Recursive Synthesis and the Foundations of Mathematics

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ABSTRACT. This paper presents mathematics as a general science of computation in a way different from the tradition. It is based on the radical philosophical standpoint according to which the content, meaning and justification of experience lies in its precise formulation. The requirement on precise, formal content discloses the relational structure of (mathematical) experience, and gives a new meaning to the 'ideal' objects beyond concrete forms. The paper also provides a systematic reason why set theory represents an ultimate stage in mathematical technology.

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1. Introduction

Philosophies of mathematics fall into established trends such as Platonism, intuitionism, constructivism, modalism, and others. They are usually studied on the basis of a special 'mathematical experience'. The radical standpoint adopted here seeks to bring the foundational study of mathematics to the global philosophical context of *experience in general.* Traditional foundational frameworks (e.g., set theory, category theory, proof theory) start from special internal fields within mathematics, and they attempt at describing the entire, present and future, mathematical enterprise from their perspective. Critically speaking, they try to 'force' the rest of mathematics into their mold. The purpose of this paper is to view mathematical foundations from the standpoint of a general structure or 'logic' of experience.

Our goal is expressed in the need of grounding the philosophy of mathematics on *expressible* and *intercommunicable* forms, rather than 'personal intuition'. We request a critical evaluation of experience, both 'natural' and 'mathematical'. The actual (formal) content of experience is to be set against the remainder that cannot be expressed in public, written forms. Philosophical and even more so, scientific, considerations should be based on, or 'effectively' connected with, formal experience. A fortiori, this ought to be expected in mathematics.

The point of departure here lies in a rigorous *articulation* of experience. We say that experience is *bounded* by its forms; each element of experience is a form, and as such is determinate and bounded (not merely finite). While the form is bounded in itself, it is – in general – related to other forms. Thus, a particular form lies in a *relational field.* The field is not there as a totality of its elements but rather as a *possibility* or mode of forms to be connected to a given one. In general, the totality would not be defined within the given formal experience.

I am concretely limited with respect to my capabilities: For example, I may easily add small numbers, say 3 and 4, without the use of an algorithmic method of counting. However, for larger numbers, we introduce new *units* that replace groups with the same number of elements. Thus, we replace ten elements by a unit, and repeat this as many times as needed to express the desired number with nine 'digits'. *Recursive synthesis*, simply described, consists in replacing the many with one, an element to be used in further syntheses. This new element is *related* to the replaced ones through the very relation of collecting (synthesis). It both 'hides' and 'opens up into' the collected elements.

Instead of beginning from the relational field of concrete, formal experience, *analytic philosophy* has based its arguments on a 'free' space of sets and individuals, placed at an ideal separation from concrete experience. Set theory, quantification theory and analytic philosophy are mutually related through their conceptual dependence on 'collections of individuals'. The first two have become the methodology of analytic philosophy, and they share the same *epistemological notion of experience*.

For recursive synthesis, the principal notion is that of collecting a multitude under a unit, i.e., forming a *totality*. Set theory has placed its notion of totality in 'pure epistemology'[1]. The problematic totalities are the infinite ones. As shown by various approaches to this field, the philosophy of mathematics hinges on the notion of infinity[2]. Here we take, however, our point of departure by asking for the precise meaning of 'finite' objects. We do not take them for granted.

The simple notion of finitude is *transitive*: A finite collection of finite collections is again finite. This notion corresponds to the simple infinity of pure epistemology, completed in itself. In fact, transitivity is a form of completion or closure. Transitivity is the essence of topology in both the classical sense (Kuratowski closure) and the Grothendieck topologies, and ties these together with modal logic (system S4). Transitivity naturally leads to the completion, as in the case of positive integers one is led to the minimal collection which contains all the repeated successors of the first element. However, the notion of 'all the successors' (in its common, non-critical sense) is not anymore given in formal experience but is epistemological. It is postulated as a well-known object of mathematical thought, but it is not formally represented [3]. On the other hand, in a purely relational setting, we are given a form f and a formal definition describing to which other forms f is related. The problem is to derive a necessary (and sufficient) description of the 'totality' of forms, thus related, without going beyond the given experience. It turns out that the synthetic 'form' of the totalities is found in their topological 'structure'.

What are then the 'concrete' forms of mathematical experience? In this paper, forms are anything that can be given a finite alphabet and finite rules of generation; in fact, a generative or 'productive' relation. For example, we may regard the formulas of a standard first-order predicate logic in this manner. (In general, we should not stipulate that the forms be 'generated' - it should be enough to check that a formal relation holds between any two *given* forms.)

Each part of mathematics has its own productive rules for the forms, and its internal relation which make the forms into a relational field. Thus, the forms in a propositional calculus are its sentences, generated in the standard manner from the logical connectives and propositional symbols. The associated relational field is obtained from the relation of implication. There is no need for individuals.

[1] Pure epistemology places its objects outside of experience and description – those are objects of which we can have *knowledge*.

[2] This does not preclude the existence of other such 'hinges'.

[3] Axiomatic set theory has to assume the existence of an infinite set and then derives the existence of a set containing the given finite sets as subsets. But in postulating a *set*, it goes beyond the *formally given*, inasmuch as sets have intuitive 'content' at all.

In a relational field, each element is *bounded* by its relational neighbourhood. Similarly, to be bounded determines a relationship between the given and the other, in other words, the boundary is always between an 'interior' and an 'exterior'. This is the natural point of contact with topology and modal logic. Indeed, modal logic has been re-appearing in philosophical and logical considerations through centuries, and recently as a general problem for the analytic logic that made Quine to discard it altogether. From our standpoint, the reason that the *ghost of modality* (as coined by Weyl [14]) keeps re-instating its relevence, is to be found in the way it is associated with the structure of relational fields.

Sentential modal logic (i. e., without quantifiers) itself is a part of the theory of relational fields. By the work of Kripke , we know that for each set with a relation, there is a corresponding 'modality' that defines a modal logic, and vice versa (the elements of the set being 'possible worlds'). In our setting, the modality is the relation which describes what forms are 'possible' for a given one. The link between modal logic (specifically, the so-called system S4) and topology has been understood since the work of MacKinsey and Tarski in the 1930's[4].

In fact, *covering systems* of forms (or propositions) — or more generally partially ordered sets — satisfying natural weak axioms were considered by Fourman and Grayson [4] to define *formal topologies*[5]. A covering relation holds between an element of a partially ordered set P and a subset $A \subset P$. Of the four axioms (equivalent to those of a Grothendieck topology) in [4], three are rather obvious[6], but the fourth — postulating the *transivity* of the concept of covering — is essential from the standpoint of topology. Transitivity corresponds to the specific axiom $MMp \rightarrow Mp$ of S4 in modal logic, as well as to the idempotency $C^2 = C$ of the Kuratowski closure operator in topology.

The axioms for topology are defined for 'parts' of a topological 'space'. In our situation, we consider *chains* in relational fields. In the most general setting, a chain is a pair (f, R) consisting of a form f and a relation connecting f to other forms. The relation R is not necessarily that of the underlying relational field; in general, it imposes additional conditions on the forms in the chain and therefore is *stronger* than the underlying relation. The category-theoretical counterpart of a chain is a projective (or 'inverse') system (often denoted as $(x_i \leftarrow x_j)$; the category theory associates with each such system the projective limit $\lim_{\leftarrow} x_i$. Similarly, we associate a *projective foundation* with a chain, determined topologically from the members of the chain. As such, the projective foundation (intuitively: limit) of the chain is nothing but its 'place'. It has no content in itself other than its topological structure. Traditionally, the projective foundation is the *set* of such chains (for example, the real numbers are given as a set of (equivalence classes of) Cauchy sequences). The topological meaning of the projective foundation is the main reason that separates our views from set theory as well analytic philosophy.

In our work, however, sets are replaced by *concepts* ('generic elements', 'parts') represented by forms in the given relational system. Even in classical, set-theoretical mathematics, sets are frequently defined as collections of elements that satisfy a formula. Usually in such situations, the set does not represent anything over and above the formula itself. However, often set-theoretical considerations go beyond formal experience and involve sets which have no formal definition (sufficient to identify them) but are justified on epistemological grounds.

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[4] A related link between sentential intuitionistic logic and modal logic (S4) was pointed out by Gödel [5] in 1933.

[5] For an exposition of this theory, see also Sigstam [13].

[6] i. e., 1) a subset covers each of its elements 2) if $a \leq b$, then $\{b\}$ covers a, and 3) the 'meet' of two sets covering an element still covers the element.

2. A Critique of 'Pure Epistemology'

A criticism of a tradition of rational thinking that had surpassed the limits of possible experience was already presented by Kant. The famous antinomies (whether well-founded or not) are reminiscent of the dilemmas of set theory. We are not thinking here of Russel's paradox, but rather results of independence, e.g., that of the Continuum Hypothesis. By going from finite formulas to objects of epistemology[7] – which are not controlled by these formulas – it is not surprising that we find antithetical statements between which we cannot decide.

Kant wanted to limit our arguments about experience to possible experience. But this domain is confused, in the sense that finitude as such is 'transitive'. Husserl's first work (*Philosophie der Arithmetik*) was characterized by the idea of bounded experience, shown by the distinction between direct and 'figurative' collecting of elements into unities. This early attempt became considered 'psychologistic', and Husserl later provided other (more philosophically 'correct') pathways toward the same goal: Phenomenological reduction, ideation, and the relationality of the *Lifeworld*. But in our opinion, the phenomenological approach still lacks (Husserl threw the same criticism against Kant) a *rational basis*. It was not based on well-defined, intercommunicable formal experience – which necessarily is concretely bounded and relational. (Nevertheless, it shares with us the understanding that a basis for philosophy is to be found in experience itself.)

The radical standpoint of *formal experience* should be contrasted with that of epistemology, which considers *objects of knowledge*, not those of experience. Thus, one may treat infinite sets as objects of knowledge, even if we cannot experience them as such. This does not mean that mathematicians do not possess a special skill of 'seeing' or visualizing infinite collections. However, the concept of set is changed when we move from finite, elementwise articulated collections to infinite ones. In the latter case it is the defining formula of the set that precedes the elements – these are present only in the sense of an epistemological 'completion'. The problem at hand is not about these completions as such, but rather pertains to their nature.

In accepting the Power Set Axiom, we leave the domain within which every element ('individual') is separately and explicitly formally definable. Whatever our countable language may be, there will be elements of $\mathcal{P}(N)$ which cannot be generated or defined by any algorithm let alone a finite finite formula of the language. This is the *decisive step* to the purely 'epistemological' sets.

Clearly, standard mathematical practice requires that there be a *'power object'*. But is it necessary to take this power object to be a set? This is a crucial question for set theory. It hinges on the power set axiom. A successful introduction of set theory requires that mathematical objects such as $\mathbb{R}$, the real numbers, be represented as sets. Higher cardinalities are obtained via the power set axiom, yielding ever-higher cardinals.

However, even the axiomatic set theory only really provides countable mathematical objects. The Löwenheim-Skolem 'paradox' has been explained away by saying that an object is uncountable because it has no mapping onto ω. The result of Löwenheim and Skolem says: Every consistent (countable) theory has a countable model. *But what do we mean by an uncountable model?* How can we even prove that it is *genuinely* uncountable? (In other words, not merely in the sense that the model lacks an enumerative

[7] i.e., in this situation to objects (sets) which are stated to be there, but which have no *explicit* formal definition, such as the elements of the power set of an infinite set.

mapping.) One needs a foundation on which to establish such genuine uncountability in the first place. But how does one prove that this foundation itself is genuinely uncountable?

The only way in which set theory can provide uncountable sets is by the method of proof, i. e., by showing that for a particular set S, the cardinality of S is greater than that of $\mathbb{N}$. On the other hand, it is possible to construct *countable models* in which such a proof is valid. Indeed, a *critical* examination *shows* that there are *no other* models to begin with. In the beginning, we have to start with a countable language, and the very first uncountable set is obtained from a *statement* proving its uncountability. However, this kind of uncountability is *virtual*: We have no other way of establishing that the first-mentioned model is uncountable than the theorem itself. But in fact the theorem only states that the model does not have an enumerating mapping. Let us assume that the first uncountable set has been obtained by applying the Power Set construction to a countable set. This construction adds a countable collection of canonical elements of the power set plus the assurance that the new set is uncountable, i. e., there is no contradicting enumerating mapping.

For the 'naïve' mathematician, this standard resolution of the Löwenheim-Skolem problem by a deficiency is not sufficient. Naïve thinking considers all sets in their final (completed) sense. If a model of the real numbers, say, is countable, then for the naïve thinker, it is so in an absolute sense. In the axiomatic setting, the model is uncountable because it lacks an enumerating mapping. This non-existence 'confirms' Cantor's proof of the non-countability of the real numbers. But there is, of course, such a function, necessarily *outside* of the model. But for the naïve thinker, this function should be *as good as any* — there are no models but only the absolute set of real numbers. If you can somehow enumerate a set, inside the model or not, then the set is enumerable. As the conclusion, we obtain that axiomatic set theory, rigorously speaking, cannot provide a full account of *the set* of real numbers, it merely gives partial, countable 'views' which are not always mutually consistent. No model of the real numbers cannot be proved to be genuinely uncountable, and a merely virtually uncountable model is — for the naïve thinker — already countable. So the common mathematician has to reach beyond axiomatic set theory to the naïve one, where paradoxes such as Russell's are waiting.

The Löwenheim–Skolem dilemma makes us suspicious of infinite sets *in general*. Indeed, if the only *known* genuinely infinite sets are the countable ones, we should investigate the reason for this rather special situation. Countable sets can be 'presented' through their enumeration. What makes such an object different from a mere algorithm of this construction is the possibility of treating the object as a *totality*, as a unity, instead of considering it element by element, produced from the algorithm. (And the totality is distinct from the form which is the algorithm.) The convenience in epistemology is to consider the totality a set, amenable to the constructions used with finite sets (such as the power set), again producing other sets (of higher and higher 'cardinalities'). What is needed here — according to our radical standpoint — is not to discard these infinite unities as such. Rather, we must critically examine the *concrete* mathematical 'experience' (or rather, 'presentation') of such collections, and the relationship of the collecting unity to the elements collected.

Why do we consider set theory to have failed as a foundation for mathematics? Let us consider the following points.

1) From the beginning, it has been beset either by paradoxes or undecidable questions;
2) It adds an unnecessary purely epistemological or 'metaphysical' layer of transfinite sets to the formalism of mathematics;

3) With the Skolem-Löwenheim problem, as explained above, even the motivation of advanced set theory (as that of transfinite numbers) is in doubt. There is no way to prove that there are genuinely uncountable sets.
4) To even prove that there are countably infinite *sets*, one still needs an axiom. This is a purely epistemological assumption[8]. Also, it is the second 'leg' of transfinite set theory.
5) Set theory postulates objects (as simple as real numbers) of which one cannot say anything descriptive at all.

But does not *every* real number have a decimal representation? However, given an arbitrary (abstract) real number x, the decimal series $x_1, x_2, x_3, \ldots$ can only be given if there is an algorithm (rule) that generates the sequence. Hence, there are uncountably many (in fact, 'almost all') real numbers destined to remain unknown forever, unless our mathematical language somehow can be extended to encompass an uncountable variety of finite forms. It is mysterious how such an extension could be achieved. It is, however, even more intellectually troubling that this residual set of real numbers may not, after all, be genuinely uncountable, but nevertheless resists our attempts at describing the elements!

By the Skolem-Löwenheim result, our collection of real numbers may (assumed to) be based on a countable model. But what could such a *countable* region of inaccessibility be? It is easy to understand why in uncountable regions, some elements remain indistinguishable. On the other hand, for a countable set, we should be able to identify the elements, one after another, in an enumerative process. Nevertheless, these elements remain hidden. The reason, of course, is that we don't know where to look for them, we don't know where to start and how to continue. The countability of the model underlying this uncountable remainder is beyond the model itself. This conflicts with the naïve, absolute point of view which takes each set in a fundamental, 'final' way: If there is a method, perhaps in a larger model, to enumerate a set, then it is countable. The conlusion is we cannot speak of the set of real numbers as such, but only of *models* of real numbers, which should be disappointing to those who had expected set theory fulfil the role of a foundation for mathematics.

As a remedy to the critical remarks 1) – 5) above (and against the recommendation of [11]) we seek in recursive synthesis a foundation which provides mathematics a place in a unity of sciences rather than a position separated from the rest. But it will turn out that this foundation cannot be *mathematical*. Indeed, if mathematics is a study of general forms of recursive synthesis, then it itself has no general form, and hence no foundation within itself.

3. The Uniformization of Recursive Synthesis

The production of forms provides as such no ground for the logical movement from an argument to another; it merely adds to the multiplicity of increasingly special situations. Moreover, the finite structure of the logical space does not solve the *recursively synthetic problem of movement* beyond the limits of the bounded reason. This problem of *combination* cannot be solved by means of a multitude of natural forms; to become effective, the collecting process of synthesis must be grounded on a simple basis which reduces the *general field* of such problems.

The problem of overcoming boundedness, in whatever field, is to connect two points A, B that have been separately presented within the limits of our cognition. But this relating requires a *comparison* of A

[8] This axiom might be called metaphysical, or 'ontological' (Russell), but as its object is the mathematical realm of abstract knowledge, we call it 'epistemological'.

with B, it calls for a *ground of movement* from the former to the latter. The initial limitedness is shown in the mere *juxtaposition* of the two which is the simple synthesis placing A and B side by side without a ground. This juxtaposition is 'groundless' as such, shown in the meaning of the synthesis that opens up to this opposited juxtaposition and hence to the emptiness of its connection. The synthetic relation can be grounded only if it opens up to a *common foundation* of both parts, in other words only if we can find a *uniform grounding* of both A and B.

The idea of finding a common basis in geometrical arguments was known in the Greek tradition in its conception of *analysis and synthesis*. To connect A and B, one must bring them onto a common ground through their analysis to simple parts, shared or comparable, and then proceed by synthesis along the path thus paved, indeed by reversing the process of analysis. Uniformization begins when the fixed units of such divisions are isolated and named, and continues recursively when the results of basic operations between them become replaced by known entities of a scale. It is precisely in the introduction of the *symbolic algebra* that recursive synthesis shows itself: a group of expressions is replaced by a new 'symbol'. This is an object to be handled in the same way as the others; it is a member of a *homogeneous* field of objects. Such uniformities had been known in the Greek mathematics[9] as basic elements corresponding to various dimensions: line, 'square' (dynamis), 'cube' (kybos) and their later higher-dimensional extensions such as dynamodynamis etc. Each uniformity forms a *species* within which one computes by addition, because the various objects of any one such field are multiples of the same basic unit, in the sense that lengths, areas and volumes each admit of addition. This understanding became systematic in Viète's *logistice speciosa*, i. e., calculus of species, a general algebra of indeterminate entities to be contrasted with the *logistice numerosa* of common numerical computations. Indeed, in the *Analytic Art* Viète enounced his *law of homogeneity*, the idea of calculus with magnitudes within uniform domains (genera) for which there has been set "a series or ladder [*series seu scala*] ... of magnitudes ascending or descending by their own nature from genus to genus"[10]. At each level of the scale, the law of homogeneity permits addition and subtraction, while division and multiplication (as Viète shows in Chapter IV of the *Analytical Art*) operate *between* these recursive steps.

The *general foundation of calculus* is the uniformity of its objects with respect to each other. Thus, the result $5 + 2 = 3 + 4$ is possible with respect to a uniform domain of integers based on a simple unit. Recursive synthesis is made possible by the condition that every result of finite collecting is again an element of the domain, directly comparable and combinable with any other element. In particular, algorithmic computation based on a decimal system becomes possible by the equivalence of all collections of ten units and so on to higher and higher synthetic levels of collecting. The efficiency of the positional system lies in the uniformity of scaling: all the positions are mutually similar; the method of position moves the uniformity of the original unit (one) to that of the different levels (powers of ten).

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[9] The Greeks did not accept even rational numbers as such; instead of a 'real' fraction they had proportions of integers. Numbers other than integers and proportions were given by geometric figures, by concrete synthetic forms in which the various number constituents were joined. Each number had a definite form, obtained from simpler forms, and the questions of their 'totality' were meaningless in that context. They were natural, total forms ordered with respect to their recursive collecting of their parts ('methexis'). This hierarchy has a simple expression in the geometric scale of dimensions. The reduction of this recursive synthesis through symbolic algebra and the subsequent simplification by means of 'coordinates' constitutes a paradigmatic example of the process of uniformization. (As regards the *'geometric algebra'* of the Greeks, the reader is referred to B. L. van der Waerden: *Science Awakening*, P. Noordhoff, 1954.)

[10] *Opera Mathematica*, Georg Olms Verlag, 1970, p. 1. Translated in Klein [8] p. 322 (see also pp. 172–173).

Uniformization is essential for a mathematics of recursive synthesis. The early uses of calculus replaced the 'real objects' by uniform units which were mutually equivalent from the standpoint of calculation. The integers are uniform units of collecting, and they abstract from the non-essential features of the collections. At this level, they are effectively merely *representational*. In order to make collecting and computing effective, a uniform scale of recursive levels — powers of 10 — was introduced. By using this scale, the addition of any two integers could be reduced to repeated additions of two digits (one had to know the 'addition table' by heart). An essential element in uniformized recursive synthesis, and perhaps its simplest form, is the use of *products*. In fact, even the simplest collecting of elements under a fixed element can be characterized as a cone product of the collection and the collecting element. More important is the case of a product between two non-trivial factors. Products — or conversely division — are uniform reformulations of syntheses, and uniform scales of products (powers) make a uniformized recursive synthesis possible.

Finding a uniform foundation of basic elements ('units') is merely the first (but necessary) step in bringing a variety of given objects into framework within which comparison and calculus is possible. This was essentially understood by Descartes (discussed in more detail in the next section), and as he says, it was already exemplified in the method of analysis and synthesis in geometry. But in order to make the ensuing mathematics of synthesis effective, and not simply remaining content with haphazard combinations of units, the recursive synthesis should be uniformized, too. A mathematical field typically has a particular kind of synthetic relation to be uniformized. For example, for Viéte it was synthesis by multiplication.

If the greatest achievements in mediaeval mathematics were characterized by algebra — abstract and recursive use of multiplication and addition — the Renaissance introduced a method of multiplicative simplification which became the essential mathematical technique for hundreds of years. Differential calculus starts from *linearization*, i. e., replacing a curve or a function by a straight line or a linear function. One considers expressions of the form $f(x+h) = f(x)+ah+\epsilon(h)h$, where a is the 'derivative' of f at x and ϵh is the difference between the linear function $g : x \mapsto a(x+h)$ and f. Ideally speaking, f is replaced by a linear function in infinitesimally small neighbourhoods of x. We may call this the *first linearization* of f.

Recursively, we next target the difference, which already contains a multiplicative term. We 'linearize' the non-linear part of the difference, and obtain an expression of the form $f(x+h) = f(x)+ah+bh^2+\epsilon_2(h)h^2$, the *second linearization* of f. In general, we derive the rule for the Taylor development of f at x expressing the function as a series in the successive powers x^n of x, which now plays the role of a uniform scale.

Classical predicate calculus itself can be described in this manner! In the situation of a general relational field, still considering the classical case in which we have a finite predicate language and a model, synthesis amounts to expressing the relational neighbourhoods of the elements in the model. The distributive normal forms and constituents of Hintikka ([7]) disclose the relational and synthetic meaning hidden in the formulas of predicate logic, which traditionally treats relations from the standpoint of set theory. Constituents of degree zero are simply exhaustive combinations of atomic formulas describing which atomic properties hold for a given k-tuple of elements. For degree 1, we describe what kind of elements 'exist' (in our setting: are related to) for the given k-tuple, for example,

$$C_*^{(1)}(y,x) = B_j(x,y)\&\Pi_s \pm \exists z C_s^{(0)}(x,y,z),$$

is a constituent of degree 1 for the pair (x, y), where $B_j(x, y)$ is a combination of atomic formulas and their negations that hold for the pair. Each constituent is a conjunct ('product') of constituents of a lower level. The above formula leads to the general rule for defining constituents of any degree. The constituents C^k represent *uniformized units of description* in the sense that *any* formula $\varphi(x)$ of the underlying language is equivalent to a unique disjunction of such constituents $C^d(x)$, where d is the 'quantifier depth' of φ. What is more, for a given element a of the model, there is a unique chain (series) $\ldots \leftarrow C_i^d(x) \leftarrow C_j^{d+1}(x) \leftarrow C_k^{d+2}(x) \leftarrow \ldots$ of constituents which represents the maximal description of a by using formulas with one free variable.

4. The Idea of Simplification in Descartes and Boole

An answer to the problem of a complete uniformization of both planar and spatial forms is obtained by a simplification known as the *introduction of coordinates.* The projection of a geometric figure onto a line is a *valuation* which 'simplifies' the *geometric point* to a number and thus makes the points comparable with each other. The figures considered are not anymore restricted to the complexes of natural forms; the permitted figures are now complexes of the basic valuations (coordinate symbols) and 'constants'. The recursive hierarchy of forms – for which scales were used by Viète – has now been transformed into a homogeneous field of calculus in which individual operations are simply 'coordinatewise arithmetic operations'.

While coordinate representations had been used earlier, the uniformization of geometry by adding the simplifying coordinate valuations to symbolic algebra is connected with Descartes. However, the case of geometry is merely a particular example; the *Regulae*[11] shows a more general understanding of recursive synthesis and its uniformization as the basis of human knowledge in general. What is at stake here is the long-disputed deeper meaning of his *'mathesis universalis'* as the basis of the true method. With respect to this method, the sciences of Arithmetic and Geometry are "nothing but ripened fruits" (373) from the principles of the deeper mathesis that is nothing like the common mathematics, *'mathematica vulgaris'* (374).

For Descartes,the required true universal mathesis is not merely "a certain general science that explains all that can be investigated concerning order and measure" (378); what is required is the *systematic uniformization* of that recursivity which is needed for the overcoming of our boundedness. This was not made clear in the full generality even in Descartes. But he understood that in order to make the movement of thought "continuous and nowhere interrupted" (Rule VII) the parts must be arranged in series with simple relations the connectivity of which is immediate and clear. Uniformization consists in finding the units which provide the common ground of the things considered, the unit is "the basis and foundation of all the relations" (462). The method of simplification is included in Rule XVII: The problems must be transformed until they are reduced to finding "certain magnitudes" (459); indeed, the relations between things are to be *reduced to those between lines* "because I could find nothing more easily pictured to my imagination" (441)[12].

[11] *Regulae ad Directionem Ingenii*, in *Oeuvres*, Chapter X. We have used the translation by Marion: *Regles Utiles et Clares pour la Direction de l'Esprit en la Recherche de la Verite*, Martinus Nijhoff, 1977.

[12] But even the case of measurement which as such requires mediation is reducible to order (452) for which the relation of terms is a direct one (451), simple recursion in which a term points at the objects immediately closed in.

One needs a systematic method of *comparing* concepts, down to the 'last elements' without remaining uncertainty. The ultimate method of comparison is to reduce concepts in a way similar to Descartes' coordinization in geometry, replacing the real values by 'truth values'.

The solution to the problem of conceptual simplification by means of valuations was to be found by Boole. His work is nowadays associated with the notion of 'Boolean algebra', a system in which certain rules crystallized by Boole are valid. However, in order to understand its essential relation to the uniformization of synthesis, we must consider the original formulation of Boole's thoughts in terms of 'elective symbols' which correspond to so-called characteristic functions, valuations from the 'universe' of discourse to the simple 'algebra' $\{0, 1\}$.

The truth-functional method of Post and Wittgenstein reduces the propositional forms to finitely many distributions of these simple elements, and the 'content' of the propositions in those forms is left out of consideration. Thus, from the *point of view of propositional truth* the form is fully given as the synthesis of those distributions, as for example the form $p \to q$ is analyzed into the truth distribution $1, 0, 1, 1$ over the four cases $(0,0), (1,0), (0,1), (1,1)$. The synthesis which it provides for the propositional form, say $p \to q$, from the four binary valuations, is in the simplest view a synthesis through a mere union, discrete collection. It provides a method of testing the validity of the propositional form by going through all the cases, one by one. The truth-functional method is in full accordance with the effective mathesis of the Cartesian method, the universal technique of uniformizing recursive synthesis that analyzes with respect to a simple unit and puts together results layer by layer in uniform steps. Propositional truth-functional calculus carries with itself the ambiguity of both ends, of computation and of logic in the traditional sense, as if it were at the same time a *method*, a *'clavis universalis'* into the classical forms of logic. But what it effectively and concretely does is a manipulation of binary assignments; to each *binary input* there corresponds a *binary output*. No wonder that Boole was accused by some of his contemporaries of deducing correct results by unfounded methods! The pre-Boolean world was still unaccustomed to the thought that logical forms might be faithfully handled by transforming them (in terms of 'elective functions') into the simpler realm of zeros and ones, although the method is internally consistent. But these critics were right insofar that what really is handled here is the binary 'image' of the traditional logical forms and nothing beyond it; with this step the classical logic became effectively replaced by the new binary logic.

In general, Leibniz's logic is still a 'concept logic' based on the containment of concepts: The calculus of *'continentibus et contentis'* understands the statement A *est* B as the containment of A by B instead of the modern interpretation as 'predication' by B, $B(A)$. This is properly speaking neither extensional nor simply intensional but a combination of both. Indeed, not only is the 'concept' A in B, but according to the extensional interpretation every individual (i. e., complete concept) 'in A' belongs to B. But this means *all possible individuals* [13], not merely the 'real' ones. Thus, A refers to the conceptual ground 'consisting' of all possible extensions of A. By a theorem of Leibniz [14] 'A is B' is equivalent to the statement 'if X is in A, then X is in B', in other words the ground of A is a ground of B. Quantification was still directly related to its older sense as 'supposition'. The ground is not extensional in the sense of

[13] as pointed out by W. Lenzen [9], esp. pp. 168–170. See also his *Das System der Leibnizschen Logik*, de Gruyter, Berlin, 1990.

[14] in *Opuscules et fragments inédits de Leibniz*, edited by L. Couturat, Paris, 1903, (reprinted by Georg Olms, Hildesheim, 1961), p. 260.

a set or a class, but rather an 'extensive' variable, and for its conceptual quantification, the statement *some A is B* is then given as the identity $AY = B$, where Y is an indefinite concept compatible with A. Thus, even quantification remains within the field of conceptual and *formal composition*.

The entire Leibnizian project, with its emphasis on the conceptual understanding of logical forms, with its understanding of calculus as given by rules of transformation, and with its ideal of a *Characteristica Universalis* reflecting the true structure of thoughts and concepts, resists the recursive uniformization of Descartes. For this very reason, his *Analysis Situs* and more generally the ideal of *Characteristica Geometrica* could not advance to an 'effective' level, because by employing such primitive concepts as point and surface and their intrinsic relations, it could not have reached the facility that only a deeper analysis into simple elements can bring before us. It is due to this reason that Grassman's *'Ausdehnungslehre'* provided a workable geometric calculus because it presents a construction of its objects based in his 'exterior' product. But Grassman's calculus was not a geometric topic in the sense of Leibniz; it was not designed as a calculus of 'native' geometric forms and their congruences. Only at the end of the 19th century did such a calculus become effectively realized through the *division* of those original forms into polyhedra of 'simplices', combinatorially uniform elements of analysis, and through the replacement of their mutual relations by their indices of 'incidence': a number which is either 1, 0, or -1. In this way, the *geometric form* became 'valuated' into a matrix of indices of incidence along the same lines as in the method of 'truth tables' with respect to *propositional forms*. It is on this basis that a systematic and even an automated process can assign 'topological invariants' to the original form.

5. Uniformization in set theory

We have considered the uniformization of recursive synthesis in several particular cases. In counting, the natural numbers provided a general foundation for collecting finite sets. According to Klein [8], the Greek conception of number in its deepest sense is that of *ordering* a set of elements by keeping them together in a synthetic unity, shown in the figurate numbers of the Pythagoreans. In abstract counting, the order is inessential, what matters is the finite *set* as the basis of the cardinality. General set theory both continues counting into the transfinite and at the same time seeks to provide a uniform synthetic foundation for all mathematical objects.

The method of uniformization obtains its most universal and radical expression in the theory of sets that provides the ultimate grounding by final elements, mutually equivalent and indivisible points. The objects of this theory are sets or 'classes' composed of these atomic individuals, the synthesis of the composition being the simple collecting of the elements, as in the pure form of finite synthesis. But what is important here is that set theory does not stop with finite collections; it extends the principle of gathering a set over all multitudes of individuals, and postulates in this process the existence of at least one non-finite set, an object with 'infinitely many' distinguishable elements. In order to proceed at all beyond merely being a theory of finite sets – which would by no means reduce it to a triviality – it has to assume the existence of a set that essentially is the 'set' of natural numbers, as the basis of its induction and its generation of 'transfinite' ordinal and cardinal numbers.

The uniformity of the set-theoretical synthesis from the foundation of natural numbers is expressed in the 'cumulative hierarchy'. It represents the 'iterative concept' of sets. In the hierarchy, sets are constructed out of the 'empty' set by the repeated use of set-theoretical operations. This depends on the first and already existing ground of natural numbers, without which it could not carry its iteration beyond finite stages. (More precisely, one may use any infinite set as a foundation, but the existence of an infinite set implies that of natural numbers.) We may as well proceed by applying the power set operation to the foundation. With the cumulative hierarchy, the recursive synthesis of mathematical objects is then uniformized by the foundation and the power set operation iteratively applied to the foundation. Let us notice that the 'usual' mathematical objects are already obtained in the first few levels of the hierarchy. In the same way as the standard decimal notation provides a uniformized synthetic representation of natural numbers, and power series gives one for the analytic functions, the cumulative hierarchy provides a framework for *all* mathematical objects.

Uniformization is the *ultimate justification* of set theory. We have discussed above the role of uniformization as the 'technology' of recursive synthesis, and interpreted Descartes' *Regulae* in this light as the universal recipe for systematically solving any problem [15]. Therefore, set theory is not merely a useful approach to the foundations of mathematics, but even stipulated by the logic of technology, given our recursively synthetic conception of experience. Synthesis from a common foundation, following a uniform method, makes all objects thus represented mutually comparable. But as with another example of uniformization — the gene technology which provides such a foundation for biology — the question is *what other* objects ('monsters') might become produced alongside with those originally desired? The set-theoretical synthesis produces (via the power set operation) sets with no definition; indeed it stipulates the existence of objects which will forever remain 'in the dark side'. Depending on the model, even 'virtual sets' (Cohen reals, etc.) can be found.

Perhaps *ethical guidelines* should be applied here, too. They would delimit our powers of set-theoretical construction, as in the predicative analysis, eliminating the bogus syntheses that come with an unfettered freedom. On the other hand, even the question of foundation (natural numbers, infinite set, etc.) is problematic. To provide a complete foundation of units for all mathematical objects (or at the minimum for the objects of classical analysis), set theory has to go beyond all finite levels to a totality which simultaneously realizes all these levels at once.

6. The 'utopia' of the set-theoretical infinite

The existence of the foundation by the simultaneous being-there of natural numbers is absolutely necessary for the development of transfinite numbers. Indeed, although natural numbers can be generated from the empty set $\emptyset$ by an iterated formation of sets, so that the numbers $0, 1, 2, \ldots$ are given as the sets $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ and so on, there is no infinite generation without the prior existence of an *already* infinite set ω, the ground and supply of all the further development in set theory. Therefore, insofar as the otherwise merely finite theory is relatively free of philosophical problems, it is *this assumption of the first infinity* that requires our attention.

As a simple synthesis, ω is not given as a projective ground (foundation), because the latter depends on things beyond set theory and is already a *complex* synthesis involving the notions of fundamental neighbourhood and rule (Cf. the next section of this paper.)

[15] Our description may be compared with that in [1], which contrasts 'geometric thinking' with 'algebraic manipulation'. The promise of the algebraic machinery - especially since Descartes introduced his algebraic coordinates - is described as an "Faustian offer": "...when you pass over into algebraic calculation ... you stop thinking geometrically ... you stop thinking about the meaning." ([1], p. 7). One sells one's soul in return for obtaining the elements necessary (in our analysis) for comparing and manipulating all concepts in a uniform, algebraic fashion. And here, of course, the 'algebra of sets' is the ultimate reduction of the 'geometry' of recursive synthesis.

It is a *simple* synthesis of its elements, but the synthesis is not concretely (formally) given – it is not a place for this synthesis (not being localized at anything concrete). Therefore, it is 'placeless'; we call it *utopian* (topos, 'place').

In the name of scientific honesty, one should in the context of set theory do away with all interpretations of infinity, as was precisely postulated in Hilbert's formalism of signs. While Cantor's theory purported to ground all mathematical objects – and therewith the vast realm of transfinite numbers – once and for all by setting a ground of a self-subsistent infinite, the Hilbertian pure play of finite symbols would eliminate the need of the resulting inaccurate understanding of the infinite by leaving it *uninterpreted*. As the meaning of the infinite in mathematics was not yet "inexhaustibly (*restlos*) explained"[16], one should replace – following the achievement of Weierstraß in the case of mathematical analysis – the uses of the infinite with "finite processes that perform the same task" (162). The "final explication" of the "essence of the infinite" had become necessary for the "honour of human understanding" (163). However, the explication of the infinite by the "contentless' use of finite operational symbolism through which its role is reduced to "that of an idea" (190) does not do anything to relieve us from the need to assume the principle of infinity and in fact rather leaves it in the darkness. The result of this symbolism is again *as if* the problem of the infinite had been resolved – by looking away from it – but it has merely been relegated from mathematics qua mathematics to the *practicing subject*. It is the practical author and the reader of mathematical works who has to provide the understanding of that pure symbolism and of setting it in concrete operational movements, most importantly in the sense of inductive operations which implicitly prepostulate the infinite in the form of the 'and so on' etc. In fact, what Weierstraß obtained for differential and integral calculus was not reproduced by Hilbert, because the quintessential topo-logical content of the Weierstraßian 'reduction' is neglected by Hilbert. The so-called (ϵ, δ)-definition of continuity and limits produces this topo-logical content, while in Hilbert the infinite is reduced to the 'ideality' of the pure symbol, which lacks the dynamical character of the Weierstraßian version of analysis. Again, the ideal element is 'as if' it were infinite, although all that is concretely given is a finite symbol.

On the other hand, the assumption of an infinite set beyond concrete experience as a 'pure' synthesis might not lead to any *mathematical harm* inasmuch as even the wildest imagination must be accompanied by a sound mathematical practice. Nevertheless, it leads to questions that cannot be decided on the basis of concrete experience, in the same way as the assumption of a *divinity beyond our solid experience* leads to problems that must remain unresolved. The Renaissance scientist could add the existence of a God beyond human powers, and without falling into contradictions with everyday experience could assume various mutually inconsistent features of the postulated God. The set-theorist has likewise been left with questions that cannot be solved on the basis of set-theoretical evidence alone, the most notable of these problems being the well-known question of the 'cardinality' of the continuum.

By placing the primary infinite object beyond concrete experience, set theory realizes the ultimate ideal of uniformization by entering a *utopia*, a *place which is nowhere*; it is not localized as a projective ground. The set-theoretical infinite set is a *concrete beyondness* vis-à-vis every possible experience. In its original form in Cantor, this utopian thinking is not restricted to a mere empty *displacement* of the ground of the infinite, but extends to an ideal of 'transfinite numbers', a well-ordered linear hierarchy of everincreasing ordinal numbers which – with their rigorously defined arithmetical rules – extend the

[16] Hilbert [6], p. 161.

realm of finite ordinals. For Cantor the pure infinite cannot be expressed in mathematics. However, it was in the spectrum of the actually (and not merely potentially) infinite ordinal numbers that mathematics approached the 'total' infinite. The first transfinite number to follow natural numbers is the ordinal corresponding to ω, i. e., the totality of the natural numbers. The ordinals to follow this first transfinite one are $\omega + 1$, $\omega + 2$, ... and so on to ever larger transfinites. Cantor sought to demonstrate that *infinite numbers* as such are not an *inconsistent utopia* – as had been thought previously by many – but an *actual fact* by *representing them in concreto* in the form of the transfinite sequence of concrete ordinal numbers.

However, the very problem of constructing 'transfinite' numbers, i. e., numbers which are 'larger' than the ordinary natural numbers, presents in itself no special difficulty. Any object distinct from numbers, e. g. this table or that book can be declared 'transfinite' as soon as as we relate it in a suitable way to the natural numbers. But it would be needless to think of such a tangible object as being 'beyond' or 'behind' the whole sequence of numbers, in the same sense that the first transfinite ordinal is waiting there, in its utopia, to be met after running through the classical number sequence. Indeed, this concrete object would be *with us*, together with the first integers. And nevertheless it would be in some definite sense 'beyond' the whole collection of natural numbers by simply not being one of them. This table or this book is here; I do not have to run through the sequence of numbers in order to catch a glimpse of it. Therefore, my placing it after the sequence is an *imaginative emplacement* that has nothing to do with purely mathematical considerations: the actual relation of this concrete object with the numbers is merely a relation *alongside with* other relations and the image of 'beyondness' is my imaginative contribution, an ideal of 'coming after'[17].

The ordinal number $\omega+1$ is given *explicitly* and not merely implicitly as e. g. the number 2^{73} of which we do not immediately have a 'full' presentation. We do not need an infinite development of the explicit expression $\omega+1$ in the same way as 2^{73} requires a finite development (to express it as a decimal number); indeed, the expression '$\omega + 1$' is not only the most precise formula for this number; it is concretely given in its explicitness here and now. As in the case of this table or some other 'natural' object, $\omega + 1$ is *by the side* along with the positive integers and not behind them; it is a 'number' which can be represented as a pair $< 1, 2, 3, \ldots ; 1 >$, in which the difference between 'coming after' and 'lying beside' is one of imagination, an addendum that corresponds to the utopia of the infinitely far.

The representation $< 1, 2, 3, \ldots ; 1 >$ does not introduce a 'new' infinity beyond and above the first infinity ω; indeed, it merely shows the old infinity with a *new form* having two ends rather than one. In the same way, the representation of 2ω as $< 1, 2, 3, \ldots ; 1, 2, 3, \ldots >$ only involves the same original openness and a new *entirely finite form* given by the brackets $<;>$ and so on to more and more complex forms. In fact, each 'countable' ordinal may be represented in a form of a 'tree' with only finite 'chains',

[17] The original motivation for Cantor's theory seems to have arisen from infinite iterations of certain operations for sets of real numbers. Those examples are also sources of the original 'pictures' of ordinal numbers. In a simplified form, we may take a converging sequence such as $1, 1/2, 1/3, \ldots$ with the limit 0 to represent a natural 'model' for the ordinal number ω. After infinitely many eliminations of the first term of the sequence, i. e., after removing $1, 1/2, 1/3$ etc. one after another, we are still left with the limit term. In this sense, it comes 'after' all the diminishing positive terms of the sequence. But in a different sense it is present there in the very same sense as the other terms that are concretely given. However, *as such* this limit term is nothing but a real number; it becomes meaningful as the limit or the first term beyond the series only in the context of that series. In other words, this situation is present only when the total collection with the limit is considered. It is the 'open' series that is the focal point here; the limit 0 is merely one of its *finite determinations* as the end point of the region of openness.

obtained recursively as a finite form molding the first infinity already present in ω. In those forms, there is no more 'infinity' than in the first; what separates them is the complexity of their finite form. The interval $[0, 1]$ is another 'formed' infinity in which the topo-logical structure is given by the rule of its subdivisions. In the same way as the move from I^1 to the second power I^2 does not add a new infinity but merely *complicates* the division of the same original openness, so does the move from ω to $\omega + \omega$ just add a new finite dimension.

Constructivism preserves the talk of sets, but declares that construction is the criterion of existence, thereby cutting the utopian elements of Cantor's set theory down to the experience with concrete content. In Brouwer and Weyl the topo-logical structure of openness of the continuum is present in the notions of 'free becoming' and in the representation of real numbers as 'choice sequences' that are in the state of free prolongement in the same way as any decimal representation of a non-rational real number is only given up to a finite initial segment together with the rule of its further expansion. Gounding is given by 'rules', the collection of which is never complete but always under creation and incrementation.

But the application of the rules of the constructivist and the very understanding of their idea is left to the *human subject*. By failing to thematize the openness of the generation of its objects, it has pushed the infinite – in the way of Hilbert's symbolism – away from the mathematical thematic to the practicing mathematician. It has to presuppose the 'first' infinite of an infinite series as the very principle of its rules, and it sufficiently describes it by the rule of simple becoming from n to $n + 1$. But while this may be considered satisfactory for the natural numbers, passing by the topo-logical content shows itself in the poverty of its description of more geometric objects such as intervals and circles etc. By not taking the model of its generative principle from geometry (as Frege during his last stage) but rather from the discrete generation of sequences, the 'almost invisible' topo-logical structure of the natural number series easily became ignored in its *philosophy of the mere rule*.

There is a way to do justice to the classical mathematical intuition of objects such as lines and circles, without taking away anything of their content, preserving their concrete presence without advancing to an unreachable 'beyond' of the Cantorian set theory. While the traditional finitism or constructivism may rigorously exhibit only a partial realization of the continuum, and the classical set theory enters – instead of the concrete infinity of the continuum – the path of abstraction beyond experience, the examination of our concrete intuition of the continuum and other geometric objects discloses their topo-logical structure in which the infinite openness (divisibility) is given as a concrete content, at once, and not as a mere possibility of unlimited division. The interval $[0, 1]$ as the projective limit is *not a set of individual elements* (as in Cantor) nor lacks any real numbers (as in Brouwer), but it is the projective foundation of the relation of betweenness (or that of inclusion between subintervals). Instead of amending the serial philosophy with openness ('choice sequences')[18] – a difficult marriage between determinism and freedom – we replace successor rules with relations. The series are now objects derived from the relation. We may still say that there *are* 2^ω real numbers in the sense that they form a projective foundation similar to that of $(2^n \leftarrow 2^{n+1})$. However, at this conceptual level, we may *not* say that some real numbers exist because they have recursive developments and others do not.

[18] An individual series is still always connected with a rule which selects the consecutive successors for the given relation.

7. Some Main Notions for a New Foundation

In this section, we briefly examine notions involved in developing a 'new foundation' based on the above philosophical considerations. What follows will merely indicate the path to be taken in that work. A detailed development of basic mathematics on the foundations of recursive synthesis will be presented (attempted) in another article.

A. Extension Relations. Instead of sets, we consider *formal concepts*. In the basic case, these are given by a *form* f and a formal relation R that defines extensions to the form. Traditionally, extension has been separated from the form: the extension of F has been said to consist of individuals x such that $F(x)$ (i.e., x 'satisfies' F). However, in our situation there are *no* individuals distinct from forms. The 'extension' of a form consists of forms related to the former through the given extension relation.

For each relation R, and for each f, there is the relational neighbourhood $R[f]$ of g with respect to R. However, it is not a set. Inasmuch as R is formal, $R[f]$ is the formal *concept* of an object g such that fRg. For example, let S be the successor relation of the natural numbers. Then $S[1]$ is the concept of the successor of 1, which is realized by a unique number. On the other hand, let R be the inclusion relation between open rational intervals of the form $]r, q[$. Thus, $]r, q[R]s, t[$ if, and only if, $r < s < t < q$. The neighbourhood of $]r, q[$ with respect to R is the concept of an open subinterval of $]r, q[$. Again, 'interval' does not directly imply a set; here a rational interval is a primitive form composed of 5 symbols. It is important to notice that the forms discussed here are the mathematical objects constructed (or rather 'formulated') within a formal language.

For us, a formal language is a finite alphabet equipped with a productive relation (which distinguishes well-formed expressions within the strings generated from the alphabet). A simple *formal system* $\mathcal{F}$ is a formal language together with an internal extension relation between the forms. To consider a particularly simple example, the formal system of decimal series of real numbers in $[0, 1[$ consists of the alphabet $< 0, 1, 2, \ldots, 9 >$, the production relation R for which fRg iff f is of the form $0.\alpha_1 \cdots \alpha_n$, and g is of the $f\alpha$, where α is a symbol of the alphabet. The internal extension relation is the same as the production relation.

The extension relation gives a rule which, however, is not computable in general. In the case of integers, it is the effective rule $n \leftarrow n + 1$, as in the case of recursive series. On the other hand, the extension relation for real numbers does not yield an effective functional rule, but nevertheless is given by a well-defined and effectively decidable form: $]r_1, r_2[R]s_1, s_2[$ iff $r_1 < s_1 < s_2 < r_2$.

A note on Russell's paradox. In order to attempt deriving Russell's paradox in our setting, we could consider a binary relation R and form the concept F defined via $Fx \equiv \neg xRx$. Here F is is a unary relation (property) of forms. To get the paradox, we should have $Fg \equiv gRF$ for forms g; this implies $FRF \equiv \neg FRF$ as in the set-theoretical case. This, of course, can be done, witnessed by the set-theoretical case and its membership relation ($\in$). However, the paradox requires that R be applicable to F, while F (as a form) has been constructed using the symbol R. Our requirement that all forms (and hence relations) be constructed from the given finite alphabet helps to avoid this type of circularity.

Set theory forces a single relation where there should be distinct ones. It diagonalizes forth Russell's paradox by equating (in the above example) the *relation* R and the *property* of satisfying $\neg xRx$.

Circularity in general is *a* motivation behind the study of non-wellfounded sets, including cycles such as $\Omega = \{\Omega\}$[19]. However, already the formulation of the *Axiom of Antifoundation* (every graph has a decoration by sets) indicates the naturality of graphs and general relations vís-a-vís the membership relation. In a relational formal system, sets are not needed.

B. Chains. We often denote extension relations by arrows $\leftarrow$. Given such a relation $\leftarrow$, we may consider objects h such that there is g with $f \leftarrow g$ and $g \leftarrow h$. More generally, we may consider sequences $f_1, \ldots, f_n$ such that $f \leftarrow f_1, \ldots, f_{n-1} \leftarrow f_n$. Such objects f_n are called transitively related to f. The *transitive completion* of $\leftarrow$, denoted by $\leftarrow^*$, is again defined by giving its extension relation. Thus, we define

$$\leftarrow^n \quad \Leftarrow \quad \leftarrow^{n+1}$$

by the rule: if $f \leftarrow^n g$ and $g \leftarrow h$, then $f \leftarrow^{n+1} h$. The new relation is the path relation derived from $\leftarrow$.

Alternatively, we may define the concept of path. Let $<>$ denote the empty path. The concept of $\leftarrow$-path is defined by the relation R for which fRg iff f is of the form $< f, f' >$, where $f \leftarrow f'$. Thus, a $\leftarrow$-path (based at $<>$) is of the form

$$<<< \ldots < f_1, f_2 >, \ldots, f_n >,$$

which may be simplified to the form $< f_1, \ldots, f_n >$.

Relations of a formal system $\mathcal{F}$ can be ordered by setting $R' < R$ if $fR'g$ implies fRg. An R-chain is simply a relation $R' < R$. We may call a chain R' *projective* if $fR'g$,$fR'h$ implies $gR'k$ and $hR'k$ for some k. Next, we define a notion which derives its motivation from combinatorial topology. We call a chain R' *closed* if xRy implies yRz if there is such a z, and similarly for x: uRx is there is such a u. The concept of a *closed* R-chain leads to that of the *projective foundation* of $\mathcal{F}$, denoted here by $\hat{\mathcal{F}}$.

C. Projective Foundation. We agree with [12] [20] that mathematical objects are posited. Indeed, recursive synthesis proceeds by collecting multitudes under unifying elements. Given a list $f_1, \ldots, f_n$ of forms, we may add a new form g and a relation R such that gRf_i for each i. We call $c =< g, R >$ a *cone* of the given forms relative to R. The relational neighbourhood $R[g]$ is then the formal concept of the given list. Note that for a given R, f, the concept $< f, R >$ already is the cone of $R[f]$. (We also denote by $< g, R >$ the sub-chain of R *starting* at g.)

On the other hand, multitudes are also given by means of chains. Given a chain $c =< f, R >$, the cone $\hat{c}$ of c collects the chain through a relation $\hat{R}$ that relates $\hat{c}$ to each finite part $f, f_1, \ldots, f_n$ of the chain. However, let us note (following a typical mathematical procedure of 'completions') that it is natural to choose for $\hat{c}$ the chain itself. The chain is related to each of its finite (initial) parts through forgetting the rest of the chain (projection). We call $\hat{c}$ the projective limit or foundation of the chain.

In general, the chain relation R allows for multiple 'successors' of forms in the chain. Hence, we have a multitude of chains as distinct specifications of the given concept. The projective limit $\hat{c}$ enables us to speak about the 'totality' of such chains. While we can have distinct closed R-chains c_1,c_2, they are both attached to $\hat{c}$.

[19] See, in particular, Barwise and Moss [2].

[20] esp. p. 188. Here, however, mathematical objects are not primarily positions in patterns but rather are posited for the sake of recursive synthesis.

For example, in the case of the natural numbers $\mathcal{N} =< 0, S >$ (where S denotes the successor relation), the projective foundation $\hat{\mathcal{N}}$ 'consists' of a unique closed chain (the relation S itself). In the case of dyadic rational intervals $]i/2^k, (i+1)/2^k[$ contained in $[-1, 1]$ – denote this system by $\mathcal{Q}$) – the closed chains correspond to real numbers: For a dyadic rational interval $]r, s[$, the closed chains through the interval form a subconcept which corresponds to the interval $[r, s]$ in the real line $[-1, 1]$ which is the projective foundation $\hat{\mathcal{Q}}$.

The 'elements' of the concept $\hat{R}$ should be associated with minimal closed R-chains R' as principles or *rules of choice*: Such an element R' implies, for any x, exactly one y such that $xR'y$ provided there is such a y for R. Sometimes such principles can be found, e.g., in the case of 'constructive' real numbers such as $\sqrt{2}$ which give an explicit projective foundation. However, in general there is no element of this kind. In set-theoretical thinking, it is customary to assume the set of all such elements, or, the set of the associated series. For us, the projective foundation does not consist of 'points' (although we may distinguish specific, definable points). We may only indicate its topological structure by means of the fundamental neighbourhoods $R'[y]$ of extensions from y (more precisely: the concept of (R', y)-extensions.

Projective foundation is the formal notion of 'infinite recursive synthesis'. Given a chain $c =< f, R >$, the projective foundation $p = \hat{c}$ (the 'cone' of the chain[21]) has *covers* determined by the form f and the relation R which describes how covers are refined. First of all, the trivial cover corresponds to f itself, and a cover of f is composed of all 'parts' corresponding to f' such that fRf', and so on. From the viewpoint of classical set-theoretical intuition, such a part is the 'collection' of subchains starting from f', and minimal closed subchains may be called 'points'. The connection with classical topology is obtained from the topology of the 'space of points' defined by these covers, but here it is enough to consider the *concept of cover* derived directly from the given relation. For a point p, the *fundamental neighbourhoods* are precisely the parts corresponding to the forms of the chain p itself.

The above definition of cover by means of a refinement relation is directly associated with the set-theoretical notion of covers.

As an example, consider a relational system where the forms are finite decimal series and the relation connects a form $(x_1, \ldots, x_n)$ with any extending form $(x_1, \ldots, x_n, x_{n+1})$. The associated projective foundation then 'consists' of all infinite decimal series, but should not be thought of as a set. It is an object determined by its topology, for which a canonical open 'part' is a part consisting of infinite series extending a given finite one, say having k elements. A canonical cover consists of such open parts, defined for a given finite k. The essential point here is that while there are many covers, they are all covers of the *same* object.

Covers are to be treated as concepts in the same way as other sets (in the classical sense). However, it is possible to consider covers by means of *conditions* instead of extension relations. A natural example is given by the cover of the real line consisting of all intervals $]x, y[$ such that $|x^2 - y^2| < \epsilon$. Here one has an instance of the *splitting of concepts*: In the set theory, any object is a set. This uniformization is a main advantage of the set theory. However, when we leave the set theory, we may experience the splitting of formerly unified concepts.

[21] In category theory, projective limits may be considered examples of 'cones'. For recursive synthesis, a cone is a paradigm example of collecting a multitude under a unit.

D. Specification. Functions (or 'maps') are here defined between two concepts. A function $A \to B$ and more generally a relation between A, B is a formal concept R such that xRy for some pairs (x, y), where x (resp. y) is a *specification* of A (resp. B). Recall that every concept has its relation of extension (which may be vacuous). Any concept related to the former via this extension is called its specification. Note that specification as such does not make a difference between the element and part of a concept (in the classical sense). Also note that for any extension relation, the transitive completion of the relation gives the associated transitive specification, as in the case of natural numbers.

Let us consider the specification of real numbers. First, we define the concept of a real number as the concept of a decimal sequence. On the other hand, the concept of the real line, is the projective limit $\hat{\mathcal{R}}$ of the concept of a strictly decreasing sequence of rational intervals. (For all purposes considered here, it is sufficient to define the corresponding extension relation as follows: $]r', s'[$ is a strict successor (subinterval) of $]r, s[$ if $r < r' < s' < s$ and $s' - r' < 2^{-1}(s - r)$.)

We derive the general concept of a sequence from the trivial relation R_* which relates any two forms. We represent it as the triple $(*, R_*, \to)$, where $*$ denotes the trivial form ('any-form'). Then any given sequence (concept) is a specification of this universal sequence-concept: Any relation (resp. form) is a natural specification of the trivial relation (resp. form).

The classical 'elements' of the projective limit $\hat{\mathcal{R}}$ are the decreasing sequences of the above kind. Here, we have the concept of an $\mathcal{R}$-sequence, and the elements are its specifications. What is the specification relation in this case? Any particular sequence must be given by a *rule* that identifies, for each member of the sequence, the successor with respect to the extension ($\mathcal{R}$). Such a *selection* is a subrelation of $\mathcal{R}$, minimal in the sense that each element has *exactly* one successor.

E. Conclusion. Set theory was described above as a natural uniformization based on the set-theoretical element. Traditionally, set-theoretical mathematics builds its objects from a given foundation of basic elements. The recursive synthesis of relational fields can be considered in two opposite directions: One which 'closes' in the collected elements under a synthetic element and another in which the given element is 'opened' up to those elements. In the latter direction, the ground is not pre-established but rather topologically given as a projective foundation.

The work presented here may not be the only means for obtaining the goal set in the introduction. The program of obtaining and developing classical mathematics in a predicative framework, from a countable foundation, started by Weyl and continued by Feferman et al., using more and more flexible primitives, may ultimately meet our goals in a common end result: Finitely and rigorously determined, and at the same adequate, mathematics.

8. References

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