TARTAR CONJECTURE AND BELTRAMI OPERATORS

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ABSTRACT. We study the set $\mathcal{H}^p(E)$ of $W^{1,p}$ homogeneous Gradient Young measures supported on a given closed set $E \subset \mathbf{M^{2\times 2}}$ in relation with the K-quasiconformality of E. We prove that if the set E has no rank-one connections and it is K-quasiconformal at infinity $\mathcal{H}^p(E)$ is trivial for every $p > \frac{2K}{K+1}$. If the set is K-quasiconformal at every point we also get results for the quasiconvexification of the distance function. The results are shown to be sharp by constructing sets E K- quasiconformal at every point and $W^{1,q}$ gradient Young measures supported in E for every $q < \frac{2K}{K+1}$. The main tool is the invertibility of Beltrami Operators.

1. INTRODUCTION

The study of the oscillation of sequences of gradients of Sobolev functions is a central topic in the Calculus of Variations and Non-Linear Partial Differential Equations. In this note we examine which subsets of the space of 2 by 2 matrices $\mathbf{M^{2\times 2}}$ "forbid" the oscillation of sequences of gradients approaching them. In precise words; Let E be a subset of $\mathbf{M^{2\times 2}}$ and $1 \leq p \leq \infty$. Is the set stable in the sense that every sequence $\{f_j\}$ such that

$$\operatorname{dist}_E(Df_j) \to 0 \text{ in } L^p \tag{1.1}$$

satisfies that $\{Df_j\}$ is compact in L^p?. The natural tool to study this problem are Young measures, which have proved to be an efficient devise when trying to capture properties of oscillating sequences. In particular we will need the following type of Young measures,

Definition 1.1. Let $1 \leq p \leq \infty$. A probability measure $\nu \in \mathcal{M}(\mathbf{M^{2\times 2}})$ is a $W^{1,p}$ Homogeneous Gradient Young measure if there exists a sequence $\{f_j\}$, weakly convergent in $W^{1,p}(\Omega, \mathbb{R}^2)$, such that for every $\varphi \in C_0^\infty(\mathbf{M^{2\times 2}})$

$$\varphi(Df_j) \overset{\star}{\rightharpoonup} \int_{\mathbf{M^{2\times 2}}} \varphi(\lambda) d\nu(\lambda) \text{ in } L^\infty(\Omega).$$

The set of $W^{1,p}$ Homogeneous Gradient Young measures supported in a closed set E is denoted by $\mathcal{H}^p(E)$.

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[0] *Mathematics Subject Classification:* 30C62, 49J45, 35J45.

[0] D.F, is supported by the Academy of Finland (Project # 53292)

The sequence $\{Df_j\}$ is called the *generating* sequence. A standard covering argument shows that the above definition does not depend on the domain Ω. It follows form ([P, Proposition 6.12]) that if the sequence $\{|Df_j|^p\}$ is known to be weakly convergent in $L^1(\Omega)$ we are concerned with the following question,

Question 1.2. Which closed sets $E \in \mathbf{M}^{2\times 2}$ satisfy that

$$\mathcal{H}^p(E) = \{\delta_A : A \in E\}? \tag{1.2}$$

When (1.2) holds we say that $\mathcal{H}^p(E)$ is trivial. Gradient Young measures are closely related to the so-called quasiconvex hulls. If E is compact E^{qc}, the quasiconvex hull of E is a crucial notion in the study of martensitic phase transitions [BJ],[Mü]. For unbounded sets the quasiconvex hull has turned out to be an important object to be understood in a number of different problems; Singular solutions of elliptic systems [S1], stability of the conformal set [Y],[Y1], G-closure problems and control [F], [P1], linear growth minimizing problems [FoM], etc. For unbounded sets, the most used definition of quasiconvex hull is the following:

Definition 1.3. Let $E \in \mathbf{M}^{2\times 2}$ be a closed set and $1 \leq p < \infty$. Then $E^{p,qc}$, the p-quasiconvex hull of E, is defined by

$$E^{p,qc} = \{A \in \mathbf{M}^{2\times 2} : Q\mathrm{dist}_E^p(A) = 0\}.$$

The above definition makes sense after recalling that for a continuous function f we have that

Definition 1.4. Let f a be continuous function. Then Qf,the quasiconvexification of f, is defined by

$$Qf(A) = \inf_{\varphi \in C_0^\infty(\mathbb{D})} \int_{\mathbb{D}} f\big(A + D\varphi(z)\big)dz.$$

Thorough discussions on quasiconvexification of functions and their relation with the lower semicontinuous envelope of the related functionals can be found in [D],[Mü],[P].

In our setting, the definition of p-quasiconvex hull immediately yields the following question.

Question 1.5. Which closed sets E have trivial p-quasiconvex hull?

Let us start with question 1.2. It goes back at least to Tartar that a necessary condition on a set E to have trivial $\mathcal{H}^p(E)$ is that for every A and B in E

$$\det(A - B) \neq 0. \tag{1.3}$$

A set satisfying (1.3) is called a set without rank-one connections. In 1982 Tartar conjectured that this condition might be also sufficient (see [T]). However, later on Tartar himself provided an example of four matrices (independently discovered by other people) without rank-one connections but supporting a $W^{1,\infty}$ Homogeneous Gradient Young measure different from a Dirac delta (see [Mü, section 2.5] for a detailed description of this example).

On the other hand, in [S1] Šverák proved that if instead of (1.3) we have that

$$\det(A-B)>0 \text{ for every } (A,B) \text{ in } E\times E, \tag{1.4}$$

the weak continuity of the determinant implies that $\mathcal{H}^2(E)$ is trivial. In the same work he showed that for some sets (1.4) and (1.3) are equivalent. In particular, this is true for connected sets (precisely we have that the determinant does not change sign), and for the the unions of orthogonal wells, fundamental in the study of martensitic phase transitions.

Since a result of Zhang in [Z] states that for a compact set E, $\mathcal{H}^p(E)=\mathcal{H}^\infty(E)$, Šverák arguments work for each $1\le p\le\infty$. In addition, the compactness of E implies that

$$E^{p,qc}=\{A\in\mathbf{M^{2\times 2}}:A=\int_{\mathbf{M^{2\times 2}}}\lambda\nu(\lambda),\ \nu\in\mathcal{H}^p(E)\}. \tag{1.5}$$

Thus, compact sets satisfying (1.4) have trivial quasiconvex hulls.

In the papers [Z1], [Z2] Zhang extended Šverák result to unbounded sets. The prize to pay was that the set E needs to satisfy condition (1.4) in a stronger way. He considered sets E such that, in addition to (1.4), satisfy that the quantity K_E defined by,

$$K_E=\sup\left\{\frac{\|A-B\|^2}{\det(A-B)}:(A,B)\in E\times E\right\} \tag{1.6}$$

is finite (although he used complex notation, see section 2). Recall that the set of K-quasiconformal matrices is defined by

$$Q(K)=\{A\in\mathbf{M^{2\times 2}}:\|A\|^2\le K\det(A).\} \tag{1.7}$$

This motivated us to name a set E such that $K_E\le K$, K-quasiconformal at every point. Therefore, K-quasiconformality at every point is the natural quantitative version of (1.4).

An important feature of these type of sets is that the answers to questions (1.2) and (1.5) depend on p. Zhang discovered the existence of a certain threshold $p(K_E)$ such that for $p>p(K_E)$ the set $\mathcal{H}^p(E)$ is trivial. Unfortunately, in his proof the relation between p and K_E depends on the norm of the Beurling Ahlfors transform, and this norm remains as perhaps the most challenging open problem in planar quasiconformal geometry.

In this note we get around this difficulty by means of the invertibility of the Beltrami Operators proved by Astala, Iwaniec and Saksman in [AIS]. We obtain that $p(K_E) = \frac{2K_E}{K_E+1}$, the Weyl exponent for quasiregular mappings determined by Astala in [A]. In fact, we present a more general condition than (1.6) depending on the complex dilatation of the matrices in E which still implies the triviality of $\mathcal{H}^p(E)$ (see Remarks 3.7, 4.4 and condition (3.10)). It is remarkable that our proofs make no appeal to the weak continuity of the determinant. Therefore, we recover the previous results of Šverák and Zhang using a new viewpoint to deal with questions (1.2) and (1.5). In addition, our method yields a characterization of the behaviour of the distance function and its quasiconvexification, extending the work of Iqbal and Zhang [I] [Z3].

In a more recent article [Z4] Zhang proved that condition (1.4) and certain control on how the set E behaves near ∞ were enough to establish the triviality of $\mathcal{H}^2(E)$. We introduce the notion of sets asymptotically K-quasiconformal.

Definition 1.6. A closed set $E \subset \mathbf{M^{2\times 2}}$ is said to be asymptotically K-quasiconformal if

$$\lim_{M\to\infty} \sup\{\frac{\|A\|^2}{\det A}, A \in E \setminus B(0,M)\} \leq K. \tag{1.8}$$

and

$$\lim_{M\to\infty} \inf\{\det A, A \in E \setminus B(0,M)\} > 0. \tag{1.9}$$

We prove that for sets E satisfying (1.4) and asymptotically K-quasiconformal $p(E) = \frac{2K}{K+1}$ as well. The basic ingredient is a higher integrability argument. However, here we need to use the weak continuity of the determinant since we do not have control of the quasiconformality of the set when $\det(A-B)$ is close to 0. This also prevents us from understanding the relaxation of the distance function in this case.

In the same paper [Z4] it is asked if (1.4) toguether with the asymptotic K-quasiconformality for some K imply the triviality of $\mathcal{H}^1(E)$. We present sets E such that $K_E < \infty$ but for $q < p(K_E)$ $\mathcal{H}^q(E)$ is not trivial. Since they contain 0 our sets are trivially asymptotically K-quasiconformal. Hence the question is solved in negative. In a future project we plan to investigate how some version of the method of convex integration applied to our examples gives new very weak quasiregular mappings.

If we look at our results as a whole, we see two things. Firstly, regularity results for elliptic PDEs give restriction on the kind of sets that can support non trivial Gradient Young measures. Secondly, if we examine unbounded sets E, the scale of exponents for which the p-quasiconvex hull behaves naturally coincides with the range of the exponents for which the associated PDE can be solved. If we are below these exponents very weak solutions give surprising p-quasiconvex hulls and surprising p-quasiconvex hulls give very weak solutions.

Since in the plane sharp results for elliptic PDEs are avalaible due to Astala's Theorem [A] we could get sharp results when studying sets growing linearly at infinity. In case we want to understand sets growing non-linearly at infinity we should look for/prove regularity results for the associated non-linear elliptic PDE.

The paper will be organized as follows. Section 2 will review some basic notation and facts on Beltrami Operators. We will prove a corollary of Astala, Iwaniec, Saksman theorem on the invertibility of Beltrami Operators to obtain the most general conditions on the complex dilatations of the matrices in a set E guaranteeing the triviality of $\mathcal{H}^p(E)$. In section 3 we deal with sets asymptotically K-quasiconformal. We present coercivity results for the distance function, and higher integrability results for homogeneous Gradient Young measures supported in such a set E. We also show that for asymptotically K-quasiconformal sets the characterization 1.5 for the p-quasiconvex hull holds provided p stays in the right range. In section 4 we analyze sets without rank-one connections. Firstly, when they are quasiconformal at every point , and then when we only have control of their quasiconformality at infinity. We also present a result characterizing sequences of gradients approaching to unions of these sets. In Section 5 we will construct examples of sets K-quasiconformal at every point and such that $E^{qc,q} = co(E)$ for every $1 \leq q < \frac{2K}{K+1}$. They will show the optimality of our results and answer the questions raised by Zhang.

Acknowledgments. It is pleasure to thank Kari Astala for useful discussions.

2. Notation and Preliminaries

In the paper we will be living in two different worlds, the domain of the functions under consideration, i.e the complex plane $\mathbb{C}$, and the range of their derivatives, i.e the space of two by two matrices $\mathbf{M^{2\times 2}}$. In $\mathbb{C}$, $\mathbb{D}$ will be the unit disc and $\mathbb{D}(a,r)$ the disc centered at a with radius r. The supports of a function f and of a measure ν will be denoted by $\mathrm{spt}(f)$ and $\mathrm{spt}(\nu)$, respectively. For preimages of Borel Sets we will make the standard abuse of notation. For example, if P is a measurable function the set $\{z \in \mathbb{D} : P(z) \geq M\}$ will be denoted by $\{P(z) \geq M\}$. The hyperbolic distance between two points $\{z_1, z_2\}$ in $\mathbb{D}$ is denoted by $\mathrm{dist}_h(z_1, z_2)$. The length element in the hyperbolic distance is

$$ds = \frac{2}{1-|z|^2}|dz|.$$

Let $A \in \mathbf{M}^{\mathbf{2}\times\mathbf{2}}$. Then $\|A\|$ will stand for the operator norm of A and $|A|$ for the Hilbert-Smith norm. The set

$$CO_+ = \{A \in \mathbf{M}^{\mathbf{2}\times\mathbf{2}} : A = \rho R, \rho > 0, R \in SO(2)\}$$

is called the set of conformal matrices. Similarly,

$$CO_- = \{A \in \mathbf{M}^{\mathbf{2}\times\mathbf{2}} : A = CH, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C \in CO_+\}$$

is the set of anticonformal matrices. The function $\Pi_{CO_\pm} : \mathbf{M}^{\mathbf{2}\times\mathbf{2}} \to CO_\pm$ will denote the orthogonal projections on $CO_\pm$. By means of the natural identification between matrices in $SO(2)$ and complex numbers with modulus 1,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \cos(\theta) + i\sin(\theta) \tag{2.1}$$

the projections $\Pi_{CO_\pm}$ induce complex coordinates for matrices; $A = (A_+, A_-) : A_+, A_- \in \mathbb{C}$ (see [Al]). Since $A_\pm$ are complex numbers $|A_\pm|$ denote their absolute values. Let us remark that (2.1) implies that $|\Pi_{CO_\pm}(A)| = \sqrt{2}|A_\pm|$. In complex coordinates we have the identities,

$$\begin{aligned} &\det(A) = |A_+|^2 - |A_-|^2 \ , \ \|A\| = |A_+| + |A_-| \text{ and} \\ &|A|^2 = 2(|A_+|^2 + |A_-|^2). \end{aligned} \tag{2.2}$$

Given two matrices A, B we say that $[A, B] = \{tA + (1-t)B : t \in [0,1]\}$ is a rank-one segment if $\det(A - B) = 0$. In conformal coordinates the latter condition is equivalent to $|A_+ - B_+| = |A_- - B_-|$. Complex coordinates give rise to the concept of complex dilatation of a matrix A, $\mu_A = \frac{A_-}{A_+} \in \mathbb{C}$. Let $k = \frac{K-1}{K+1}$. Then it follows from (2.2) that $A \in Q(K)$ if and only if $|\mu_A| \leq k$. Let $E \subset \mathbf{M}^{\mathbf{2}\times\mathbf{2}}$. Then $co(E)$ denotes the closed convex hull of E.

Let us turn now our attention to mappings. The space $C_0^\infty(\mathbb{D})$ is the space of infinitely differentiable functions with compact support. The space of vector valued Sobolev mappings is denoted by $W^{1,p}(\mathbb{D}, \mathbb{C})$ or $W^{1,p}(\mathbb{D}, \mathbb{R}^2)$ indistinctly. If $f \in W^{1,2}(\mathbb{D}, \mathbb{C})$ and $Df \in Q(K)$ almost everywhere in $\mathbb{D}$, f is said to be K-quasiregular. If in addition f is an homeomorphism it is said to be K-quasiconformal. We will also use complex coordinates to study the Jacobian derivative of a Sobolev mapping f;

$$Df(z) = \big(\partial_z f(z), \partial_{\overline{z}} f(z)\big), \mu_f = \frac{\partial_{\overline{z}} f(z)}{\partial_z f(z)}, J_f(z) = \det(Df(z)).$$

In case we are dealing with functions supported in the whole space the complex derivatives of a (non-homogeneous) Sobolev function are related by the Beurling-Ahlfors transform

$$S(\partial_{\overline{z}} f) = \partial_z f,$$

where S is a Calderon-Zygmund Operator with Fourier multiplier equal to $\frac{\bar{\xi}}{\xi}$. Hence, the general theory of these operators says that S is bounded from $L^p(\mathbb{C})$ to $L^p(\mathbb{C})$ if $1 < p < \infty$. One can interpret the boundness of S as a coercivity result for the PDE

$$\partial_{\overline{z}} f(z) = h$$

where $h \in L^p(\mathbb{C})$. This is the simplest example of the so-called Beltrami equations. In general, each function $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$ determines an elliptic equation. The function μ is called a Beltrami coefficient and the equation the Beltrami equation. For a given μ the associated Beltrami equation is

$$\partial_{\overline{z}} f - \mu \partial_z f = h,$$

were h in $L^p(\mathbb{C})$. These equations are well understood thanks to the work of Astala, Iwaniec and Saksman [AIS]. The key result in [AIS] is the coercivity of the so-called Beltrami Operator $I - \mu S$. (See [AIS, Lemma 14])

Theorem 2.1. *Let $g \in L^p(\mathbb{C}, \mathbb{C})$ and $\mu \in L^\infty(\mathbb{C}, B(0,k))$. Let $1 + k < p < 1 + \frac{1}{k}$. Then,*

$$\int_{\mathbb{C}} |g(z)|^p dz \leq C(k,p) \int_{\mathbb{C}} |g(z) - \mu(z) S(g(z))|^p dz.$$

This theorem will be the main tool used in this paper. However, in some places we will need more general versions. The main ingredient in the proof of Proposition 4.5 is the analogous result to Theorem 2.1 where the associated equation is nonlinear. Astala, Iwaniec and Saksman also established the following invertibility result for non-linear Beltrami operators ([AIS, Theorem 1]).

Theorem 2.2. *Let H be a measurable function $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that H is k-Lipschitz ($k < 1$) in the second variable and $H(w, 0) = 0$ for almost every w. Let $B(g) = g - H(z, S(g))$. Then $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is an invertible operator provided that $1 + k < p < 1 + \frac{1}{k}$.*

To get the optimal results with respect to the question studied in this work it is better to express Theorem 2.1 in the following more general version.

Corollary 2.3. *Let $K > 1$ and let $\mu \in L^\infty(\mathbb{C})$ such that there exists a complex number $\mu_0 \in \mathbb{D}$ with $dist_h(\mu(z), \mu_0) \leq \log(K)$. Let $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Then the operator $(I - \mu S)^{-1}$ is bounded from $L^p(\mathbb{C})$ into itself.*

Notice that if $K = \frac{k+1}{1-k}$ and $\mu_0 = 0$ the corollary just rephrase Theorem 2.1.

Proof:

The proof of Theorem 2.1 in [AIS] uses $|\mu(z)| \leq k$ only to obtain that for every K-quasiconformal homeomorphic solution to the homogeneous Beltrami equation

$$\partial_{\overline{z}} f = \mu \partial_z f,$$

$w = |J_f|^{1-\frac{p}{2}}$ is an A_p-weight for $2 \leq p < 1 + \frac{1}{k}$ see [AIS, Lemma 6]. Therefore it is enough to prove that if μ is as in the Corollary 2.3 every quasiconformal solution to

$$\partial_{\overline{z}} F = \mu \partial_z F$$

gives rise to the corresponding A_p-weights . This is a consequence of the appropriate linear change of variables. Consider the linear mapping $A_0(z) = z + \mu_0 \overline{z}$. Then the chain rule gives that

$$\mu_{F\circ A_0^{-1}}(A_0(z)) = \frac{\mu_F(z) - \mu_0}{1 - \mu_F(z)\overline{\mu_0}} 1.$$

The point is that $T(z) = \frac{z-\mu_0}{1-z\overline{\mu_0}}$ is a Möbius transformation from the disc to the disc mapping μ_0 to 0. Hence, it is an isometry with respect to the hyperbolic metric of the disc. Thus, for every $w \in \mathbb{C}$, $\mathrm{dist}_h(\mu_{F\circ A_0^1}(w), 0) \leq \log K$ and hence

$$\|\mu_{F\circ A_0^{-1}}\|_\infty \leq k,$$

i.e $F \circ A_0^{-1}$ is K-quasiconformal. Then if we set $f = F \circ A_0^{-1}$, we have shown that $F(z) = f(A_0(z))$ where f is K-quasiconformal. Since A_p properties are not affected by linear change of variables $|J_F|^{1-\frac{p}{2}}$ is also an A_p-weight. Similarly, $|J_{F^{-1}}|^{1-\frac{p}{2}}$ is an A_p-weight because $|J_{f^{-1}}|^{1-\frac{p}{2}}$ is an A_p- weight. $\square$

The above theorem is almost sharp if we look at the regularity of the solutions to the Beltrami equations in terms of the range of the Beltrami coefficient μ. Take two complex numbers $w_1, w_2 \in \mathbb{D}$ such that $\mathrm{dist}_h(w_1, w_2) = 2\log(K)$. Pick the point w_0 in the hyperbolic geodesic γ joining w_1 and w_2 such that $\mathrm{dist}_h(w_0, w_1) = \mathrm{dist}_h(w_0, w_2) = \log(K)$. Consider the Möbius transformation from the disc into itself T_{w_0} such that $T_{w_0}(w_0) = 0$. Since $T_{w_0}(\gamma)$ is an hyperbolic geodesic passing through 0 it must be an straight line. Moreover, $\mathrm{dist}_h(T_{w_0}(w_1), 0) = \mathrm{dist}_h(T_{w_0}(w_2), 0) = \log(K)$. Hence, composing with a rotation we can assume that there is a Möbious transformation T mapping w_1 and w_2 to k and $-k$, respectively. It follows from the ideas in [F1] that one can saturate the bounds for the integrability of the gradient of quasiregular mappings with mappings f such that $\mu_f \in \{k, -k\}$ almost everywhere. Let f be one of these mappings.

Now if A is a linear mapping the chain rule gives

$$\mu_{f\circ A^{-1}}(A(z)) = \frac{\mu_f(z) - \mu_A}{1 - \mu_f(z)\overline{\mu_A}} \frac{A_+}{\overline{A}_+}.$$

This implies that we can find a matrix A_0 such that $\mu_{f\circ A_0^{-1}} = T^{-1}(\mu_f(z))$. Therefore the mapping $F(z) = f(A_0^{-1}z)$ satisfies that $\mu_F(z) = \{w_1, w_2\}$ almost everywhere and F behaves as badly as f.

3. Quasiconvex hulls and Quasiconformality of sets

Quasiconvex hulls have arisen as the right analogous to convex hulls in the vectorial calculus of variations. After proper interpretation the quasiconvex hull of a closed set E expresses the possible macroscopical effects of a system constrained by the set E at the microscopical level. There are several definitions in use of quasiconvex hulls [Z4],[Y1]. The Definition 1.3 is the least restrictive, whereas the characterization (1.5) is the most. In other words for a general set E we always have

$$\{A \in \mathbf{M^{2\times 2}} : A = \int_{\mathbf{M^{2\times 2}}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\} \subset E^{p,qc},$$

and the other existing definitions of p-quasiconvex hull (see [Z4]) give sets in between.

In [Z4] Zhang presented examples where the above inclusion is strict. His sets are asymptotically close to rank-one lines. The following less subtle example illustrates the idea behind them.

Example 3.1. The set E will be contained in the plane of all diagonal matrices in $\mathbf{M^{2\times 2}}$. Hence we use the identification

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = (d_1, d_2).$$

In this notation let $E = \{(d_1, 0) \cup (0, d_2) \text{ with } 0 \le d_1, d_2 < \infty\}$. Then $E^{p,qc} = co(E)$, for every $1 \le p < \infty$. Nevertheless if $\nu \in \mathcal{H}^2(E)$, $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) \in E$.

Proof:

Let $A \in co(E)$. Then $A = (d_1, d_2)$ with $0 < d_1, d_2 < \infty$. We use that finite quasiconvex functions are convex along rank-one segments, and that in the diagonal plane vertical segments are rank-one segments. Therefore for every natural number j

$$\mathrm{Qdist}_E^p(A) \le \theta_j \mathrm{dist}_E^p\big((d_1, j)\big) + (1-\theta_j)\mathrm{dist}_E^p\big((d_1, 0)\big) = \theta_j \mathrm{dist}_E^p(d_1, j)$$

where $\theta_j j = d_2$. Since $\mathrm{dist}_E(d_1, j) = \min\{d_1, j\}$

$$\mathrm{Qdist}_E^p(A) \le \frac{d_1^p d_2}{j}$$

for every j, whence $co(E) \in E^{p,qc}$. The claim is proved because $E^{p,qc}$ is always included in $co(E)$.

However. if $A \in co(E) \setminus E$ was the centre of mass of a measure $\nu \in \mathcal{H}^2(E)$, by [Mü, Theorem 1], ν would satisfy Jensen inequality for separately convex functions with quadratic growth. The function $f(d_1, d_2) = \max(d_1, 0)\max(d_2, 0)$ yields a contradiction. □

Observe that what we have done in this example is finding a sequence $\{g_j\} \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$, which yields the infimum in the definition of Qdist^p, but such that $\mathrm{dist}_E(A + Dg_j) \in \{0, d_1\}$, and $\lim_{j\to\infty} \int_{\mathbb{D}} |Dg_j(z)|^2 dz = \infty$. When the sequence giving the infimum in the definition of the quasiconvexification is bounded in $W^{1,p}$ the characterization (1.5) for the p-quasiconvex hull holds.

Lemma 3.2. *Let $E \in \mathbf{M^{n\times m}}$ be a closed set such that for every matrix $A \in \mathbf{M^{n\times m}}$ there exists a constant C_A such that for every function $\phi \in C_0^\infty(\Omega, \mathbb{R}^m))$*

$$\int_\Omega |D\phi(z)|^p dx \leq C_A\Big(1 + \int_{\mathbb{D}} dist_E^p\big(A + D\phi(z)\big)dz\Big).$$

Then

$$E^{qc,p} = \{A \in \mathbf{M^{2\times 2}} : A = \int_{\mathbf{M^{2\times 2}}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\}.$$

Proof:

This is well-known. Under our assumptions one can arguee as in [P, chapter 4, section 3]. More precisely, the claim is a corollary of Theorem 4.4, Corollary 4.5 and Corollary 4.7. □

Beltrami Operators yield natural sets satisfying the above coercivity condition. Using them as the main tool we can show that the examples containing rank-one lines or asymptotic to them [Z4] are quite extremal. Roughly speaking, if a set $E \subset \mathbf{M^{2\times 2}}$ is bounded away from the rank-one lines the both previous notions of p- quasiconvex hulls agree for a range of exponents depending on how much we are off the rank-one lines. This can be made quantitatively precise.

Proposition 3.3. *Let $M > 0$, $K > 0$ and $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Set $E = B(0, M) \cup Q(K)$. Then for every $A \in \mathbf{M^{2\times 2}}$ there exists a constant C_A depending on A,p,M and K such that for every function $\phi \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$*

$$\int_{\mathbb{D}} |D\phi(z)|^p dz \leq C_A\Big(1 + \int_{\mathbb{D}} dist_E^p\big(A + D\phi(z)\big)dz\Big)$$

Proof:

Let $v(z) = Az + \phi(z)$. By smoothness there exists $R(v) > 0$ such that $|Dv(z)| \leq R$. Thus, the so-called Measurable Selection Lemma ([E, page 236]) provides us with a measurable function $P : \mathbb{D} \to E \cap B(0, 2R)$ such that for every z

$$|Dv(z) - P(z)| = \mathrm{dist}_E\big(Dv(z)\big). \tag{3.1}$$

We declare

$$\mu(z) = \begin{cases} \frac{P(z)_-}{P(z)_+} & \text{if } z \in \mathbb{D} \text{ and } |P(z)| \geq M \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We only need a bound for the L^p norm so it is enough to control $D\phi$ in some ball independent of ϕ centered at ∞. Since $\phi(z)$ has zero boundary values its zero extension belong to $W^{1,p}(\mathbb{C},\mathbb{C})$. In addition, it follows from the definition of μ that $\|\mu\|_\infty \leq k$. Hence, we can use the key Theorem 2.1 and the L^p boundness of the Beurling transform to obtain

$$\int_{\mathbb{D}} |D\phi(z)|^p dz \leq C(k,p) \int_{\mathbb{D}} |\partial_{\overline{z}}\phi(z) - \mu\partial_z\phi(z)|^p dz. \tag{3.3}$$

Therefore, if we find a constant C such that

$$\int_{\mathbb{D}} |\partial_{\overline{z}}\phi(z) - \mu\partial_z\phi(z)|^p dz \leq C\Big(1 + \int_{\mathbb{D}} \operatorname{dist}_E^p\big(Dv(z)\big)dz\Big)$$

the Proposition will be proved. Let us plug in the definition of μ. Then,

$$\begin{aligned}\int_{\mathbb{D}} |\partial_{\overline{z}}\phi(z) - \mu\partial_z\phi(z)|^p dz \leq & \int_{\{P(z)\leq M\}} |\partial_{\overline{z}}\phi(z)|^p dz \\ & + \int_{\{P(z)\geq M\}} |\partial_{\overline{z}}\phi(z) - \mu\partial_z\phi(z)|^p dz.\end{aligned} \tag{3.4}$$

Now, we estimate each term separately: For the first term we have

$$\begin{aligned}\int_{\{|P(z)|\leq M\}} |D\phi(z)|^p dz \leq C(p)\Big(&\int_{\{|P(z)|\leq M\}} \operatorname{dist}_E^p\big(Dv(z)\big) + \\ & |P(z)|^p dz + |A|^p\Big) \\ \leq C(p,A,M)(1 + &\int_{\{|P(z)|\leq M\}} \operatorname{dist}_E^p\big(Dv(z)\big)dz);\end{aligned} \tag{3.5}$$

For the second term, use (3.1) to get

$$\begin{aligned}\int_{\{|P(z)|\geq M\}} |\partial_{\overline{z}}\phi(z) - \mu\partial_z\phi(z)|^p dz \leq & \int_{\{|P(z)|\geq M\}} |\partial_{\overline{z}}v(z) - \mu\partial_z v(z)|^p dz \\ & + \int_{\{|P(z)|\geq M\}} |A_- - \mu(z)A_+|^p dz \\ & \leq C\Big(1 + \int_{\mathbb{D}} \operatorname{dist}_E^p(Dv(z))dz\Big).\end{aligned} \tag{3.6}$$

We have only added and subtracted the appropriate quantities and used the respective domains of integration. Combining (3.5) and (3.6) with (3.4) and (3.3) we obtain the desired result. □

Corollary 3.4. *Let $K > 1$ and $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Let $E = B(0,M) \cup Q(K)$. Then*

$$E^{qc,p} = \{A \in \mathbf{M^{2\times 2}} : A = \int_{\mathbf{M^{2\times 2}}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\}$$

Proof:
The proof readily follows from Lemma 3.2 and Proposition 3.3. □

One of the remarkable features of quasiregular mappings is the automatic improvement of their regularity. Precisely, a mapping $f \in W^{1,q}_{loc}(\mathbb{D},\mathbb{R}^2)$ with $q > \frac{2K}{K+1}$ and such that $Df(z) \in Q(K)$ belongs to $W^{1,p}_{loc}(\mathbb{D},\mathbb{R}^2)$ for every $p < \frac{2K}{K-1}$. In [AF] this was generalized to Gradient Young measures. In that paper, it was proved that $\mathcal{H}^q(Q(K)) = \mathcal{H}^p(Q(K))$ for $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. In this work we shall need the following sharper result. The idea of the proof is push the sequence to the quasiconformal cone when is close to the cone. In this way, we find another generating sequence $\{D\varphi_j\}$ which is either quasiconformal or close to the ball.

This implies that the elements of the new generating sequence satisfy Beltrami equations with the right hand sides uniformly bounded in $L^\infty(\mathbb{D})$. Then by means of cut-off functions the new sequence is shown to have local higher integrability. Since the Young measure is homogeneous, this is enough to imply the correct integrability for the measure.

Proposition 3.5. *Let $M > 0$, $K > 1$ and $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. Set $E = B(0,M) \cup Q(K)$. Then*

$$\mathcal{H}^q(E) = \mathcal{H}^p(E).$$

Proof:
Let $\nu \in \mathcal{H}^q(E)$. A well known property of gradient Young measures (See [P, Proposition 8.15]) is that there exists generating sequence $\varphi_j \in W^{1,\infty}(\mathbb{D},\mathbb{R}^2)$ such that $|D\varphi_j|^q$ is equiintegrable. Therefore

$$\int_{\mathbb{D}} \operatorname{dist}^q_E\big(D\varphi_j(z)\big)dz \to 0.$$

It is easy to see that this implies that,

$$\int_{\{|D\varphi_j(z)|\geq 2M\}} \operatorname{dist}^q_{Q(K)}(D\varphi_j(z))dz \to 0. \tag{3.7}$$

Then, let $P_j(z)$ be the measurable function obtained from the Measurable selection Lemma such that $\operatorname{dist}_{Q(K)}\big(D\varphi_j(z)\big) = |D\varphi_j(z) - P_j(z)|, P_j(z) \in Q(K)$ (we need here that $\varphi_j \in W^{1,\infty}(\mathbb{D},\mathbb{R}^2)$). Declare $\mu_j(z) = \mu_{P_j(z)}\chi_{\{|D\varphi_j(z)|\geq 2M\}}(z)$ and $h_j = (\partial_{\overline{z}}\varphi_j - \mu_j\partial_z\varphi_j)\chi_{\{|D\varphi_j(z)|\geq 2M\}}$. Arguing as in the proof of Proposition 3.3, we find a constant C such that $|\partial_{\overline{z}}\varphi_j - \mu_j\varphi_j|^q \leq C\operatorname{dist}^q_{Q(K)}(D\varphi_j)$. Hence (3.7) implies that

$$\|h_j\|_{L^q(\mathbb{D})} \to 0.$$

By means of the invertibility of the Beltrami Operators and the Cauchy transform we find a sequence $\{F_j\} \in L^q(\mathbb{C})$ such that

$$\partial_{\overline{z}} F_j - \mu_j \partial_z F_j = h_j \tag{3.8}$$

in the whole complex plane $\mathbb{C}$, and $\|DF_j\|_{L^q(\mathbb{C})} \leq C\|h_j\|_{L^q(\mathbb{D})}$ (For a related argument see [AF, Theorem 1.2]). Thus the sequence $\{D(\varphi_j - F_j)\}$ also generates the measure ν. Set $g_j = \varphi_j - F_j$. Combining (3.8) and the definition of h_j we get that in $\mathbb{D}$

$$\partial_{\overline{z}} g_j - \mu_j \partial_z g_j = \partial_{\overline{z}} \varphi_j \chi_{\{|D\varphi_j(z)|<2M\}}. \tag{3.9}$$

That is, g_j satisfies locally a non homogeneous Beltrami equation with a right hand side bounded by $2M$ and compactly supported.

In order to apply Theorem 2.1 we need to extend g_j to the whole plane. Formally, let $\eta_1 \in C_0^\infty(\mathbb{D})$ be a cut-off function such that $\eta(z) = 1$ if $|z| \leq \frac{1}{2}$. Then, by the chain rule

$$\partial_{\overline{z}} \eta_1 g_j - \mu_j \partial_z \eta_1 g_j = \eta_1(\partial_{\overline{z}} g_j - \mu_j \partial_z g_j) + g_j(\partial_{\overline{z}} \eta_1 - \mu_j \partial_z \eta_1),$$

in the distributional sense.

Let us denote the new sequence of right hand sides by $\tilde{h}_j$. Since Dg_j is uniformly bounded in L^q, the Sobolev Embedding Theorem implies that g_j is uniformly bounded in $L^{q^*}(\mathbb{D}, \mathbb{C})$, where q^* is the Sobolev conjugate exponent of q (notice that $\mathbb{D}$ is an extension domain).

Since η_1 is compactly supported, $\|\eta_1(\partial_{\overline{z}} g_j - \mu_j \partial_z g_j)\|_{L^p(\mathbb{C})} \leq 2M|\operatorname{spt}(\eta_1)|^{\frac{1}{p}}$ for every p and therefore $\tilde{h}_j$ is uniformly bounded in $L^{q^*}(\mathbb{C})$. Now we need to consider two different cases:

If $q^* \geq \frac{2K}{K-1}$, given that $\partial_{\overline{z}}(\eta_1 g_j) = (I - \mu_j S)^{-1}\tilde{h}_j$, we can use once more Theorem 2.1 to obtain that $\{D(\eta_1 g_j)\}$ is uniformly bounded in $L^p(\mathbb{C})$ for every $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Therefore $\{Dg_j\}$ is uniformly bounded in $L^p(\mathbb{D}(0, \frac{1}{2}))$.

If $q^* < \frac{2K}{K-1}$, the above argument only gives that the sequence $\{Dg_j\}$ is uniformly bounded in $L^{q^*}\big(\mathbb{D}(0, \frac{1}{2})\big)$. Next, we take a second cut-off function $\eta_2 \in C_0^\infty\big(\mathbb{D}(0, \frac{1}{2})\big)$ $\eta_2(z) = 1$ if $|z| \leq \frac{1}{4}$. Proceeding as above, we find that $\eta_2 g_j$ satisfies a non homogeneous Beltrami equation with the same Beltrami coefficient μ_j but with a new right hand side H_j,

$$H_j(z) = \eta_2(\partial_{\overline{z}} g_j - \mu_j \partial_z g_j) + g_j(\partial_{\overline{z}} \eta_2 - \mu_j \partial_z \eta_2).$$

The key is that now $g_j \in W^{1,q^*}\big(\mathbb{D}(0, \frac{1}{2})\big)$. Since $q^* > 2$, the Sobolev embedding gives us that there are representatives of g_j which are uniformly Hölder continuous (again we use that discs are extension domains). There is no loss of generality assuming that $g_j(0) = 0$ so in particular the g_j are uniformly bounded in $L^\infty\big(\mathbb{D}(0, \frac{1}{4})\big)$. Therefore, since we already knew that $\partial_{\overline{z}} g_j - \mu_j \partial_z g_j$ were uniformly bounded in $L^\infty(\mathbb{D})$ we obtain that the sequence $\{H_j\}$ is uniformly bounded in $L^p(\mathbb{C})$ for each $p > 1$. Now as above the invertibility of the Beltrami Operators gives that $\{D(\eta_2 g_j)\}$ is uniformly bounded in $L^p(\mathbb{C})$ for $p < \frac{2K}{K-1}$,

which implies that $\{Dg_j\}$ is uniformly bounded in $L^p(\mathbb{D}(0,\frac{1}{4}))$. Lastly, the homogeneity of ν implies that both $\{Dg_{j|\mathbb{D}(0,\frac{1}{2})}\}$ and $\{Dg_{j|\mathbb{D}(0,\frac{1}{4})}\}$ generate ν. This gives the desired fact: $\nu \in \mathcal{H}^p(E)$ as well.
□

Let us put the above theorems together and extend the result to sets asymptotically K-quasiconformal.

Theorem 3.6. *Let $E \in \mathbf{M^{2\times 2}}$ be a set asymptotically K-quasiconformal in the sense of definition 1.6 and let $\frac{2K}{K+1} < q, p < \frac{2K}{K-1}$. Then E enjoys the following properties:*

i) *For every $A \in \mathbf{M^{2\times 2}}$ there exists a constant $C(K,A)$ such that for every $\varphi \in C_0^\infty(\mathbb{D},\mathbb{R}^2)$*
$$\int_{\mathbb{D}} |D\phi(z)|^p dz \le C(K,A)\Big(1+\int_{\mathbb{D}} dist_E^p(A+D\phi(z))dz\Big).$$
ii) $E^{qc,p} = \{A \in \mathbf{M^{2\times 2}} : A = \int_{\mathbf{M^{2\times 2}}} \lambda\nu(\lambda),\ \nu \in \mathcal{H}^p(E)\}$.
iii) $\mathcal{H}^q(E) = \mathcal{H}^p(E)$.

Proof:
Let $\frac{2K}{K+1} < p$. Pick $\epsilon > 0$ such that $\frac{2K}{K+1} < \frac{2(K+\varepsilon)}{K+\varepsilon+1} < p$. If E is asymptotically K-quasiconformal, the definition provides us with $M(\varepsilon)$ such that $E \subset B(0,M(\varepsilon)) \cup Q(K+\varepsilon)$. Therefore we can apply Proposition 3.3, Corollary 3.4 and 3.5 to such a set and such a p. The result follows □

Remark 3.7. Theorem 3.6 concerns the quasiconformal set of matrices $Q(K)$. However it should be clear, that Corollary 2.3 immediately yields the following more general result. Let p and K as above. Let E a closed set such that $K = \frac{k+1}{1-k}$. Then if

$$\lim_{M\to\infty}\Big\{\inf_{\mu_0\in\mathbb{D}}\{\sup_{\|A\|>M, A\in E} \operatorname{dist}_h(\mu_A,\mu_0)\}\Big\} \le \log(K) \tag{3.10}$$

then Theorem 3.6 holds.

4. Sets with trivial quasiconvex hulls

Let E be a set such that for $A, B \in E$, $\det(A-B) > 0$. In [Z2], Zhang discovered that an enlightening way to express this fact is to describe E as the graph of a Lipschitz function $H : \Pi_{CO_+}(E) \mapsto CO_-$. Precisely, H is defined by $H(A) = \Pi_{CO_-} \circ \Pi^{-1}_{CO_+}(A)$. Furthermore, it is an straightforward calculation to see that E is K-quasiconformal at every point (1.6) if and only if the Lipcschitz norm of H is at most $k = \frac{K-1}{K+1}$. We combine Beltrami Operators with the ideas of Zhang and Iqbal [Z3], [I] from the case $K = 1$ to obtain the following theorem.

Theorem 4.1. *Let $K > 1$ and let $E \subset \mathbf{M}^{2\times 2}$ be K-quasiconformal at every point in the sense of (1.6). Let $\frac{2K}{K+1} < p$. Then E has the following properties:*

i) *$\mathcal{H}^p(E)$ is trivial.*
ii) *$E^{p,qc} = E$.*
iii) *There exists a constant $C(p,k)$ such that*

$$dist_E^p(A) \leq C(p,k)\, Q dist_E^p(A).$$

Proof:
Let $H : \Pi_{CO_+}(E) \mapsto CO_-$ be the k Lipschitz function such that E is the graph of H. We extend H, by means of Kirzsbraun extension theorem to a Lipschitz map, still denoted by H, defined on the whole conformal set $CO_+(2)$.

Let $A \in \mathbf{M}^{2\times 2}$ and let $P(A)$ be such $|A - P(A)| = \mathrm{dist}_E(A)$. Then

$$|A_- - H(A_+)| = |A - P(A) + P(A)_+ - A_+ + H(P(A)_+) - H(A_+)|.$$

The Lipschitz condition implies that

$$\begin{aligned} |A_- - H(A_+)| &\leq |A - P(A)| + (1+k)|P(A)_+ - A_+| \\ &\leq (2+k)\mathrm{dist}_E(A). \end{aligned} \tag{4.1}$$

Let $A \in \mathbf{M}^{2\times 2}$ and $\{\varphi_j\} \in C_0^\infty(\mathbb{D}, \mathbb{R}^2)$ be a sequence such that

$$\lim_{j\to\infty} \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz = \mathrm{Qdist}_E^p(A).$$

A rescaling argument as in [Z3] shows that we can assume that

$$\varphi_j \rightharpoonup 0 \text{ in } W_0^{1,p}(\mathbb{D}, \mathbb{R}^2). \tag{4.2}$$

Let φ_j, φ_i be two elements in the sequence. Using the estimate (4.1) at almost every point z we get:

$$\begin{aligned} \int_{\mathbb{D}} |A_- + \partial_{\overline{z}}\varphi_j(z) - H(A_+ + \partial_z\varphi_j(z))|^p dz + \int_{\mathbb{D}} |A_- + \partial_{\overline{z}}\varphi_i(z) \\ - H(A_+ + \partial_z\varphi_i(z))|^p dz \\ \leq (2+k)^p \Big(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz + \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_i(z))dz. \Big) \end{aligned}$$

By the convexity of the pth-power the left hand side of the inequality is larger than

$$\frac{1}{2^{p-1}} \int_{\mathbb{D}} |\partial_{\overline{z}}(\varphi_j - \varphi_i)(z) - H(A_+ + \partial_z\varphi_j(z)) + H(A_+ + \partial_z\varphi_i(z))|^p.$$

The important point here is choosing appropriate sequence of Beltrami coefficients. Let

$$\mu_j(z) = \begin{cases} \frac{H(A_+ + \partial_z \varphi_j(z)) - H(A_+ + \partial_z \varphi_i(z))}{\partial_z(\varphi_j(z) - \varphi_i(z))} & \text{if } \varphi_j(z) - \varphi_i(z) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Putting the above equations together we arrive to the following inequality:

$$\begin{aligned} &\int_{\mathbb{D}} |\partial_{\overline{z}}(\varphi_j - \varphi_i)(z) - \mu_j(z)\partial_z(\varphi_j - \varphi_i)(z)|^p dz \\ &\leq 2^{p-1}(2+k)^p \Big(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz + \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz \Big). \end{aligned}$$

The Lipschitz condition on H implies that $\mu_j(z) \leq k$ and $(\varphi_j - \varphi_i)(z)$ belongs to $C_0^\infty(\mathbb{D}, \mathbb{R}^2)$. Hence we can apply Theorem 2.1. Together with the boundness of the Beurling-Alhfors transform this yields

$$\begin{aligned} \int_{\mathbb{D}} |D(\varphi_j - \varphi_i)(z)|^p dz \leq C(k,p) \Big(&\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz \\ &+ \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz \Big). \end{aligned} \tag{4.4}$$

Now we compute the limit when i tends to ∞. On the right hand side we obtain $Qdis_E^p(A)$. On the left hand side, we use that the functional $I(\varphi) = \int_{\mathbb{D}} |D\varphi_j - D\varphi|^p$ is sequentially weakly lower semicontinuous in $W^{1,p}$ due to the convexity of $F_z(A) = |D\varphi_j(z) - A|^p$ for every z. We obtain

$$\begin{aligned} &C(k,p) \Big(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz + Qdis_E^p(A) \Big) \\ &\geq \liminf_{i \to \infty} \int_{\mathbb{D}} |D(\varphi_j - \varphi_i)(z))|^p dz \geq \int_{\mathbb{D}} |D\varphi_j(z)|^p dz. \end{aligned} \tag{4.5}$$

Finally, we use the following norm estimate. Let M_1, M_2 two matrices in $\mathbf{M^{2\times 2}}$ and E a closed set. Then

$$|M_1|^p \geq \frac{1}{2^{p-1}} \mathrm{dist}_E^p(M_2) - \mathrm{dist}_E^p(M_1 + M_2). \tag{4.6}$$

Take $M_1 = D\varphi_j$ and $M_2 = A$ at every z in $\mathbb{D}$. Rearranging the inequality 4.5 we get

$$\frac{1}{2^{p-1}} \mathrm{dist}_E(A) \leq C(k,p) \Bigg(\int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\varphi_j(z))dz + Qdis_E^p(A). \Bigg)$$

Letting $j \to \infty$ the claim (iii) is proved.

The definition of p-quasiconvex hull (1.3) immediately yields (ii). Let $\nu \in \mathcal{H}^p(E)$. Then, by general principles we know the existence of a sequence $\{\phi_j\} \in W^{1,\infty}(\mathbb{D}, \mathbb{R}^2)$ generating ν and such that

$$\lim_{j \to \infty} \int_{\mathbb{D}} \mathrm{dist}_E^p(A + D\phi_j(z))dz = 0.$$

Applying the argument of the proof to $\{\phi_j\}$, in particular inequality (4.4), gives that the sequence $\{D\phi_j\}$ is a Cauchy sequence in L^p. This implies that ν is a Dirac Delta and hence proves (i). □

Remark 4.2. Sets quasiconformal at every point enjoy further interesting properties. It was proved by Zhang that Theorem 4.1 iii) implies that if we denote by E_ϵ the epsilon neighborhood of E there exists a constant C such that $E_\epsilon^{p,qc} \subset E_{C\epsilon}$.

In a different direction Theorem 4.1 implies that the only mappings $\varphi \in W^{1,p}(\mathbb{D},\mathbb{C})$ affine in the boundary and such that $D\varphi \in E$ almost everywhere are affine. Šverák proved in [S] that for E K-quasiconformal at every point and compact there exists $p > 2$ such that every Lipschitz mapping φ such that $D\varphi(z) \in E$ a.e.$z \in \mathbb{D}$ actually belongs to $W^{2,p}_{loc}(\mathbb{D},\mathbb{C})$. He studied the behaviour of the quotients $\varphi_h = \frac{\varphi(x+ah)-\varphi(x)}{h}$ with a any unit vector and proved that they satisfy Cacciopoli inequalities. Our notation simplifies his proof and extends the result, removing the compactness assumption and yielding the precise value of p. The idea is that E being K-quasiconformal at every point implies that for every h φ_h are K-quasiregular mappings. This observation yields rather easily the following result.

Let $\frac{2K}{K+1} < q$ and $\varphi \in W^{1,q}_{loc}(\mathbb{D},\mathbb{C})$ be such $D\varphi \in E$, E K-quasiconformal at every point. Then $\varphi \in W^{2,p}_{loc}$ for every $p < \frac{2K}{K-1}$ and $D\varphi$ is locally $\frac{1}{K}$ Hölder continuous. The argument is a simple integration by parts and Astala's Theorem [A].

If we have control on the quasiconformality of E only at ∞ we can not estimate the norm of the Lipschitz function H whose graph is E and hence the above proof does not work. However, the results from the previous sections and the weak continuity of the determinant when the integrability is large enough, immediately yield the following weaker theorem.

Theorem 4.3. *Let $E \subset \mathbf{M^{2\times 2}}$ be an asymptotically K-quasiconformal set (Definition 1.6) such that*

$$\det(A-B) > 0, \text{ for all } A,B \in E. \tag{4.7}$$

Then for every $q > \frac{2K}{K+1}$,

1) *$\mathcal{H}^q(E)$ is trivial.*
2) *$E^{q,qc} = E$.*

Proof:

By Theorem 3.6 i) it is enough to prove 1). But by Theorem 3.6 ii) $\mathcal{H}^q(E) = \mathcal{H}^p(E)$ for some $p > 2$. The determinant is weakly continuous in $W^{1,p}$ thus every $\nu \in \mathcal{H}^p(E)$ satisfies the so-called minor relations:

$$\det\Big(\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda)\Big) = \int \det(\lambda) d\nu(\lambda).$$

Then we can apply the arguments from [S] to conclude that ν must be a Dirac mass. □

Remark 4.4. Corollary 2.3 gives generalizations of the above results. In Theorem 4.1 we could replace the requirement of the K-quasiconformality of E at every point by requiring that E belongs to the graph of a Lipschitz function H such that, for every A, B the complex number $\frac{H(A)-H(B)}{A-B}$ belongs to some hyperbolic ball in the unit disc with radius $\log(K)$. Similarly, the asymptotic quasiconformality can be replaced by the condition (3.10) in the Remark 3.7. This should be compared with the infinite rank-laminates constructed in section 5.

The content of the last result of the section is that if we are studying Gradient Young measures supported in a union of graphs of Lipschitz functions we can use the Non-linear Beltrami Operators to obtain a generating sequence with gradients almost everywhere in the graphs.

Proposition 4.5. *Let $\{H_i\}_{i=1}^N$ with $H_i : CO_+(2) \to CO_-(2)$ be a collection of k- Lipschitz functions with $k < 1$. Denote by $E_i \subset \mathbf{M^{2\times 2}}$ the graph of H_i. Let $q > 1+k$ and $E = \cup_{i=1}^n E_i$. Then every $\nu \in \mathcal{H}^q(E)$ can be generated by gradients Dg_j such that $Dg_j(z) \in E$ for a.e.z.*

Proof:
Let $\nu \in \mathcal{H}^q$ and $D\varphi_j$ be the usual equiintegrable generating sequence. Then as always we have that $\lim_{j\to\infty}\int_{\mathbf{M^{2\times 2}}} \operatorname{dist}_E^q(D\varphi_j(z))dz = 0$. By the pointwise estimate (4.1) we obtain that

$$\lim_{j\to\infty}\int_{\mathbf{M^{2\times 2}}} |\partial_{\overline{z}}\varphi_j(z) - H_j(\partial_z\varphi_j(z), z)|^p dz = 0, \tag{4.8}$$

where $H_j(A, z) = H_{i(z)}(A)$ for some i. The existence and measurability of H_j is guaranteed by the Measurable Selection Lemma.

Denote $h_j = \partial_{\overline{z}}\varphi_j(z) - H_j\big(\partial_z\varphi_j(z), z\big)$. We want to correct φ_j to obtain another sequence satisfying the homogeneous elliptic PDE determined by H_j and with the same oscillating behavior. It clearly suffices finding for every j a solution F_j to the equation

$$\partial_{\overline{z}}F_j - H_j(\partial_z\varphi_j + \partial_z F_j, z) + H_j(\partial_z\varphi_j, z) = -h_j \tag{4.9}$$

Then $g_j = F_j + \varphi_j$ would satisfy

$$\partial_{\overline{z}}g_j - H_j(\partial_z g_j, z) = 0$$

in $\mathbb{D}$. To solve the equation (4.9) we observe that for almost every z $\hat{H}_j(A, z) = H_j(\partial_z\varphi_j(z) + A, z) - H_j(\partial_z\varphi_j(z), z)$ is a k-Lipschitz map and $\hat{H}_j(0, z) = 0$. Hence, we can apply Theorem 2.2 and obtain that the operator $I - \hat{H}_j(S, z)$ is invertible for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Then we can follow the proof of [AF, Theorem 1.2] with $I - \hat{H}_j(S, z)$ in the place of $I - \mu_j S$ to obtain a solution to (4.9) in the whole complex plane. Furthermore, the solution comes with the estimate $\|DF_j\|_p \leq$

$C\|h_j\chi_{\mathbb{D}}\|_p$. From here on we can repeat the arguments in [AF] line by line obtaining that the sequence $\{Dg_{j_{|\mathbb{D}}}\}$ is a generating sequence for ν. □

5. Examples

In this section we present examples of connected sets E without rank-one connections, and even K- quasiconformal at every point, but with non-trivial p-quasiconvex hulls. Precisely, for every $1 < q < 1 + k$ we construct non trivial measures $\nu \in \mathcal{H}^q(E)$. We use the two known methods to construct Homogeneous Gradient Young measures. The first, known in the literature as the Riemann-Lebesgue Lemma, consists in homogenizing a function with affine boundary values. It is of interest to observe that the appropriate function is a so-called very weak quasiregular mapping.

Example 5.1. Let $E \subset \mathbf{M^{2\times 2}}$ be defined by

$$E \equiv \{A : A_- = k|A_+|, A_- \in \mathbb{R}\}$$

and $1 < q < 1 + k$. Then

$$E^{q,qc} = co(E).$$

Proof:

The example lives in the space of reflected symmetric matrices. Throughout all the example we will use conformal coordinates. In these coordinates a matrix A is the composition of a reflection and a symmetric matrix if and only if its anticonformal coordinate is real. Consider the quasiconformal inversion,

$$f(z) = \overline{z}|z|^{\frac{2K}{K+1}}.$$

A direct computation shows that the mapping belongs to the class $W^{1,q}(\mathbb{D}, \mathbb{C})$ and that $Df(z) \in E$ a.e.z. Moreover on the boundary of the unit disc $f(z) = \overline{z}$. Since f is affine on the boundary of the unit disc, we can apply the classical theorem from Gradient Young Measures theory [P, Lemma 8.2], which says that the probability measure ν_f defined by

$$\int_{\mathbf{M^{2\times 2}}} \varphi(\lambda) d\nu_f(\lambda) = \int_{\mathbb{D}} \varphi(Df(z))dz$$

belongs to $\mathcal{H}^q(E)$, q being the integrability of Df. Moreover, the centre of mass of ν_f is equal to the affine boundary values. Therefore, in conformal coordinates we have that $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu_f(\lambda) = (0, 1)$. Thus, $(0, 1)$ is in $E^{q,qc}$. Similarly, $(0, \rho) \in E^{q,qc}$ for every $\rho > 0$.

To deal with the rest of $co(E)$ we notice that $A \in co(E)$ if and only if $A_- \geq k|A_+|$. By drawing a picture on the diagonal plane, one sees that if $A = (A_+, A_-)$ with $A \in co(E)$ and $A_+ > 0$ A belongs to a rank-one segment between the set E and the half line $\{(0,\rho),\rho<0\}$. Hence if $A\in co(E)$ we can find positive numbers $t,\rho,\tilde{B_+}$ such that

$$(|A_+|,A_-)=t(0,\rho)+(1-t)(\tilde{B_+},k\tilde{B_+}), \tag{5.1}$$

with $\tilde{B_+}=\rho-k\tilde{B_+}$. The last follows because we must have that $\det(\tilde{B_+},\rho-k\tilde{B_+})=0$. It is easy to see that this implies that A belongs to a rank-one segment between $(0,\rho)$ and $(\frac{A_+}{1-t},k\frac{A_+}{1-t})$ with t as in (5.1). Since $Qdist^q$ is convex along rank-one segments we obtain that $A\in E^{q,qc}$. □

This example already answers in negative a question raised in [Z4]. There it was asked if that E being asymptotically quasiconformal for some K is enough to guarantee $E^{1,qc}=E$. As observed in [Z3] the result is true if $K=1$ but the above example shows that it is false in general.

The next example is devoted to showing that our Theorem 4.1 is almost sharp. There are sets, consisting only on two lines, saturating the bounds between the integrability of the approaching sequence and the quasiconformality of the set. We use the second method to construct Homogeneous Gradient Young measures, the so-called p-laminates. The literature on these measures is already extensive (see in the case $p=\infty$ [P], [K]). Laminates in relation with integrability issues were first considered in [F]. In fact, the next example might be considered dual to the one in [F].

Let us recall that in the plane of diagonal matrices we use the notation

$$\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix}=(d_1,d_2).$$

We also recall the definition of laminates and prelaminates.

Definition 5.2. The family of prelaminates $\mathcal{PL}$ is the smallest family of probability measures on $\mathbf{M^{n\times m}}$ such that

(1) $\mathcal{PL}$ contains all Dirac masses in $\mathbf{M^{n\times m}}$.
(2) Let $\nu=\sum_{i=1}^k\lambda_i\delta_{A_i}\in\mathcal{PL}$ and let $A_1=\lambda B+(1-\lambda)C$ where $\lambda\in[0,1]$ and $[B,C]$ is a rank-one segment. Then the probability measure $\sum_{i=2}^k\lambda_i\delta_{A_i}+\lambda_1(\lambda\delta_B+(1-\lambda)\delta_C)\in\mathcal{PL}$.

Definition 5.3. Let ν be a probability measure on $\mathbf{M^{n\times m}}$ and $1\le p<\infty$. Then ν is said to be a p-laminate if there exists a sequence of prelaminates ν_j such that

a) $\sup_j\int_{\mathbf{M^{n\times m}}}|\lambda|^p d\nu_j(\lambda)<\infty$,
b) $\nu_j\overset{\star}{\rightharpoonup}\nu$ in $\mathcal{M}(\mathbf{M^{n\times m}})$.

Example 5.4. Let $E\subset D$, where D is the diagonal plane, be defined by

$$E=\{(a,Ka)\text{ if }a\ge 0,\ (Ka,a)\text{ if }a<0.\}$$

Let $q < 1 + k = \frac{2K}{K+1}$. Then

$$E^{q,qc} = co(E).$$

Proof:

For the proof we construct a laminate ν with centre of mass $(-1, 1)$. Since a dual construction was explained in detail in [F], we shall briefly sketch the main steps in the construction of ν. Consider the sequence $A_n = (-n, n)$. Then we will build prelaminates ν_n supported in $E \cup A_n$ and then take the weak star limit when $n \to \infty$. A computation of the integrability of limit measure ν will yield the result. The key are the following equalities:

$$\begin{aligned}(-n,n) &= \tfrac{K}{n(K+1)+K}(\tfrac{n}{K},n) + (1-\tfrac{K}{n(K+1)+K})(-(n+1),n)\\ (-(n+1),n) &= \tfrac{K}{(n+1)(1+K)}(-(n+1),\tfrac{-(n+1)}{K})\\ &\quad +(1-\tfrac{K}{(n+1)(1+K)})(-(n+1),n+1).\end{aligned} \tag{5.2}$$

In the picture 1 we use the following notation for the auxiliary matrices $B_n = (\frac{n}{K}, n), C_n = (-(n+1), n), D_n = (-(n+1), \frac{-(n+1)}{K})$ Since the convex combinations in (5.2) happen along rank-one segments (see the picture 1) we can use the closedness of prelaminates under splitting, this is property 2 in the definition 5.2. The basic idea is as follows. The matrix A_1 is the centre of mass of a laminate ν_1 supported on B_1 and C_1. Then δ_{C_1} can be splitted into a measure supported in A_2 and D_1. This gives as the measure ν_2 supported on $\{B_1, D_1, A_2\}$. However, δ_{A_2} can be splitted in a measure supported on B_2 and C_2. Further, δ_{C_2} can be splitted into a measure supported on D_2 and A_3 so we have a prelaminate ν_3 supported on $\{B_1, D_1, B_2, D_2, A_3\}$. The picture 1 and the equalities (5.2) indicate how the construction can be iterated to obtain the required sequence of measures ν_n. It is easy to check that these prelaminates will enjoy the following properties. Firstly, they can be written as $\nu_n = \mu_n + \lambda_n \delta_{A_n}$, where

$$\lambda_n = \Pi_{i=1}^{n-1}(1 - \frac{K}{i(K+1)+K})(1 - \frac{K}{(i+1)(1+K)}) \tag{5.3}$$

and ν_n is supported in $E \cap B(0, n|(1, \frac{1}{K})|)$. Secondly, the centre of mass is fixed, $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) = (-1, 1)$ for every n. The measure ν is then defined as the weak star limit of the measures ν_n. The integrability of ν follows from standard arguments in measure theory and the basic properties of the logarithms. The details are explained in the dual construction in [F]. Let us state the result; For every $q < \frac{2K}{K+1}$

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^q d\nu(\lambda) < \infty,$$

but

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^{\frac{2K}{K+1}} d\nu(\lambda) = \infty.$$

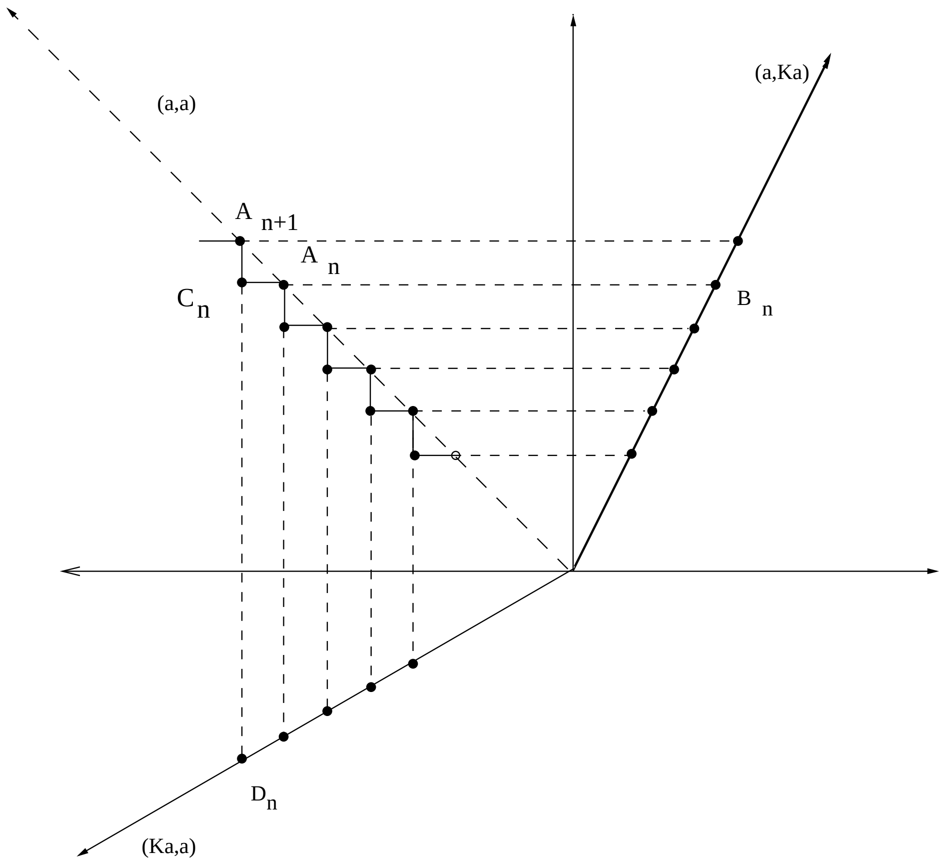

FIGURE 1. *Black dots denote the support of the measure ν_n, and the white dot is the centre of mass.*

Thus we have proved that $(-1,1) \in E^{q,qc}$. It is clear that our construction remains the same if we multiply everthing by a scalar $\rho > 0$, whence it follows that $(-\rho, \rho) \in E^{q,qc}$. As in the previous example this suffices to conclude that $E^{q,qc} = co(E)$ as desired. $\square$

Remark 5.5. Let μ_1, μ_2 such that $\mathrm{dist}_h(\mu_1, \mu_2) = 2\log(K)$. The affine change of variables used in section 2 gives us the natural plane of matrices where a q-laminate supported in two lines with dilatations μ_1, μ_2 lives. The changes of variables is given by the unique matrix A such such that if $\mu_f \in \{k, -k\}$, $\mu_{f\circ A} \in \{\mu_1, \mu_2\}$. Here of course, $k = \frac{K+1}{K-1}$.

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