MAPPINGS OF FINITE LENGTH DISTORTION

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Abstract

General properties of mappings of finite metric distortion and of finite length distortion are studied. Uniqueness, equicontinuity, boundary behavior and removability of singularities are obtained under minimal additional assumptions.

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1 Introduction

For $x \in E \subset \mathbb{R}^n$ and a mapping $\varphi : E \to \mathbb{R}^n$, we set

$$L(x,\varphi) = \limsup_{y\to x, y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|}, \tag{1.1}$$

and

$$l(x,\varphi) = \liminf_{y\to x, y\in E} \frac{|\varphi(y)-\varphi(x)|}{|y-x|}. \tag{1.2}$$

Below we assume that D is a domain in $\mathbb{R}^n$, $n \geq 2$, and all mappings $f : D \to \mathbb{R}^n$ are continuous.

Definition 1. A mapping $f : D \to \mathbb{R}^n$ is said to be of **finite metric distortion**, abbr. $f \in$ **FMD**, if f has the Lusin (N)–property and

$$0 < l(x,f) \leq L(x,f) < \infty \qquad \text{a.e.} \tag{1.3}$$

Recall that a mapping $f : X \to Y$ between measurable spaces (X, Σ, μ) and (X', Σ', μ') is said to have the **(N)–property** if $\mu'(f(S)) = 0$ whenever $\mu(S) = 0$. Similarly, f has the **($\mathbf{N}^{-1}$)–property** if $\mu(S) = 0$ whenever $\mu'(f(S)) = 0$.

A path γ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where Δ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int_\gamma \rho(x)|dx| \geq 1$$

for each $\gamma \in \Gamma$. The **modulus** $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)$$

interpreted as $+\infty$ if $adm\,\Gamma = \varnothing$. Thus, every family Γ which contains a constant path is of infinite modulus. We say that a property P holds for **almost every (a.e.)** path γ in a family Γ if the subfamily of all paths in Γ for which P fails has modulus zero.

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable path, then there is the unique increasing length function l_γ of Δ onto a length interval $\Delta_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subpath $\gamma|_{[t_0,t]}$ of γ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t,t_0]})$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping, and suppose that the path $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is an unique increasing function $L_{\gamma,g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that

$$L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t) \quad \text{for all } t \in \Delta. \tag{1.4}$$

We say that a mapping $f : D \to \mathbb{R}^n$ has the **(L)–property** if the following two conditions hold:

(L_1) for a.e. path γ in D, $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable and the function $L_{\gamma,f}$ has the (N)–property;

(L_2) for a.e. path $\tilde{\gamma}$ in $f(D)$, each lifting γ of $\tilde{\gamma}$ is locally rectifiable and the function $L_{\gamma,f}$ has the (N^{-1})–property.

A path γ in D is a **lifting** of a path $\tilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. Note that the condition (L_2) applies only to paths $\tilde{\gamma}$ which have a (maximal) lifting.

Definition 2. A mapping $f : D \to \mathbb{R}^n$ is said to be of **finite length distortion**, abbr. $f \in$ **FLD**, if f is FMD and has the (L)–property.

The class of all FLD mappings includes all nonconstant quasiregular mappings and, in turn, every FLD mapping f belongs to the following class fore some Q which is determined by f.

Definition 3. Let $Q(x,y) = (Q_1(x), Q_2(y))$ be a pair of measurable functions $Q_1 : D \to [1,\infty]$ and $Q_2 : D' \to [1,\infty]$. We say that $f : D \to \mathbb{R}^n$, $n \geq 2$, $f(D) = D'$, is a **Q–mapping** if

$$M(f\Gamma) \leq \int\limits_D Q_1(x)\rho^n(x)\,dm(x) \tag{1.5}$$

and

$$M(\Gamma) \leq \int\limits_{D'} Q_2(y)\rho_*^n(y)\,dm(y) \tag{1.6}$$

for every family Γ of paths in D and all $\rho \in \operatorname{adm}\Gamma$ and $\rho_* \in \operatorname{adm} f\Gamma$.

If, in addition, f is discrete and open we say that f is a **Q-covering**. The more restrictive notion of a **Q–homeomorphism**, when f is a homeomorphism and $Q_2 \equiv \infty$, has been recently introduced in [MRSY$_1$]–[MRSY$_2$].

Recall that a mapping $f : D \to \mathbb{R}^n$ is **open** if the image of every open subset of D under f is an open set in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is **discrete** if the preimage $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points. B_f denotes the **branch set** of a mapping f, i.e., a set of all points $x \in D$ at which f fails to be a local homeomorphism.

2 Preliminaries

We adopt the following conventions. Given a set $E \in \mathbb{R}^n$ and a path $\gamma : \Delta \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If γ is locally rectifiable we set

$$l(\gamma \cap E) = |E_\gamma| \tag{2.1}$$

where

$$E_\gamma = l_\gamma(\gamma^{-1}(E)). \tag{2.2}$$

Here $|A|$ means the length (Lebesgue) measure and $l_\gamma : \Delta \to \Delta_\gamma$ is as in Section 1. Note that

$$E_\gamma = \gamma_0^{-1}(E) \tag{2.3}$$

where $\gamma_0 : \Delta_\gamma \to \mathbb{R}^n$ is the natural parametrization of γ and that

$$l(\gamma \cap E) = \int\limits_{\Delta} \chi_E(\gamma(t))|dx| = \int\limits_{\Delta_\gamma} \chi_{E_\gamma}(s)ds. \tag{2.4}$$

We say that $\gamma \cap E$ is measurable on γ if E_γ is measurable in Δ_γ.

2.5. Remark. The definition of the modulus immediately implies that:

1) a.e. path in $\mathbb{R}^n$ is rectifiable;

2) given a Borel set B in $\mathbb{R}^n$ of measure zero,

$$l\,(\gamma \cap B) = 0 \tag{2.6}$$

for a.e. rectifiable path γ in $\mathbb{R}^n$;

3) for every Lebesgue measurable set E in $\mathbb{R}^n$, there exist Borel sets B_* and B^* in $\mathbb{R}^n$ such that $B_* \subset E \subset B^*$ and $|B^* \setminus B_*| = 0$, and thus by 2), E_γ and χ_{E_γ} is a measurable set, respectively, function in the length interval Δ_γ for a.e. γ in $\mathbb{R}^n$.

The following lemma extends Theorem 33.1 in [Vä$_1$] from Borel sets to arbitrary sets, cf. also Theorem 3 in [Fu] and based on 3).

2.7. Lemma. *Let E be a set in a domain $D \subset \mathbb{R}^n$, $n \geq 2$. Then E is measurable if and only if $\gamma \cap E$ is measurable for a.e. path γ in D. Moreover, $|E| = 0$ if and only if*

$$l\,(\gamma \cap E) = 0 \tag{2.8}$$

on a.e. path γ in D.

Proof. Suppose first that E is measurable. Then by 3) in Remark 2.5, $\gamma \cap E$ is measurable for a.e. path γ in D.

For the other direction, let C be a closed cube in D with edges parallel the coordinate axes. By the assumption, $\gamma \cap E$ is measurable for a.e. line segments γ joining opposite faces of C and parallel to the edges. Thus, the Fubini theorem, E is measurable.

Next suppose that $|E| = 0$. Then there is a Borel set B such that $|B| = 0$ and $E \subset B$. By Remark 2.5, (2.6) and hence (2.8) holds for a.e. path γ in D.

The sufficiency of 2.8 follows from the corresponding result in Theorem 33.1 in [Vä$_1$] by virtue of 3) in Remark 2.5. This completes the proof.

Given a Lebesgue measurable function $\rho : \mathbb{R}^n \to [0, \infty]$, there is a Borel function $\rho^* : \mathbb{R}^n \to [0, \infty]$ such that $\rho^* = \rho$ a.e. in $\mathbb{R}^n$, see e.g 2.3.5 in [Fe] and [Sa], p. 69. This suggests an alternative definition of the modulus. A Lebesgue measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ is **extensively admissible** for a path family Γ in $\mathbb{R}^n$, abbr. $\rho \in \text{ext adm}\,\Gamma$, if

$$\int\limits_{\gamma} \rho(x)|dx| \geq 1 \tag{2.9}$$

for a.e. $\gamma \in \Gamma$. The **extensive modulus** $\overline{M}(\Gamma)$ of Γ is defined as

$$\overline{M}(\Gamma) = \inf \int\limits_{\mathbb{R}^n} \rho^n(x)dm(x) \tag{2.10}$$

where the infimum is taken over all $\rho \in \text{ext adm}\,\Gamma$. It is easy to see that

$$\overline{M}(\Gamma) = M(\Gamma). \tag{2.11}$$

3 Mappings of finite metric distortion

A map $\varphi : X \to Y$ between metric spaces X and Y is said to be **Lipschitzian** provided

$$\text{dist}(\varphi(x_1), \varphi(x_2)) \leq M \cdot \text{dist}(x_1, x_2) \tag{3.1}$$

for some $M < \infty$ and for all x_1 and $x_2 \in X$. The map φ is called bi–lipschitz if, in addition,

$$M^*\text{dist}(x_1, x_2) \leq \text{dist}(\varphi(x_1), \varphi(x_2)) \tag{3.2}$$

for some $M^* > 0$ and for all x_1 and $x_2 \in X$. Later on X and Y are subsets of $\mathbb{R}^n$ with the Euclidean distance. We denote by $B^n(a, r)$ the Euclidean ball $|x - a| < r$ in $\mathbb{R}^n$ with center a and radius r. We also let $B^n(r) = B^n(0, r)$ and $\mathbb{B}^n = B^n(1)$.

3.3. Lemma. *Let $E \subset \mathbb{R}^n$ be a measurable set and let a mapping $\varphi : E \to \mathbb{R}^n$ satisfy the condition $L(x, \varphi) < \infty$ a.e. Then there is a countable collection of compact sets $C_k \subset E$ such that*

$$|E \setminus \bigcup_{k=1}^{\infty} C_k| = 0 \tag{3.4}$$

and $\varphi|_{C_k}$ *is Lipschitzian for every* $k = 1, 2, \ldots$.

Proof. Since $L(x, \varphi) < \infty$ a.e., we have by 3.1.8 in [Fe] that there is a countable collection of measurable sets $C_i \subset E$ such that $|E \setminus B| = 0$, $B = \bigcup_{i=1}^{\infty} C_i$, and $\varphi|_{C_i}$ is Lipschitzian for every $i = 1, 2, \ldots$. By regularity of the Lebesgue measure, we have that, for every $i = 1, 2, \ldots$, there is a countable collection of closed sets $C_{ij} \subset C_i$ such that $|C_i \setminus \bigcup_{j=1}^{\infty} C_{ij}| = 0$. Moreover, for every fixed $i, j, l = 1, 2, \ldots$, the set $C_{ijl} = C_{ij} \cap \overline{B^n(l)}$ is compact and $C_{ij} = \bigcup_{l=1}^{\infty} C_{ijl}$. Thus, the countable collection C_k, $k = 1, 2, \ldots$, consisting of the sets C_{ijl}, $i, j, l = 1, 2, \ldots$, is a desired collection.

For $f : D \to \mathbb{R}^n$ and $E \subset D$, we use the **multiplicity functions**

$$N(y, f, E) = \operatorname{card}\{x \in E : f(x) = y\}, \tag{3.5}$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \tag{3.6}$$

3.7. Proposition. *Let* $f : D \to \mathbb{R}^n$ *be an FMD mapping. Then:*

(i) f *is differentiable a.e. and*

$$J(x, f) \neq 0 \quad \text{a.e.} \tag{3.8}$$

(ii) f *has* (N^{-1})*–property;*

(iii) the change of variables formula

$$\int_E g(f(x))|J(x, f)|\, dm(x) = \int_{\mathbb{R}^n} g(y)N(y, f, E)\, dm(y) \tag{3.9}$$

holds for every measurable function $g : \mathbb{R}^n \to [0, \infty)$ *and every measurable set* $E \subset D$.

Proof. (i) f is differentiable a.e. by Rademacher–Stepanoff's theorem because $L(x, f) < \infty$ a.e., see e.g. 3.1.9 in [Fe]. Since $l(x, f) > 0$ a.e. we have that

$$l(f'(x)) > 0 \quad \text{a.e.} \tag{3.10}$$

where, for a given linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, $l(T)$ denote the minimum of the modulus $|Tz|$ over all unit vectors $z \in \mathbb{R}^n$. Hence (3.8) follows from (3.10).

(ii) By the Ponomarev result for a.e. differentiable mappings f, the condition (3.8) is equivalent to (N^{-1})–property, see Theorem 1 in [Pon_2], p. 412.

(iii) By Lemma 3.3 $f|_{C_k}$, $k = 1, 2, \ldots$, is Lipschitzian where we may assume that C_k are mutually disjoint bounded Borel sets. By Kirszbraun's theorem, see 2.10.43 in [Fe], each $f|_{C_k}$ can be extended to a Lipschitz map of $\mathbb{R}^n$. Thus, (3.9) follows from 3.2.5 in [Fe] by countable additivity of integral, (3.4), the (N)–property and (ii).

3.11. Remark. As it is clear from the proof, the differentiability of f a.e. is equivalent to the condition

$$L(x,f) < \infty \quad \text{a.e.} \tag{3.12}$$

If f is differentiable a.e., then the conditions

$$l(x,f) > 0 \quad \text{a.e.} \tag{3.13}$$

(3.8), (3.10) and (N^{-1})–property are all equivalent.

3.14. Corollary. *A mapping $f : D \to \mathbb{R}^n$ is of finite metric distortion if and only if f is differentiable a.e. and has (N)– and (N^{-1})–properties.*

3.15. Corollary. *Compositions of FMD mappings are FMD mappings. In particular, the FMD mappings remain FMD mappings under compositions with nonconstant quasiregular mappings. The inverse mapping f^{-1} of a FMD homeomorphism f is a FMD homeomorphism.*

Denote by $J(D)$ the collection of all subdomains G of D such that $\overline{G} \subset D$ is compact. Given a mapping $f : D \to \mathbb{R}^n$, $G \in J(D)$ and $y \in \mathbb{R}^n \setminus f(\partial G)$, let $\mu(y,f,G)$ is the **topological index** of the triple (y,f,G), see e.g. [RR], [Re$_1$] and [FG]. A mapping $f : D \to \mathbb{R}^n$ is said to be sense–preserving (sense–reversing) if $\mu(y,f,G) > 0$ ($\mu(y,f,G) < 0$) for all such triples (y,f,G) with $y \in f(G) \setminus f(\partial G)$. It is well–known that, if f is one–to–one, then f is either sense–preserving or sense–reversing and, moreover, $\mu(y,f,G) \equiv 1$ or $\mu(y,f,G) \equiv -1$, respectively for all $y \in fG$, see e.g. [RR], p. 133–134.

The **branch set** B_f of a mapping $f : D \to \mathbb{R}^n$ is the set of all points in $D \subset \mathbb{R}^n$ at which f fails to be a local homeomorphism. Note that the topological dimension of the branch set B_f for a discrete and open mapping f is not more than $n-2$ by the Chernavskii theorem, see [Ch$_1$] and [Ch$_2$]. Hence B_f does not separate D by the Menger–Urysohn theorem, see e.g. [HW], p. 48. Thus, every discrete and open mapping is either sense–preserving or sense–reversing, cf. also Theorem 9 in [TY], p. 336. By Lemma 2.14 in [MRV] every discrete, open and sense–preserving mapping f satisfies $J(x,f) = 0$ whenever $x \in B_f$ is a point of differentiability of f and hence we have the following consequence of (i) in Proposition 3.7.

3.16. Proposition. *For every discrete open FMD mapping $f : D \to \mathbb{R}^n$,*

$$|B_f| = |f(B_f)| = |f^{-1}(f(B_f))| = 0. \tag{3.17}$$

It is well-known that every light sense–preserving mapping is open and discrete, see [TY], p. 333. Consequently, we have the next, see [RR], p. 333.

3.18. Corollary. *If an FMD mapping $f : D \to \mathbb{R}^n$ is light and sense–preserving, then*

$$J(x,f) > 0 \quad \text{a.e.} \tag{3.19}$$

3.20. Lemma. *Let $f : D \to \mathbb{R}^n$ be an FMD mapping. Then there is a countable collection of compact sets $C_k^* \subset D$ such that $|B| = 0$ where $B = D \setminus \bigcup_{k=1}^{\infty} C_k^*$ and $f|_{C_k^*}$ is one–to–one and bi–lipschitz for every $k = 1, 2, \ldots$.*

Proof. First, let C_k are as in Lemma 3.3. By Proposition 3.7(i) and regularity of Lebesgue's measure we can always replace C_k by compact sets where f is differentiable, $J(x, f) \neq 0$ and $l(x, f) > 0$, see Remark 3.11. Then every C_k is the union of countable collection of Borel sets where f is one–to–one, see [EG], p. 94. Hence there is a countable collection of Borel sets C_k' such that $f|_{C_k'}$ is one–to–one and Lipschitzian for every $k = 1, 2, \ldots$ and $|D \setminus \bigcup_{k=1}^{\infty} C_k'| = 0$. By regularity of Lebesgue's measure we may assume that C_k' are compact sets. Finally, since $l(x, f) > 0$ on C_k' and hence $L(y, f_k^{-1}) < \infty$ on $f(C_k')$, we are able to apply Lemma 3.3 to f_k^{-1} where $f_k = f|_{C_k'}$ for deriving the bi–lipschitz property of $f|_{C_k^*}$.

4 (L)–property and ACL

In the final section, we will give examples showing that the (L)–property does not imply openness and discretness. Later on, we often apply the following simple remark. We recall that if a family of paths contains a degenerate path, then its modulus is infinite.

4.1. Remark. (a) If $f : D \to \mathbb{R}^n$ satisfies the (L_2)–condition, then $f^{-1}(y)$ cannot contain a nondegenerate path. We call a mapping with the latter property **weakly light.**

(b) If f is weakly light, then the correspondence $L_{\gamma,f} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ between the natural parameters of locally rectifiable paths γ and $\tilde{\gamma} = f \circ \gamma$ is a homeomorphism.

In view of Remark 4.1, (L)–property implies **absolute continuity on paths,** abbr. **ACP**, i.e., $L_{\gamma,f}$ is absolutely continuous on closed subintervals of Δ_γ for a.e. path γ in D, see e.g. 2.10.13 in [Fe]. In particular, (L)–property implies **absolute continuity on arcs,** abbr. **ACA**, and **absolute continuity on lines,** abbr. **ACL**, because we have obvious inclusions:

$$ACP \subset ACA \subset ACL. \tag{4.2}$$

Now, let a mapping $f : D \to \mathbb{R}^n$ be weakly light. Then the following notion is well defined because of Remark 4.1. We say that f is **absolute continuity on paths in the inverse direction,** abbr. **ACP^{-1}**, if $L_{\gamma,f}^{-1}$ is absolutely continuous on closed subintervals of $\Delta_{\tilde{\gamma}}$ for a.e. path $\tilde{\gamma}$ in $f(D)$ and for each lifting γ of $\tilde{\gamma}$. By 2.10.13 in [Fe] the (L_2)–condition implies ACP^{-1}.

4.3. Proposition. *A mapping $f : D \to \mathbb{R}^n$ has the (L)–property if and only if f is weakly light and*

$$f \in ACP \cap ACP^{-1}. \tag{4.4}$$

Below $f \in$ACL^p means that the mapping $f : D \to \mathbb{R}^n$ is ACL and its partial derivatives belong to $L^p(D)$, $p \geq 1$.

4.5. Remark. It is known that :

1) if $f \in ACL^n$, then $f \in ACP$, see Theorem 3 in [Fu] and Theorem 28.2 in [Vä$_1$];

2) if $f \in ACL^p$, $p > n-1$, is an open mapping, then f is differentiable a.e. in D, see Lemma 3 in [Vä$_2$];

3) for every $n \geq 2$, there is a function $f \in ACL^n$ which is nowhere differentiable, see [Se], p. 371;

4) if $f \in W^{1,n}$ is open, then f has the (N)–property, see [MM];

5) there is a homeomorphism $f \in W^{1,p}$ for all $p < n$ which does not possess the (N)–property, see [Pon$_1$], p. 140.

Combining 1), 2) and 4) with Proposition 4.3 and Corollary 3.14 we have the folowing statement, see also [Ma], p. 9.

4.6. Theorem. *Let $f : D \to \mathbb{R}^n$ be a homeomorphism such that $f \in ACL^n$ and $f^{-1} \in ACL^n$. Then f is of finite length distortion.*

By [GI], a mapping $f : D \to \mathbb{R}^n$ is said to be of **finite distortion** if $f \in W^{1,n}_{loc}$ and

$$|f'(x)| \leq K(x) \cdot J(x,f) \qquad \text{a.e.} \tag{4.7}$$

for some finite valued function $K(x) : D \to [1, \infty)$. As it was established in [HK], if in addition f is a homeomorphism and $K \in L^{n-1}$, then $f^{-1} \in W^{1,n}$. Thus, we have the next conclusion.

4.8. Corollary. *Every homeomorphism of finite distortion with $K \in L^{n-1}_{loc}$ is a mapping of finite length distortion.*

In the literature, the condition $f \in W^{1,n}_{loc}$ is sometimes replaced by the weaker condition $f \in W^{1,1}_{loc}$, see e.g. [KO], [KKM$_1$] and [KKM$_2$].

4.9. Remark. By arguments to Lemma 6 in [Pol] and the above Heinonen–Koskela result, the generalization of the Poletskii lemma can be obtained:

1) if $f : D \to \mathbb{R}^n$ is a discrete and open mapping of finite distortion with $|B_f| = 0$ and

$$K \in L^{n-1}_{loc}, \tag{4.10}$$

then f is ACP^{-1};

2) by Remark 4.5, 1), a discrete open mapping f of finite distortion with $|B_f| = 0$ and $K \in L^{n-1}_{loc}$ has the (L)-property;

3) by Remarks 4.5, 2) and 4), and Corollary 3.14, a discrete and open mapping f of finite distortion with $|B_f| = 0$ and $K \in L^{n-1}_{loc}$ is of finite length distortion;

4) in view of [MV$_1$]–[MV$_2$], a mapping f of finite distortion with $|B_f| = 0$ and

$$K \in L^p_{loc}, \qquad p > n-1, \tag{4.11}$$

is of finite length distortion;

5) in particular, a mapping f of finite distortion with $|B_f| = 0$ and

$$K(x) \le Q(x) \in BMO_{loc} \tag{4.12}$$

is of finite length distortion.

4.13. Corollary. *Every nonconstant quasiregular mapping is a mapping of finite length distortion.*

5 Sobolev classes

Let a mapping $f : D \to \mathbb{R}^n$ be differentiable at a point $x \in D$. The **outer dilatation** of f at x is defined by

$$K_O(x,f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \ne 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{5.1}$$

and at the rest points we set $K_O(x,f) = \infty$. Similarly, the **inner dilatation** of f at x is defined as

$$K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f) \ne 0 \\ 1, & \text{if } f'(x) = 0 \end{cases} \tag{5.2}$$

and $K_O(x,f) = \infty$ at the rest points. The **maximal dilatation**, or in short the **dilatation**, of f at x is

$$K(x,f) = \max(K_O(x), K_I(x)), \tag{5.3}$$

cf. [MRV]. Here $f'(x)$ denotes the Jacobian matrix of f and $J(x,f)$ is its determinant. Note that, say by 1.2.1 in [Re$_1$],

$$K_I(x) \le K_O(x)^{n-1} \tag{5.4}$$

and

$$K_O(x) \le K_I(x)^{n-1}, \tag{5.5}$$

and hence $K_O(x), K_I(x)$ and $K(x)$ are simultaneously finite or infinite.

5.6. Lemma. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion. If $y \mapsto N(y,f,C)$ is integrable in $\mathbb{R}^n$ for every compact set C in D and if*

$$K_O(x,f) \in L^q_{loc}, \quad q \ge 1/(n-1), \tag{5.7}$$

then $f \in W^{1,s}_{loc}$ where $s = nq/(1+q)$.

Proof. Since f satisfies (L_1), f is ACL and it suffices to establish the local L^s–inequality of the partial derivative $\partial_i f$, $i = 1, 2, \ldots, n$. For this let C be a compact set in D. We obtain from (3.9) with $g \equiv 1$ and from the Hölder inequality that

$$\begin{aligned} \|\partial_i f\|_s &\leq \|K_O^{1/n}(x,f)\|_p \cdot \|J^{1/n}(x,f)\|_n \\ &\leq \|K_O(x,f)\|_q^{1/n} \cdot \left(\int\limits_{\mathbb{R}^n} N(y,f,C)dm(y) \right)^{1/n} \end{aligned} \tag{5.8}$$

where $1/s = 1/n + 1/p$ and $p = qn$.

In view of (iii) in Proposition 3.7 the Jacobian $J(x,f)$ of a FMD mapping f is locally integrable in D if and only if the multiplicity function $N(y,f,C)$ is integrable for every compact set C in D. Thus, by Theorem 1.3 in [KKM$_1$], p. 137, see also Lemma 3.1 in [KKM$_2$], p. 174, and Corollary 3.14 we obtain the following.

5.9. Corollary. *If in addition f is sense–preserving and $K_O(x,f) \in BMO_{loc}$, then f is discrete and open.*

A discrete open mapping $f : D \to \mathbb{R}^n$ has the bounded multiplicity $N(y,f,C)$ on every compact set $C \subset D$.

5.10. Corollary. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion. If f is discrete and open and*

$$K_O(x,f) \leq K < \infty \quad \text{a.e.,} \tag{5.11}$$

then f is a quasiregular mapping. For $n \geq 3$ and $K = 1$, the mapping f is a Möbius transformation.

6 The uniqueness theorem

We use Corollary 5.10 to establish uniqueness results for FLD mappings with prescribed characteristics. We employ the normalized Jacobian matrix

$$M_f(x) = f'(x)/|J(x,f)|^{1/n} \tag{6.1}$$

and the symmetrized normalized Jacobian matrix

$$G_f(x) = M_f^*(x)M_f(x) \tag{6.2}$$

where $M_f^*(x)$ denotes the transpose of $M_f(x)$. Note that $M_f(x)$ of f is defined by (6.1) a.e. for a mapping f of FMD. We set $M_f(x) = I =$ identity at the rest of the points in D. We call $M_f(x)$ and $G_f(x)$ the **matrix dilatation** and the **dilatation tensor** of the mapping f at x, respectively, see [A$_1$], [A$_2$].

If a FMD mapping f is sense–preserving, then $M_f(x)$ is a **unimodular** matrix, i.e., $\det M_f(x) = 1$ for all $x \in D$. Moreover, if f and g are FMD mappings and if $f \circ g$ is well–defined, then the composition rule

$$M_{f\circ g}(x) = M_f(g(x))M_g(x) \tag{6.3}$$

holds a.e. The dilatation tensor $G_f(x)$ of the mapping f at x is symmetric, positive definite and unimodular.

The **special linear group** $SL(n)$ is the multiplicative group of all $n \times n$ unimodular matrices over $\mathbb{R}$. The collection of all the symmetric positive definite matrices of $SL(n)$ is denoted by $S(n)$. Note that $S(n), n \geq 2$, is not a group because the product of symmetric matrices need not be symmetric (see [Bell], p. 24). For a matrix $M \in SL(n)$, set

$$G = M^* M \in S(n) \tag{6.4}$$

where M^* is the transpose of M. The matrix G is called the **symmetrization** of M. The following statement follows immediatelly from the definition of the symmetrization.

6.5. Proposition. *Let M_1 and $M_2 \in SL(n)$, $n \geq 2$, and let G_1 and G_2 be the symmetrizations of M_1 and M_2, respectively. Then $G_1 = G_2$ if and only if $M_2 = UM_1$ where $U \in \mathcal{O}^+(n)$.*

Here $\mathcal{O}^+(n)$ denotes the group of $n \times n$ orthogonal matrices U, defined by $U^*U = I = UU^*$, with determinant equal to 1. In other words, G determines the corresponding M from (6.4) up to left rotations.

6.6. Lemma. *Let f, $g : D \to \mathbb{R}^n$, $n \geq 3$, be sense–preserving homeomorphisms of finite length distortion. If*

$$G_f(x) = G_g(x) \qquad \text{a.e.,} \tag{6.7}$$

then

$$f = h \circ g \tag{6.8}$$

where h is a Möbius transformation of $\overline{\mathbb{R}^n}$.

Proof. Indeed, $h = f \circ g^{-1}$ is a FLD homeomorphism. By Proposition 6.5 $M_f(x) = U(x)M_g(x)$ with $U(x) \in \mathcal{O}^+(n)$ a.e. and hence by the chain rule (6.3) and (N)– and (N^{-1})–properties we have

$$M_h(x) = U(g^{-1}(y)) \in \mathcal{O}^+(n) \qquad \text{a.e.,} \tag{6.9}$$

i.e., $K_h(y) = 1$ a.e. By Corollary 5.10, h is a Möbius transformation.

Note that, for $n = 2$, Lemma 6.6 is valid with a conformal mapping h.

6.10. Theorem. *Suppose that discrete and open mappings f, $g : D \to \mathbb{R}^n$, $n \geq 3$, are of finite length distortion. If f and g have a.e. equal dilatation tensors, then $f = h \circ g$ where h is a Möbius transformation of $\overline{\mathbb{R}^n}$.*

Proof. By the Chernavskii theorem $\dim (B_f \cup B_g) \leq n - 2$, see [Ch$_1$] and [Ch$_2$], and then $\Omega = D \setminus (B_f \cup B_g)$ is a domain by the Menger-Urysohn theorem, see e.g. [HW], p. 48. Arguing locally in Ω we obtain from Lemma 6.6 that $f|_\Omega = h \circ g|_\Omega$ where h is a Möbius transformation of $\overline{\mathbb{R}^n}$ and by continuity this holds in D because Ω is everywhere dense in D.

7 FLD and Q-mappings

The following theorem extends the so–called K_0–inequality in the theory of quasiregular mappings to FLD mappings, cf. [MRV], p. 16, [Ri], p. 31, [Vu], p. 130, and [KO].

7.1. Theorem. *Let a mapping $f : D \to \mathbb{R}^n$ be of finite length distortion and let $E \subset D$ be a measurable set. Then*

$$M(\Gamma) \le \int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y) \ dm(y) \tag{7.2}$$

for every path family Γ in E and $\rho_* \in adm\, f\Gamma$ where

$$K_I(y, f^{-1}, E) = \sum_{x \in E \cap f^{-1}(y)} K_O(x, f). \tag{7.3}$$

In particular, for $E = D$ we have

$$K_I(y, f^{-1}, D) = K_I(y, f^{-1}) = \sum_{x \in f^{-1}(y)} K_O(x, f). \tag{7.4}$$

Proof. We may assume that $\rho_* \equiv 0$ outside of $f(E)$. Let C and C_k^*, $k = 1, 2, \ldots$, be as in Lemma 3.20. Setting by induction $B_0 = C$, $B_1 = C_1^*$, $B_2 = C_2^* \setminus B_1, \ldots$, and

$$B_k = C_k^* \setminus \bigcup_{l=1}^{k-1} B_l \tag{7.5}$$

we obtain the countable covering of D consisting of mutually disjoint Borel sets $B_k, k = 0, 1, 2, \ldots$ with $|B_0| = 0$. Then by Remark 2.5 $l\,(\gamma \cap B_0) = 0$ for a.e. $\gamma \in \Gamma$ and hence by (L)–property $l\,(f \circ \gamma \cap f(B_0)) = 0$ also for a.e. $\gamma \in \Gamma$.

Given $\rho_* \in \text{adm}\, f\Gamma$, set

$$\rho(x) = \begin{cases} \rho_*(f(x))|f'(x)|, & \text{for } x \in D \setminus B_0, \\ 0, & \text{otherwise.} \end{cases} \tag{7.6}$$

Arguing piecewise on B_k, we have by 3.2.5 for $m = 1$ in [Fe] and the additivity of integral that

$$\int\limits_{\gamma} \rho ds \ge \int\limits_{f \circ \gamma} \rho_* ds \ge 1 \tag{7.7}$$

for a.e. $\gamma \in \Gamma$, i.e., $\rho \in \text{adm}\, \Gamma$. Therefore,

$$M(\Gamma) \le \int\limits_{D} \rho^n(x) dm(x). \tag{7.8}$$

Note that $\rho = \sum\limits_{k=1}^{\infty} \rho_k$ where $\rho_k = \rho \cdot \chi_{B_k}$ have mutually disjoint supports. By 3.2.5 for $m = n$ in [Fe] we obtain that

$$\int\limits_{f(B_k \cap E)} K_O(f_k^{-1}(y), f) \cdot \rho_*^n(y) \ dm(y) = \int\limits_{D} \rho_k^n(x) \ dm(x) \tag{7.9}$$

where every $f_k = f|_{B_k}$, $k = 1, 2, \ldots$ is injective by the construction.

Finally, by the Lebesgue positive convergence theorem, see e.g. [Sa], p. 27, we conclude from (7.8) and (7.9) that

$$\int\limits_{f(E)} K_I(y, f^{-1}, E) \cdot \rho_*^n(y)\ \ dm(y) = \int\limits_D \sum_{k=1}^{\infty} \rho_k^n(x)\ \ dm(x) \geq M(\Gamma).$$

The next inequality is a generalized form of the K_I–inequality which is also known as Poletskii's inequality, cf. [Pol], [Ri], p. 49–51, and [Vu], p. 131.

7.10. Theorem. *Let $f : D \to \mathbb{R}^n$ be an FLD mapping. Then*

$$M(f\Gamma) \leq \int\limits_D K_I(x, f) \cdot \rho^n(x)\ \ dm(x) \tag{7.11}$$

for every path family Γ in D and $\rho \in adm\,\Gamma$.

Proof. Let B_k, $k = 0, 1, 2, \ldots$, be given as above by (7.5). By the construction and (N)–property $|f(B_0)| = 0$. Thus, by Lemma 2.7 $l\,(\tilde\gamma \cap f(B_0)) = 0$ for a.e. $\tilde\gamma \in f\Gamma$ and hence by (L)–property $l\,(\gamma \cap B_0) = 0$ also for a.e. $\tilde\gamma \in f(\Gamma)$, $\tilde\gamma = f \circ \gamma$.

Let $\rho \in \text{ext adm}\,\Gamma$ and

$$\tilde\rho(y) = \sup_{x \in f^{-1}(y) \cap D \setminus B_0} \rho^*(x) \tag{7.12}$$

where

$$\rho^*(x) = \begin{cases} \rho(x)/l(f'(x)), & \text{for } x \in D \setminus B_0, \\ 0, & \text{otherwise.} \end{cases} \tag{7.13}$$

Note that $\tilde\rho = \sup \rho_k$ where

$$\rho_k(y) = \begin{cases} \rho^*(f_k^{-1}(y)), & \text{for } y \in f(B_k), \\ 0, & \text{otherwise,} \end{cases} \tag{7.14}$$

and every $f_k = f|_{B_k}$, $k = 1, 2, \ldots$ is injective. Thus, the function $\tilde\rho$ is measurable, see e.g. Theorem I(8.5) in [Sa], p. 15.

Arguing as in (7.7) we obtain that

$$\int\limits_{\tilde\gamma} \tilde\rho ds \geq \int\limits_{\gamma} \rho ds \geq 1 \tag{7.15}$$

for a.e. $\tilde\gamma = f \circ \gamma \in f\Gamma$ and, thus, $\tilde\rho \in \text{ext adm}\, f\Gamma$. Hence (2.11) yields

$$M(f\Gamma) \leq \int\limits_{f(D)} \tilde\rho^n(y) dm(y). \tag{7.16}$$

Moreover, by 3.2.5 for $m = n$ in [Fe] we have that

$$\int\limits_{B_k} K_I(x,f) \cdot \rho^n(x)\ dm(x) = \int\limits_{f(D)} \rho_k(y)\ dm(y). \tag{7.17}$$

Finally, by Lebesgue's theorem we obtain from (7.16) and (7.17) the desired inequality

$$\int\limits_{D} K_I(x,f)\cdot\rho^n(x)\ dm(x) = \sum_{k=1}^{\infty} \int\limits_{f(D)} \rho_k(y)\ dm(y) = \int\limits_{f(D)} \sum_{k=1}^{\infty} \rho_k(y)\ dm(y) \geq M(f\Gamma).$$

7.18. Remark. If $K_I(f) = \text{ess sup } K_I(x,f) < \infty$, then (7.11) yields the standard inequality by Poletskii:

$$M(f\Gamma) \leq K_I(f) \cdot M(\Gamma) \tag{7.19}$$

for every path family in D. If $K_O(f) = \text{ess sup } K_O(x,f) < \infty$ and E is a Borel set with $N(f,E) < \infty$, then we have from (7.2) the usual form of the K_O–inequality:

$$M(\Gamma) \leq N(f,E) \cdot K_O(f) \cdot M(f\Gamma) \tag{7.20}$$

for every path family in E.

Finally, combinig Theorems 7.1 and 7.10 we obtain.

7.21. Corollary. *Every FLD mapping f is a Q-mapping with*

$$Q(x,y) = (K_I(x,f), K_I(y,f^{-1})) \tag{7.22}$$

where $K_I(y,f^{-1})$ is defined in (7.4).

8 On FLD homeomorphisms

By Corollary 7.21 every FLD homeomorphism f is a $Q(x)$–homeomorphism with $Q(x) = K_I(x,f)$. The theory of Q–homeomorphisms in [MRSY$_1$]–[MRSY$_2$] yield several results for FLD homeomorphisms.

8.1. Theorem. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a FLD homeomorphism. If*

$$\limsup_{r\to 0} \frac{1}{|B^n(r)|} \int\limits_{B^n(r)} K_I(x,f) dm(x) < \infty, \tag{8.2}$$

then f has an extension to $\mathbb{B}^n$ which is a FLD homeomorphism.

8.3. Theorem. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a FLD homeomorphism with*

$$K_I(x,f) \leq Q(x) \in BMO(\mathbb{B}^n \setminus \{0\}). \tag{8.4}$$

Then f has a FLD homeomorphic extension to $\mathbb{B}^n$.

8.5. Corollary. *Isolated points are removable for FLD homeomorphisms with $K_I(x,f) \le Q(x) \in BMO$ and, in particular, if $f:\mathbb{R}^n \to \mathbb{R}^n$, then f has a homeomorphic extension to $\overline{\mathbb{R}^n}$ and $f(\mathbb{R}^n) = \mathbb{R}^n$.*

The removability theorems for the inverse mappings of FLD homeomorphisms can be formulated under much weaker conditions.

8.6. Proposition. *Let E be a nondegenerate continuum in $\mathbb{B}^n$. Then there exists no FLD homeomorphism of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$ with*

$$K_I(x,f) \in L^1(\mathbb{B}^n \setminus E). \tag{8.7}$$

The next theorem extends the Gehring–Martio result on the boundary correspondence from quasiconformal mappings to FLD homeomorphisms, cf. [GM], p. 196.

8.8. Theorem. *Let $f : D \to D'$ be a FLD homeomorphism between uniform domains D and D' with*

$$K_I(x,f) \le Q(x) \in BMO(D). \tag{8.9}$$

Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.

8.10. Corollary. *If a domain D in $\mathbb{R}^n$ is uniform but not a Jordan domain, then there is no FLD homeomorphism of D onto $\mathbb{B}^n$ with $K_I(x,f) \le Q(x) \in BMO(D)$.*

In the following we state some mapping properties of FLD homeomorphisms.

8.11. Theorem. *Let D be a domain in $\mathbb{R}^n, D \ne \mathbb{R}^n$, and $f : D \to \mathbb{R}^n$ a FLD homeomorphism. If there exist a point $b \in \partial D$ and a neighborhood U of b such that $K_I(x,f)|_{D\cap U} \in L^1$, then $f(D) \ne \mathbb{R}^n$.*

8.12. Theorem. *Let E be a nondegenerate continuum in $\mathbb{B}^n$, $D = \mathbb{B}^n \setminus E$, and let $f : D \to \mathbb{R}^n$ be a FLD homeomorphism. If there exist a point $x_0 \in \partial D \cap \mathbb{B}^n$ and a neighborhood U of x_0 such that $K_I(x,f)|_{D\cap U} \in L^1$, then $f(D)$ is not a punctured topological ball.*

Moreover, we have the following estimations of distortions under FLD homeomorphisms.

8.13. Theorem. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ be a FLD homeomorphism with $K_I(x,f) \in L^1(\mathbb{B}^n), f(0) = 0$, $q(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n)) \ge \delta > 0$ and $q(f(x_0), f(0)) \ge \delta$ for some $x_0 \in \mathbb{B}^n$. Then, for all $|x| < r = \min(|x_0|/2, 1-|x_0|)$,*

$$|f(x)| \ge \psi(|x|) \tag{8.14}$$

where $\psi(t)$ is a strictly increasing function with $\psi(0) = 0$ which depends only on the L^1-norm of K_I in $\mathbb{B}^n$, n and δ.

Similar upper estimates are possible in terms of $K_I(y, f^{-1})$. However, the estimates in terms of $K_I(x, f)$ are more interesting.

8.15. Theorem. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ a FLD homeomorphism with $K_I(x, f) \leq Q(x) \in BMO(\mathbb{B}^n)$. If $q(\overline{\mathbb{R}^n} \setminus f(B^n(1/e))) \geq \delta > 0$, then for all $|x| < e^{-2}$*

$$q(f(x), f(0)) \leq \frac{C}{(\log 1/|x|)^{\alpha}} \tag{8.16}$$

where C and α are positive constants which depend only on n, δ, the BMO norm $||Q||_*$ of Q and the average Q_1 of Q over the ball $|x| < 1/e$.

8.17. Corollary. *Let $\mathcal{F}$ be a family of FLD homeomorphisms $f : D \to \mathbb{R}^n$, with $K_I(x, f) \leq Q(x) \in BMO(D)$, and let $\delta > 0$. If every $f \in \mathcal{F}$ omits two points a_f and b_f in $\overline{\mathbb{R}^n}$ with $q(a_f, b_f) \geq \delta$, then the family $\mathcal{F}$ is equicontinuous.*

In view of Corollary 4.8, the corresponding consequences but in the terms of the maximal dilatation $K(x, f)$ can be formulated for homeomorphisms of finite distortion by Gehring–Iwaniec. Further consequences of Corollary 7.21 will be obtained in frames of the theory of Q–mapping and first of all Q–coverings.

9 Examples and subclasses

It is easy to give examples of FLD mappings which are not discrete, open or sense–preserving, see e.g. [SY] and [Ma]. Consider one more interesting mapping below.

Example. Let C be a Cantor set in $[0, 1]$. Then $(0, 1) \setminus C = \cup I_j$ where the intervals I_j are open and disjoint. Define $f : (0, 1) \to \mathbb{R}$ as $f(x) = 0$ for $x \in C$ and

$$f(x) = \frac{|b_j - a_j|}{2} - \left| x - \frac{b_j + a_j}{2} \right| \tag{9.1}$$

for $x \in I_j = (a_j, b_j)$, $j = 1, 2, \ldots$, $f(x) = x$ for $x \leq 0$ and $f(x) = x - 1$ for $x \geq 1$. Then f is a FLD mapping provided that $|C| = 0$. In this case the mapping f is length–preserving

$$l(f \circ \gamma) = l(\gamma) \qquad \text{for every } \gamma. \tag{9.2}$$

Moreover,

$$l(x, f) = L(x, f) = 1 \qquad \text{a.e.} \tag{9.3}$$

For $|C| > 0$, f is not a FLD mapping because $l(x, f) = 0$ for all $x \in C$.

The mapping $f : \mathbb{R} \to \mathbb{R}$ has the natural extension f^* to $\mathbb{R}^n$, $n \geq 2$,

$$f^*(z, x) = (z, f(x)), \;\; z \in \mathbb{R}^{n-1}, \;\; x \in \mathbb{R} \tag{9.4}$$

with similar properties. Note that f as well as f^* is neither discrete nor open as well as sense–preserving.

General considerations of length–preserving mappings in metric spaces can be found in [Bu]. However, the main results there deal with locally injective mappings.

9.5. Proposition. *Let $f : D \to \mathbb{R}^n$ be a mapping satisfying the condition*

$$L^{-1} \cdot l\,(\gamma) \leq l\,(f \circ \gamma) \leq L \cdot l\,(\gamma) \tag{9.6}$$

for some $L \geq 1$ and for every path γ in D and let

$$l(x, f) > 0 \qquad \text{a.e.} \tag{9.7}$$

Then f is of finite length distortion.

Proof. Indeed, (9.6) implies the (L)–property and the inequality

$$L(x, f) \leq L < \infty \qquad \text{for all } x \in D. \tag{9.8}$$

Since f is L–lipschitz, f clearly satisfies the (N)–property.

9.9. Remark. Note that the condition (9.6) implies that f is weakly light. If f is light,

$$L(x, f) < \infty \qquad \text{for all } x \in D \tag{9.10}$$

and $J(x, f) \geq 0$ a.e. in D or $J(x, f) \leq 0$ a.e. in D, then by Theorem 8 in [Cr] f is open. However, the above example shows that in general the mappings in Proposition 9.5 need not be discrete, open as well as sense–preserving and hence they are not of bounded length distortion in the sense of Martio–Väisälä [MV].

As it is clear from Remark 4.5, 1) and 3), and the Rademacher–Stepanoff theorem, the (L)–property does not imply $L(x, f) < \infty$ a.e. On the other hand, it is easy to give examples of FMD mappings without the (L)–property, see e.g. [Ge].

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