Continuity of the cone functor

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Abstract. This paper looks at the continuity of a class of functors that includes as special cases the *cone functor* Γ and the *suspension functor* Σ. The purpose of the paper is to highlight a sufficient topological property satisfied by paracompact Hausdorff spaces, which guarantees the continuity. Since paracompact Hausdorff spaces constitute a large class of topological spaces studied in mathematics, we regard this as a strong result.

The impetus for the present paper came from a certain confusion encountered in the book *General Topology and Homotopy Theory* by I. M. James [Jam84]. We give a counterexample showing that the cone functor is not continuous in the category of regular spaces, as stated in the book. Although the results in this paper concern functors, the emphasis of this paper is more on classical point set topology.

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1. Introduction

Doing research on continuous functors for his post graduate studies, the author encountered certain confusion in [Jam84]. The book states that the cone functor, or indeed every functor obtained as the push-out of the cotriad

$$X \times T \leftarrow X \times T_0 \rightarrow T_0$$

is continuous in the category of regular topological spaces, assuming the pair (T, T_0) consists of a compact regular space T and its closed subspace T_0. Even for the cone[1]

$$X \times I \leftarrow X \times 0 \rightarrow 0$$

this statement needs to be restricted, see Theorem 18, and Theorems 5 and 6.

In this paper we prove that if the functor has its domain in the category of paracompact Hausdorff spaces, the functor is indeed continuous, see Theorems 9 and 12. In the proof of Theorem 9 we use a strong point set topological result, see Theorem 7. Since paracompact Hausdorff spaces constitute a large class of topological spaces studied in mathematics, we regard this as a strong result. Notice that we also replace the assumption that T is compact and regular by assuming only that T is Hausdorff.

Let us quickly go through how this paper is structured. We begin with a short description in Section 2 of what we mean by a continuous functor. In fact, throughout the paper we consider two different definitions for a continuous functor.

In Section 3 we define the push-out functor Φ and discuss it in some detail. Following [Jam84], we define two topologies for ΦX, the fine topology and the coarse topology. We also point out what parts in the book need attention.

In Section 4 we inspect the sufficient assumptions that guarantee that the natural projection associated with the fine Φ functor is a *semiproper map.* The section presents a space constructed by Tuomas Korppi for showing that even for the fine cone there exist regular spaces for which the natural projection is not a semiproper map. The main tool used in the section is accumulation.

The four positive continuity results are all listed in Section 5. For the fine Φ we restrict the analysis to paracompact Hausdorff spaces. For the coarse topology, continuity holds more generally, as we will indicate. The main tool used in the section is mapping diagrams.

[1] To simplify the notation we will always denote the set $X \times \{0\}$ by $X \times 0$.

Finally, in Section 6 counterexamples to the continuity of the fine Φ functor are constructed using the characterizing property of completely regular spaces. We give counterexamples for both continuity notions, and they both make use of the space constructed by Korppi.

The author has been in contact with professor I. M. James, who had not received any comment on the problem before. This information from James gives us sufficient reason to assume that the problem has not been addressed in any paper before.

While working on this paper I have received funding from the Graduate School of Mathematical Analysis and Logic at the University of Helsinki. Finally, I wish to thank my fellow post graduate student Tuomas Korppi whose contribution to this paper goes far beyond the ideas used in Theorems 5 and 6.

2. Two notions that define a continuous functor

Elementary category theory describes mathematical theories in terms of a class of *objects* that belong to a category and sets of *morphisms* between two objects of the same category. What pure elementary category theory lacks is any structure for its class of objects or the morphism sets; everything is discrete.

When some topological structure is given either for the objects or the sets of morphisms, we may define two notions of a continuous functor. In some special cases these definitions coincide, but in general the definitions have little in common. If the reader is familiar with adjoints, this connection is probably evident, but we have chosen not to digress here. The interested reader should turn to [Dug66], Chap. XII, for example.

Unfortunately, the author is unaware of proper terminology to distinguish between these two continuity notions. Continuous functors appear also in abstract category theory as limit-preserving functors. The author has not worked with this subject and does not know whether there is any connection with either of the two notions discussed here.

2.1. Continuous endofunctor of topological spaces. We first give a definition in the form given in [Jam84], p.47. The definition is restricted to endofunctors of topological spaces and continuous maps.

An endofunctor F of the category of topological spaces and continuous maps is said to be *continuous* if for all spaces X, Y and B and each continuous map $f : B \times X \to Y$ the induced function

$$\begin{aligned} \hat{f} : \ B \times FX &\to FY \\ (b, z) &\mapsto Ff_b(z) \end{aligned}$$

is continuous. Let us quickly go through the definition. By fixing $b \in B$ we obtain a partial map $f_b : X \to Y$. Applying F to f_b we get a map $Ff_b : FX \to FY$. This map is evaluated at the point $z \in FX$, and finally a new function $\hat{f} : B \times FX \to FY$ is constructed. The question is whether or not this function is always continuous.

This definition is especially practical in defining new G-spaces, see [Jam84], Section 4, or homotopies, see [Jam84], Section 5.

2.2. Continuity as a function. Working with deformation retracts of function spaces, the author has adopted a different, in some sense more concrete approach to continuous functors. This approach treats a functor as a family of functions between morphism sets, and the morphism sets are considered as topological spaces. In [Ati67], p.6, Atiyah uses this definition for endofunctors of topological vector spaces.

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories such that each set of morphism is given some fixed topology. In this paper we are only interested in the case in which the categories are formed by topological spaces and continuous maps, and the morphism sets are given the compact-open topology. For simplicity, we denote these sets by $\mathbf{map}(X,Y)$. We later use the same notation for $\mathrm{C}(X,Y)$ with the compact-open topology.

A functor $F:\mathcal{C}\to\mathcal{C}'$ is s *continuous* if for all X,Y in $\mathcal{C}$ the function

$$F:\mathbf{map}(X,Y)\to\mathbf{map}(FX,FY)$$

is continuous. Observe that this definition does not require that objects of $\mathcal{C}$ or $\mathcal{C}'$ are topological spaces or even that the morphisms are functions. For this paper, continuity of the composition is irrelevant; our interest is on the precomposition and postcomposition maps.

3. The cone functor Γ and the Φ functor families

The cone functor is an important functor that appears often in algebraic topology and K-theory, for example. In elementary point set topology, the cone of a space X is defined as the quotient space $\Gamma X:(X\times I)/(X\times 0)$, and for a map $f:X\to Y$ as

$$\begin{aligned}\Gamma f:\ \Gamma X &\to \Gamma Y\\ [x,t] &\mapsto [f(x),t],\end{aligned}$$

where $[x,t]$ denotes the equivalence class of the point (x,t). A standard quotient space argument guarantees the continuity of Γf, and thus establishes Γ as a well-defined endofunctor of topological spaces.

In more abstract category theory the cone functor is defined as the push-out of the cotriad

$$X\times I\leftarrow X\times 0\to 0$$

and similarly for maps. The two definitions differ only for the empty set, see [Jam84], p.17 and p.46. We will use the point set topology approach.

3.1. The fine Φ functor. In [Jam84], the cone functor is treated as a part of a more general functor family Φ, defined as the push-out of the cotriad

$$X\times T\leftarrow X\times T_0\to T_0,$$

see *loc.cit.*, p.47. The definition makes sense for any topological pair (T,T_0) and any space X, but we shall only consider the case where X is a non-empty Hausdorff space and the pair (T,T_0) consists of a Hausdorff space T and its closed subspace T_0. In this case ΦX is also Hausdorff, see [Jam84], Proposition 2.75, for example. For maps the functor Φ is defined in the natural way.

As for the cone, we prefer to use a more constructive approach. For a non-empty Hausdorff space X we form the quotient space

$$X\times T/\sim,$$

where

$$(x,t)\sim(x',t')\Leftrightarrow\begin{cases}x=x',t=t',t\notin T_0\\ t=t',t\in T_0.\end{cases}$$

To see that the two definitions agree, one first has to check that as sets this quotient and the push-out coincide. The easiest way to see that the topologies coincide is to show that the natural projection $X\times T\to\Phi X$ is a quotient map. This calls for a brief commentary drawn from category theory and follows from the fact that $X\times T_0\to T_0$ is a quotient map.

For $X=\emptyset$ the categorical construction gives us $\Phi X=T_0$, which is not a quotient space of $\emptyset\times T$. This is why we, for simplicity, assume that X is non-empty.

From this point forward we take $X\times T/\sim$ as our definition of ΦX. We therefore have the natural projection $p:X\times T\to\Phi X$, which is a quotient map by definition. We warn the reader that the natural projection need be neither open nor closed. The elements of ΦX are denoted by $p(x,t)$ or by $[x,t]$, whichever feels more natural.

One very important fact that we wish to emphasize is the existence of a natural map $t : \Phi X \to T$ which is induced by $\mathrm{pr} : X \times T \to T$. A similar projection onto X is defined only on a subset as $x : t^{-1}(T \setminus T_0) \to X$.

Let $T_1 = T \setminus T_0$. It is instructive to observe that ΦX can be expressed as the union of two disjoint subspaces, one of which is an embedded $X \times T_1$ and the other an embedded T_0.

Let us give a few simple examples. The extreme cases $T_0 = \emptyset$ and $T_0 = T$ lead to $X \times T$ and T. The cone functor is obtained by setting $T = I$ and $T_0 = 0$. Replacing T_0 by $\{0, 1\}$ we obtain the *suspension*, another classical functor.

We sometimes call the functor defined above the *fine* Φ to distinguish it from the *coarse* Φ defined on p.48 of [Jam84]. We later give a short description of the coarse topology.

We now discuss how the continuity of Φ is addressed in [Jam84]. The book reads:

> *"In general this functor Φ is not continuous. However if the pair (T, T_0) satisfies certain conditions, to be considered later in this chapter, the continuity condition is satisfied. Fortunately these conditions are satisfied in the cases which concern us the most, namely when $T = I = [0, 1] \subset \mathbb{R}$ and either $T_0 = \{0\}$ or $T_0 = \dot{I} = \{0, 1\}$."*,

see *loc.cit.*, p.47.

Actually, no explicit proof is given that uses the continuity notion given in the book. The only statement that points in this direction is Proposition 2.97, p.64, which asserts that if the pair (T, T_0) consists of a compact regular space T and its closed subspace T_0, the function

$$\Phi : \mathbf{map}(X, Y) \to \mathbf{map}(\Phi X, \Phi Y)$$

is continuous for all regular X and for all Y when the function spaces have the compact-open topology.

Unfortunately, the proof works only for compact X. The key argument is based on the fact that the precomposition map p^* is an embedding, which in turn is deduced from the fact that $p : X \times T \to \Phi X$ is a compact surjection. Here a compact map is the same as a proper map in the sense of [Bou66], p.97. But in general p is compact only if X is compact.

In Theorem 18 we show that for the space X constructed by Korppi the function

$$\Gamma : \mathbf{map}(X, \mathbb{R}) \to \mathbf{map}(\Gamma X, \Gamma \mathbb{R})$$

is *nowhere continuous.*

We note that the same remarks apply for Proposition 3.69, which is a fiberwise generalization of 2.97. Luckily, it seems that no other theorem in the book relies on either of these.

3.2. The coarse Φ functor. The coarse ΦX has the same underlying set as the fine ΦX, but the topology is coarser. We take the parts $X \times T_1$ and T_0 that form ΦX and we attach these spaces together by the obvious maps to T. To be explicit,

- if $x \in X, t \in T_0$, sets of the form $p(X \times V)$, where V is a neighborhood of t, give a neighborhood base at $[x, t]$; and
- if $x \in X, t \notin T_0$, sets of the form $p(U \times V)$, where U is a neighborhood of x and V is a neighborhood of t, give a neighborhood base at $[x, t]$.

For simplicity, we continue to assume that X is a non-empty Hausdorff space and that the pair (T, T_0) consists of a Hausdorff space T and its closed subspace T_0. Then also in this coarse case ΦX is Hausdorff, see [Jam84], Proposition 2.75. Notice that p need not be a quotient map.

4. Semiproper natural projections

The key observation of this paper is that continuity of the fine cone functor Γ and the fine Φ functor is closely connected with the question whether or not the natural projections

$$\begin{array}{ccc} X \times I & & X \times T \\ {\scriptstyle p}\big\downarrow & \text{and} & {\scriptstyle p}\big\downarrow \\ \Gamma X & & \Phi X \end{array}$$

associated with the functors are semiproper maps.

A map $f : X \to Y$ is called *semiproper* if for each compact $B \subset Y$ there is a compact $A \subset X$ such that $f(A) = B \cap f(X)$. In this paper the definition is used only for surjective maps onto Hausdorff spaces, so that it is enough to find for each compact $B \subset Y$ a compact $A \subset X$ such that $B \subset f(A)$.

The main results in this section are:

- Theorems 5 and 6 that form the counterexample given by Korppi showing that even for the fine cone functor Γ there are spaces with quite strong properties but for which the natural projection $p : X \times I \to \Gamma X$ fails to be a semiproper map.
- Theorem 9 that shows that for a non-empty paracompact Hausdorff space X the natural projection $p : X \times T \to \Phi X$ is a semiproper map.

But before proceeding to those, we would like to recall some terminology and basic results form classical point set topology.

4.1. Accumulation. The proofs of the theorems given in this section make use of the fundamental topological notion called accumulation.

By an *accumulation point of a subset A in X* we mean a point $x \in X$ such that every neighborhood of x contains points from the set A other than x. The union of a set A and its accumulation points form the closure of A, and thus a closed set contains its accumulation points.

By an *accumulation point of a sequence (a_n) in X* we mean a point $x \in X$ such that every neighborhood of x contains points a_n for infinitely many indices. In T_1 spaces, that is, when points are closed, every neighborhood of an accumulation point of a set A meets A in an infinite set. Hence, in T_1 spaces, every accumulation point of the set $\{ a_n \mid n \in \mathbb{N} \}$ associated with the sequence (a_n) is an accumulation point of (a_n).

It is not true that a continuous function $f : X \to Y$ always maps accumulation points of a set A to accumulation points of the set $f(A)$, but this is true if A contains at most one point from each fibre $f^{-1}(\{f(x)\})$ of f. This feature is so important for us later that we wish to write it down as a separate lemma.

Lemma 1. *Let $f : X \to Y$ be a continuous map and let $A \subset X$ be such that for each $x \in X$ the fibre $f^{-1}(\{f(x)\})$ contains at most one point of A. Then f maps the accumulation points of A to accumulation points of $f(A)$.*

Recall that if two sequences (x_n) and (y_n) have accumulation points x and y, then (x, y) need not be an accumulation point of the sequence (x_n, y_n). But if one of the sequences converges to its accumulation point, then the result holds, as the next lemma states.

Lemma 2. *Let (x_n, y_n) be a sequence in the product space $X \times Y$. If x is an accumulation point of (x_n) and (y_n) converges to y, then (x, y) is an accumulation point of (x_n, y_n).*

Recall that a topological space X is said to be *countably compact* if every sequence in X has an accumulation point in X. This implies that every infinite subset of A has accumulation points, meaning that X is *weakly countably compact*. Since we are concerned here solely with Hausdorff spaces, it helps to know that these concepts coincide for T_1 spaces. Countably compact spaces are sometimes confused with *sequentially compact* spaces, in which every sequence has a convergent subsequence, but remember that a sequence may well have accumulation points without any subsequence converging to them. For metric spaces each of these three notions coincides with compactness, see [SS95], p.35. Even more interesting for us is that this happens also for paracompact Hausdorff spaces, see [SS95], p.23, and glance at Theorems 5 and 9.

4.2. The natural projection need not be a semiproper map. It was not in any way obvious whether the proof of Proposition 2.97 in [Jam84] had a gap that could be fixed or whether the statement actually needed more restrictions. At first the author hoped that every compact subset of the cone could be presented as the image of some compact subset of $X \times I$. Then Korppi constructed a counterexample to this conjecture. This paper presents the original example divided into small steps, some of which are well-known facts. The cone functor Γ is one of the most simple non-trivial members of the Φ functor family.

The next result follows easily from Lemma 2 and the fact that I is first countable.

Lemma 3. *Let X be any countably compact space and let I be the unit interval. Then the projection $\mathrm{pr} : X \times I \to I$ is a closed map.*

The next lemma appears frequently in the literature in one form or another. Observe that this actually shows that the fine and the coarse topology for the cone coincide if the projection $\mathrm{pr} : X \times I \to I$ is a closed map.

Lemma 4. *If the projection $\mathrm{pr} : X \times I \to I$ is a closed map, then every neighborhood U of the set $X \times 0$ contains a neighborhood of the form $X \times [0, t[$ for some $t > 0$.*

Theorem 5. *Let X be a non-compact, countably compact and separable Hausdorff space. Then the natural projection*

$$p : X \times I \to \Gamma X$$

is not a semiproper map.

Proof. Since X is separable, we may choose a sequence (x_n) such that the associated set $\{ x_n \mid n \in \mathbb{N} \}$ is dense in X. We also choose a sequence (t_n) in I that converges to 0 with all $t_n > 0$. We denote

$$B = \{ p(x_n, t_n) \mid n \in \mathbb{N} \} \cup \{X \times 0\}.$$

Let $\mathcal{U}$ be an open cover of B. Now $X \times 0$ belongs to some $U \in \mathcal{U}$, which, by Lemmas 3 and 4, contains all but finitely many of the points in B. Hence B is compact.

Now suppose there were a compact subset K of $X \times I$, such that $B = p(K)$. Then $\mathrm{pr}_X(K)$ would be a compact subset of X containing the dense set $\{ x_n \mid n \in \mathbb{N} \}$. As a closed subset of X, $\mathrm{pr}_X(K)$ would be the whole of X, which is non-compact. Hence such a set K cannot exist. □

We have to show that spaces as in the hypothesis of the previous theorem do exist. In order to construct the counterexamples it is important that the space is in addition completely regular.

Theorem 6. *There exists a non-compact, countably compact and separable space that is locally compact and Hausdorff, and therefore completely regular.*

Proof. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the natural numbers. We choose one point a from $\beta\mathbb{N}$ such that $a \notin \mathbb{N}$. The set $\{a\}$ is closed, but not open since $\mathbb{N}$ is dense in $\beta\mathbb{N}$. We then form the space $X = \beta\mathbb{N} \setminus \{a\}$. As an open, but not closed subset of a separable compact

Hausdorff space, X itself is separable, non-compact but yet locally compact and completely regular Hausdorff space.

We have to show that X is countably compact. Since X is Hausdorff, and in particular T_1, it is enough to show that every infinite subset of X has accumulation points, meaning that X is weakly countably compact. This follows by a clever use of [Wal74], p. 71. Namely, in $\beta\mathbb{N}$ every infinite closed subset is uncountable, and this is easily seen to hold for X, also. Hence every infinite subset of X contains a non-closed subset and thus has accumulation points in X. □

The author would like to remark that all the hypotheses of Theorem 5 are satisfied by the *Novak space*, see [SS95], Example 112. Hence, the Novak space is another completely regular space for which the natural projection fails to be a semiproper map. Steen and Seebach give no indication whether or not the Novak space is locally compact. With this regard the example constructed by Korppi is more useful because it answers in the negative many educated guesses concerning the sufficient conditions under which the natural projection is a semiproper map. Local compactness is also used in Theorem 16. In fact, these spaces are both subspaces of $\beta\mathbb{N}$.

4.3. For paracompact Hausdorff spaces the natural projection is a semiproper map. We have now seen that there exists a space with quite strong properties but for which the natural projection is not a semiproper map. We next give a very general condition which guarantees that the natural projection is a semiproper map.

Recall that a topological space X is said to be *paracompact* if every open cover $\mathcal{U}$ has an open locally finite refinement cover $\mathcal{V}$, that is, an open cover $\mathcal{V}$ such that each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$ and every point $x \in X$ has a neighborhood that meets only finitely many $V \in \mathcal{V}$.

The following theorem contains the key observation needed in the proof of Theorem 9. There is no use in trying to improve this theorem by assuming only that A is not contained in any weakly countably compact subset, since a weakly countably compact subset of a paracompact Hausdorff space is compact. Moreover, Theorem 7 fails for the *open ordinal space* $[0,\Omega[$, where Ω is the first uncountable ordinal, see [SS95], Example 42, p.68. This space is a countably paracompact Hausdorff T_5 space.

Theorem 7. *Let X be a paracompact Hausdorff space and A a subset that is not contained in any compact set. Then A contains an infinite set L without accumulation points in X.*

Proof. We first reduce the claim to the situation in which A is a dense subset of a non-compact paracompact Hausdorff space X. Namely, the accumulation points of a subset of A are surely contained in the closure of A, which cannot be compact since it contains A. But as a closed subspace of a paracompact Hausdorff space, $\overline{A}$ is also a paracompact Hausdorff space.

Since X is non-compact, we may find an open cover $\mathcal{U}$ of X that has no finite subcover. Using paracompactness we find an open locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. Since $\mathcal{U}$ has no finite subcover, $\mathcal{V}$ must be infinite.

We next consider the sets $V \cap A$, where $V \in \mathcal{V}$. Since A is dense in X, the sets $V \cap A$ are all non-empty. Using the axiom of choice we construct a set L by choosing one point from each of the sets $V \cap A$. A priori nothing guarantees us that L is infinite. But suppose L were finite. Then at least one point $x \in L$ would belong to infinitely many sets $V \in \mathcal{V}$, which is impossible since $\mathcal{V}$ is locally finite. Therefore L is infinite.

We proceed to show that L has no accumulation points in X. Suppose a point $x \in X$ were an accumulation point of L. Since X is Hausdorff, and in particular a T_1 space, every neighborhood of x meets L in an infinite set. Consequently, every neighborhood of x meets infinitely many of the sets $V \in \mathcal{V}$, which again is impossible since $\mathcal{V}$ is locally finite. Therefore L has no accumulation points in X. □

For metric spaces the corresponding result is more easily proved in the form of Theorem 8, which we give only as a curiosity. An important theorem of Stone states that every metric space is paracompact. We leave it as a simple exercise for the reader to verify that Theorem 8 is the metric equivalent of Theorem 7 and to come up with an alternative proof without referring to the result of Stone.

Theorem 8. *Let X be a metric space and A a subset that is not contained in any compact subset. Then there is a sequence (a_n) in A that has no accumulation points in X.*

We are now ready to present the main positive result of this section. In particular, the result applies for metrizable spaces and CW complexes.

Theorem 9. *Let X be a non-empty paracompact Hausdorff space and (T, T_0) a pair consisting of a Hausdorff space T and its closed subspace T_0. Then the natural projection*

$$p : X \times T \to \Phi X$$

is a semiproper map.

Proof. Let $K \subset \Phi X$ be compact. We denote the compact set $t(K) \subset T$ by T', and let $T'_0 = T_0 \cap T'$. If $T'_0 = T'$, then the claim follows easily by choosing an element x of X and forming the set $\{x\} \times T'_0$. If on the other hand the set $K' = p^{-1}(K) \setminus (X \times T'_0)$ is non-empty, then it suffices to show that $\mathrm{pr}_X(K')$ is contained in some compact subset A of X, since then $K \subset p(A \times T')$, from which the claim follows since ΦX is Hausdorff.

Suppose, on the contrary, that $\mathrm{pr}_X(K')$ is not contained in any compact subset of X. Theorem 7 shows that $\mathrm{pr}(K')$ must contain an infinite set L that has no accumulation points in X. Using the axiom of choice, we choose for each point $l \in L$ a point $(l, t_l) \in K'$, and we denote this set by F. Lemma 1 shows us that F has no accumulation points in $X \times T$, for otherwise $\mathrm{pr}_X(F) = L$ would have accumulation points in X. This has two important implications. First, F is closed in $X \times T$. Second, F is non-compact, since otherwise this infinite set would have accumulation points.

But since F is contained in K', which contains no points from the set $X \times T_0$, F is homeomorphic to $p(F)$. Moreover, since $F = p^{-1}(p(F))$, $p(F)$ is a closed subset of K, and hence compact, contradicting the fact that F is non-compact. Therefore, $\mathrm{pr}_X(K')$ must be contained in some compact subset A of X. □

5. Continuity results for the fine and coarse Φ

In this section we give continuity results for both the fine and the coarse Φ. We will consider both definitions of continuity discussed in Section 2.

To use the latter continuity notion, we need to define a topology for the morphism sets. Without further ado, we decide to use the compact-open topology. The set of continuous maps from X to Y with the compact-open topology is denoted by $\mathbf{map}(X, Y)$. This is the topology generated by the elementary neighborhoods

$$\mathrm{W}(A, B) = \{\ f : X \to Y \mid f(A) \subset B\ \},$$

where A is compact and B is open.

Lemma 10. *If $f : X \to Y$ is a surjective semiproper map and Z is any topological space, then the precomposition map*

$$f^* : \mathbf{map}(Y, Z) \to \mathbf{map}(X, Z)$$

is an embedding.

Proof. The composition of two continuous maps gives a continuous map, so f^* is well-defined. Since f is surjective, f^* is injective.

The topological properties of the function follow from the set theoretical formula

$$f^{*-1}(\mathrm{W}(A,C)) = \mathrm{W}(f(A),C).$$

We see that if $\mathrm{W}(A,C)$ is an elementary neighborhood in $\mathbf{map}(X,Z)$, then its inverse image is also an elementary neighborhood and hence f^* is continuous.

To see that f^* is an embedding we need only to express every elementary neighborhood in $\mathbf{map}(Y,Z)$ in the form $\mathrm{W}(f(A),C)$ for some compact $A \subset X$ and conclude that every such set can be obtained as the inverse image of some open set in $\mathbf{map}(X,Z)$. $\square$

5.1. The fine Φ. We first present the revised version of Proposition 2.97 in [Jam84]. Since the proof is relatively short we have chosen to include it here entirely rather than just point out the parts which need attention. Please note that we replace the assumption that T is compact and regular by the assumption that T is Hausdorff. But before this, we write down a continuity result concerning the product functor $TX = X \times T$, $Tf = f \times \mathrm{id}$.

Lemma 11. *Let T be a Hausdorff space. Then the function*

$$T : \mathbf{map}(X,Y) \to \mathbf{map}(X \times T, Y \times T)$$

given by the product functor T is continuous for each Hausdorff space X and for each space Y.

The proof of Theorem 12 repeats the elegant main argument used in the proof presented by James, but we have put more emphasis on checking the arguments preceding that point. Indeed, this argument used by James has quite a nice generalization as to how certain properties of natural transformations between functors relate to continuity. The author is currently working on a more detailed study related to these questions.

Theorem 12. *Let (T, T_0) be a pair consisting of a Hausdorff space T and its closed subspace T_0. Let X be such a Hausdorff space that the natural projection $p : X \times T \to \Phi X$ is a semiproper map. Then the function*

$$\Phi : \mathbf{map}(X,Y) \to \mathbf{map}(\Phi X, \Phi Y)$$

given by the fine Φ functor is continuous for each space Y.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{map}(X,Y) \arrow[r, "T"] \arrow[d, "\Phi"'] & \mathbf{map}(X \times T, Y \times T) \arrow[d, "p_*"] \\
\mathbf{map}(\Phi X, \Phi Y) \arrow[r, "p^*"] & \mathbf{map}(X \times T, \Phi Y)
\end{tikzcd}$$

The postcomposition map p_* is easily seen to be continuous by an argument similar to the one used in Lemma 10. The function T is continuous by Lemma 11. Now since p^* is an embedding by 10, and since $p_* \circ T$ is continuous, it follows that Φ is continuous. $\square$

The question whether the fine Φ can be shown to be continuous in the sense of [Jam84] seems to be far more involved and the book gives no indication how this could be done. Observe that we should show that the function $\hat{f}$ in every commutative diagram

$$\begin{tikzcd}
B \times X \times T \arrow[r, "f \times \mathrm{id}_T"] \arrow[d, "\mathrm{id}_B \times p"'] & Y \times T \arrow[d, "p"] \\
B \times \Phi X \arrow[r, "\hat{f}"] & \Phi Y
\end{tikzcd}$$

is continuous. The first idea that comes to mind is to try to show that $\mathrm{id}_B \times p$ is a quotient map. We show this only in the following special case. Recall that a Hausdorff space is said to be *compactly generated* if only closed sets can meet every compact set in a closed subset. For more information on compactly generated spaces, please turn to [Ste67].

Theorem 13. *Let (T, T_0) be a pair consisting of a Hausdorff space T and its closed subspace T_0. Let X be such that the natural projection $p : X \times T \to \Phi X$ is a semiproper map. Suppose further that $B \times \Phi X$ is a compactly generated Hausdorff space. Then the map*

$$\mathrm{id}_B \times p : B \times X \times T \to B \times \Phi X$$

is a quotient map.

Proof. Since $B \times \Phi X$ is Hausdorff, the map $\mathrm{id}_B \times p$ is a surjective semiproper map. For brevity, we denote $f = \mathrm{id}_B \times p$. Let $F \subset B \times \Phi X$ be such that $f^{-1}(F)$ is closed in $B \times X \times T$. We need to show that F is closed in $B \times \Phi X$.

Let $K \subset B \times \Phi X$ be compact. Then there is a compact $A \subset B \times X \times T$ such that $f(A) = K$. Now $f^{-1}(F) \cap A$ is a compact subset and hence $f(f^{-1}(F) \cap A)$ is closed in $B \times \Phi X$. But since f is surjective, this set can be written as $F \cap K$. Therefore F is closed in $B \times \Phi X$, since it meets every compact set in a closed subset. □

5.2. The coarse Φ. For the sake of completeness we write down the continuity results for the coarse Φ. The easiest way to work out continuity in the sense of [Jam84] is using the neighborhood basis given for the coarse topology earlier.

Theorem 14. *Let (T, T_0) be a pair consisting of a Hausdorff space T and its closed subspace T_0. Then for all spaces X, Y and B and each continuous map $f : B \times X \to Y$ the function*

$$\begin{aligned} \hat{f} : \; B \times \Phi X &\to \Phi Y \\ (b, z) &\mapsto \Phi f_b(z) \end{aligned}$$

induced by the coarse Φ is continuous.

On the other hand, continuity in the sense of the next theorem requires some consideration. The proof needs to be done using the elementary neighborhoods, and one may start by reducing this to Lemma 11.

Theorem 15. *Let (T, T_0) be a pair consisting of a Hausdorff space T and its closed subspace T_0. The function*

$$\Phi : \mathbf{map}(X, Y) \to \mathbf{map}(\Phi X, \Phi Y)$$

given by the coarse Φ functor is continuous for each Hausdorff space X and each space Y.

6. Counterexamples to the continuity of the fine Γ functor

In this section we present counterexamples to the continuity of the fine cone functor. We first show in Theorem 16 that the fine cone is not a continuous functor in the sense of [Jam84]. In Theorem 18 we give the main counterexample in this paper.

The main tool we use in this section is the Urysohn function for a closed set A and a point b outside it. Recall that a topological space X is said to be *completely regular* if for each closed subset A of X and each point b outside A there exists a function $f : X \to I$ such that $f(x) = 0$ for all $x \in A$ and $f(b) = 1$. Some sources include in the definition the assumption that X is Hausdorff.

Let us make a short remark about the compact-open topology. If $f : X \to Y$ is a constant map, then f has a particularly simple neighborhood base, namely all the elementary neighborhoods of f. This makes the next theorem easy to prove.

Theorem 16. *Let X be a locally compact, completely regular, non-compact but yet countably compact Hausdorff space and let Y be the open interval $]-1,1[$. Then the evaluation function*

$$e : \mathbf{map}(X,Y) \times X \to Y$$

is continuous, but the induced function

$$\hat{e} : \mathbf{map}(X,Y) \times \Gamma X \to \Gamma Y$$

is not continuous.

Proof. A fundamental and well-known theorem for the compact-open topology states that the evaluation map is continuous when X is locally compact, see [Jam84], Corollary 2.99. However, as we will now see, the induced function $\hat{e}$ is not continuous at $(f, X \times 0)$ if f is the constant map $f(x) = 0$ for all $x \in X$ and $X \times 0$ is the idealized point of the cone ΓX.

Let $V' \subset Y \times I$ be the open set

$$V' = \{ \ (y,t) \mid |y| < 1 - t/2 \ \}.$$

The set V' is saturated and contains the set $Y \times 0$, and hence $p(V') = V$ is an open neighborhood of the point $\hat{e}(f, X \times 0) = Y \times 0$, and therefore some neighborhood $(f, X \times 0)$ should be mapped into V.

Let U be a neighborhood of the point $(f, X \times 0)$. According to Lemmas 3 and 4 and the remark above, U contains a neighborhood of the form $W \times p(X \times [0,t[)$ for some $t > 0$, where W is an elementary neighborhood of f.

Basic geometry, or a simple calculation shows that the point $[y, t/2] \in \Gamma Y$ does not belong to the set V if $y = 1 - t/4$, for example. The space X being non-compact and completely regular, it is a simple task to show that W contains a map g such that for at least one $x \in X$, $g(x) = y$. For such g and x we conclude that

$$\hat{e}(g, [x, t/2]) = [y, t/2] \notin V.$$

Hence $\hat{e}(U) \not\subset V$, and therefore $\hat{e}$ is not continuous. □

To cope with the increasing complexity we need a better understanding of the compact-open topology. The next lemma captures an important feature of the open sets in $\mathbf{map}(X, \mathbb{R})$, when X is completely regular. The main tool used in the proof is the Urysohn function. Observe that the technique used is deformation rather than extension, although extension would be possible, too, since the set $A \cup \{x_0\}$ is compact.

Lemma 17. *Let X be a completely regular Hausdorff space, and $f : X \to \mathbb{R}$ any continuous map. For each neighborhood V of f there is a compact set $A \subset X$ such that if $x_0 \in X \setminus A$ and $y_0 \in \mathbb{R}$, there is a map $g \in V$ such that $g(x_0) = y_0$.*

Proof. We may assume that X is non-compact. Let V be a neighborhood of f. We may assume that V is the intersection of finitely many sets of the form $\mathrm{W}(A_i, U_i)$, where A_i is compact and U_i is open.

Let A denote the union of the sets A_i. The set A is of course closed since X is Hausdorff. Since A is compact and X is not, we may choose a point $x_0 \in X$ such that $x_0 \notin A$. Let $y_0 \in \mathbb{R}$. Since X is completely regular we may pick a map $h : X \to I$ such that $h(x) = 0$ for all $x \in A$ and $h(x_0) = 1$. We form the map

$$\begin{array}{rccc} g : & X & \to & Y \\ & x & \mapsto & (y_0 - f(x))h(x) + f(x). \end{array}$$

This is continuous and $g|A = f|A$ and $g(x_0) = y_0$. The fact that $g|A = f|A$ is enough to guarantee that $g \in V$. □

We conclude this paper by showing that Propositions 2.97 and 3.69 in [Jam84] need some additional restrictions, because the cone is one of the simplest non-trivial examples of the Φ functor. Note the way we make use of the unbounded metric structure of $\mathbb{R}$. According to Theorems 5 and 6, there exists a space X as in the next theorem.

Theorem 18. *Let X be a completely regular Hausdorff space such that the natural projection*

$$p : X \times I \to \Gamma X$$

is not a semiproper map. Then the function

$$\Gamma : \mathbf{map}(X, \mathbb{R}) \to \mathbf{map}(\Gamma X, \Gamma\mathbb{R})$$

is nowhere continuous.

Proof. Let $f : X \to \mathbb{R}$ be any map. To show that Γ is not continuous at f we must find a neighborhood W of Γf such that Γ maps no neighborhood of f into W.

Let $B \subset \Gamma X$ be such a compact set that cannot be expressed as the image of a compact subset of $X \times I$. Since B is compact, also $\Gamma f(B) \subset \Gamma\mathbb{R}$ is compact. We next construct a neighborhood V of $\Gamma f(B)$ such that for each $t_0 > 0$ there is a point $y_0 \in \mathbb{R}$ such that $[y_0, t_0] \notin V$.

Since $\mathbb{R}$ is metric, it follows from Theorem 9 that there exists a compact $K \subset \mathbb{R}$ such that $\Gamma f(B) \subset p(K \times I)$. The distance from the compact set K defines an unbounded map $f(y) = d(y, K)$. Define a subset $V' \subset \mathbb{R} \times I$ as

$$V' = \{\ (y, t) \in \mathbb{R} \times I \mid t < 2/(1 + f(y))\ \}.$$

Then V' is an open subset of $\mathbb{R} \times I$ and $K \times I \cup \mathbb{R} \times 0 \subset V'$, and hence V' is saturated. Therefore the set $V = p_{\mathbb{R}}(V')$ is a neighborhood of $\Gamma f(B)$ in $\Gamma\mathbb{R}$. If $t_0 > 0$, we may choose a $y_0 \in \mathbb{R}$ such that

$$2/(1 + f(y_0)) < t_0.$$

It follows that (y_0, t_0) cannot belong to the set V' and hence $[y_0, t_0]$ cannot belong to V.

Let $W = \mathrm{W}(B, V)$. We will show that no neighborhood of f is mapped into $\mathrm{W}(B, V)$. Let U be a neighborhood of f in $\mathbf{map}(X, \mathbb{R})$. Let A be a compact set as in Lemma 17. Then the set $A \times I$ is compact, and hence B contains at least one point b such that $b = [x_0, t_0]$, where $t_0 > 0$ and $x_0 \notin A$. We may choose a map g such that $g \in U$ and $g(x_0) = y_0$. Now

$$\Gamma g(b) = \Gamma g([x_0, t_0]) = [g(x_0), t_0] = [y_0, t_0] \notin V.$$

Therefore $\Gamma g \notin W = \mathrm{W}(B, V)$. Hence Γ is not continuous at f. □

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