \tag{4.6}$$

We see that the condition (4.1) for β is just a restatement of (4.5) with this choice γ. Consequently, by Theorem 9.4.2 of [10] we know that (4.6) has a Gaussian (weak and strong) solution if and only if

$$\gamma(t,\omega) \;=\; \int_0^t r(t,s)\,\mathrm{d}\omega(s).$$

Since ξ is equivalent in law to W and ξ is a strong solution we can operate with the kernel z on both sides of (4.6). But this yields the equation (4.2). Also, we see that a solution ζ of (4.2) must be equivalent in law to Z. Consequently,

$$\zeta_t \;=\; \int_0^t z(t,s)\,\mathrm{d}\xi_s$$

is a strong solution to (4.2) and its uniqueness follows from the uniqueness of ξ. The claims (i) and (iii) follow.

(ii) In view of Theorem 1(ii) and Lemma 4 this claim is obvious. □

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