SOBOLEV EXTENSION DOMAINS

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In this doctoral thesis we study properties of Sobolev functions in irregular domains. The thesis consists of the following three independent papers:

(A) Traces and Sobolev extension domains, Reports of the Department of Mathematics, University of Helsinki, Preprint 263, August 2000.
(B) Sobolev extension domains on metric spaces of homogeneous type, to appear in Real Anal. Exchange.
(C) Maximal inequality in (s, m)-uniform domains, to appear in Ann. Acad. Sci. Fenn. Ser. A I Math.

Extension results for Sobolev functions are important tools in analysis. It is well known that if $\Omega \subset \mathbb{R}^n$ is a domain with a Lipschitz boundary, then it is a $W^{1,p}$-extension domain for all $1 \leq p \leq \infty$. That is, there exists a bounded linear extension operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

with $E|_\Omega u = u$ for all $u \in W^{1,p}(\Omega)$. By a Lipschitz boundary we mean that the boundary is locally the graph of a Lipschitz continuous function. This theorem was proved by A. P. Calderón in the case $1 < p < \infty$ [Cal61]. E. M. Stein extended it to the cases $p = 1, \infty$ [Ste70, Theorem 5, p. 181].

P. W. Jones generalized the result of Calderón and Stein to the locally uniform domains [MS79] or by the terminology of Jones to the (ε, δ) domains [Jon81, Theorem 1, p. 73]. He also proved that if $\Omega \subset \mathbb{R}^2$ is finitely connected, then it is a $W^{1,p}$-extension domain for all $1 \leq p \leq \infty$ if and only if it is locally uniform [Jon81, Theorem 3, p. 74].

For arbitrary $W^{1,p}$-extension domains necessary and sufficient (geometric) conditions are not known. The case $p \geq n$ is better understood than the case $1 \leq p < n$. P. Koskela proved that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, $p > n$, then it is locally quasiconvex [Kos90, Theorem 6.1, p. 27], so every pair of points in Ω can be joined by a curve whose length is comparable to the Euclidean distance between these points. He also showed that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, $n - 1 < p < \infty$, then there exists a constant $C > 0$ such that

$$|\Omega \cap B^n(x, r)| \geq C|B^n(x, r)|$$

for all $x \in \bar{\Omega}$ and all sufficiently small $r > 0$ [Kos90, Theorem 6.5, p. 28]. Here $|A|$ is the Lebesgue n-measure of a set A and $B^n(x,r)$ is the n-dimensional open ball centered at x and with a radius $r > 0$.

It is well known that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, $p > n$, then there exists an imbedding

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\bar{\Omega}). \tag{1}$$

Here $C^{0,1-\frac{n}{p}}(\bar{\Omega})$ is the space of continuous functions u in Ω with

$$\sup_{x \in \Omega} |u(x)| + \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x-y|} < \infty.$$

Koskela proved that if the imbedding (1) holds for $p > n$ then Ω is a $W^{1,q}$-extension domain for all $q > p$ [Kos98, Theorem A, p. 369]. This result almost gives a characterization for the Sobolev extension domain in the case $p > n$. He also showed that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,n}$-extension domain, then it is a $W^{1,q}$-extension domain for all $q > n$ [Kos98, Theorem B, p. 370]. For properties of the imbedding (1) we refer to the work of Koskela, F. Reitich and S. M. Buckley [BK96], [KR93].

The case $p = \infty$ goes back to H. Whitney [Whi34]. A bounded domain $\Omega \subset \mathbb{R}^n$ is a $W^{1,\infty}$-extension domain if and only if it is quasiconvex, see [Zob99]. V. G. Maz'ya has an example, a shark domain, which shows that in the plane a $W^{1,p}$-extension domain, $1 \leq p < 2$, is not necessarily quasiconvex [Maz85, 1.5.2, p. 73].

A. Jonsson and H. Wallin have widely studied traces of functions to proper subsets of $\mathbb{R}^n$, see [JW84]. Their results imply that if $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,p}$-extension domain, $n-1 < p < \infty$, and if its boundary $\partial\Omega$ is m-regular, $0 < m < n$, then there exists a trace operator from $W^{1,p}(\Omega)$ to a Besov space defined on the boundary. We say that the boundary of Ω is m-regular if there exists a constant $C > 0$ such that

$$C^{-1} r^m \leq H^m\big(B^n(x,r) \cap \partial\Omega\big) \leq C r^m$$

for every $x \in \partial\Omega$ and every $0 < r \leq \operatorname{diam}(\partial\Omega)$. Here H^m is the m-dimensional Hausdorff measure. We study the opposite direction in the paper (A). We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary is m-regular, $0 < m < n$. If there exists a trace operator from $W^{1,p}(\Omega)$, $1 < p < \infty$, to a Besov space defined on the boundary, and if Lipschitz continuous functions are dense in $W^{1,p}(\Omega)$, then the domain is a $W^{1,p}$-extension domain. Since Lipschitz continuous functions are dense in bounded Sobolev extension domains, our result gives a characterization for bounded extension domains in terms of traces. Note that in the plane extremely weak geometric conditions imply that Lipschitz continuous functions are dense in the Sobolev space, see [Lew87] and [SSS94].

The Sobolev spaces in metric spaces have been studied during the last ten years. Assume that (X,d,μ) is a metric measure space. P.

Hajłasz [Haj96] defined that $u \in L^p(X,\mu)$ belongs to the Sobolev space $M^{1,p}(X,d,\mu)$, $1 \le p \le \infty$, if there exists a non-negative $g \in L^p(X,\mu)$ such that

$$|u(x)-u(y)| \le d(x,y)\big(g(x)+g(y)\big)$$

for μ-almost every $x, y \in X$. Here g is a generalized gradient of u. In the Euclidean case we just write $M^{1,p}(\Omega)$ without a metric and a measure. If $1 < p \le \infty$ and $\Omega \subset \mathbb{R}^n$ is an arbitrary domain, then $M^{1,p}(\Omega) \subset W^{1,p}(\Omega)$. If $1 < p \le \infty$ and if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain or $\Omega = \mathbb{R}^n$, then the classical Sobolev space $W^{1,p}(\Omega)$ coincides with $M^{1,p}(\Omega)$ and $g = M\nabla u$. Here $M\nabla u$ is the Hardy-Littlewood maximal operator of the weak gradient of u. In the case $p = 1$ this does not hold since the Hardy-Littlewood maximal operator is not bounded in $L^1(\Omega)$.

Hajłasz and O. Martio have studied in [HM97] traces of Sobolev functions using the theory of Sobolev spaces on metric spaces. They proved that if a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies a weak geometric condition, called the $A(c)$-condition, then there exists a linear bounded extension operator

$$E : M^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

with $Eu|_\Omega = u$ for all $u \in M^{1,p}(\Omega)$ [HM97, Theorem 9, p. 240]. They showed that in a bounded uniform domain $\Omega \subset \mathbb{R}^n$ the spaces $W^{1,p}(\Omega)$ and $M^{1,p}(\Omega)$ coincide for $1 < p \le \infty$ [HM97, Lemma 14, p. 243]. This gives a new proof for Jones' extension theorem.

In the paper (B) we study extension properties of Sobolev functions on metric measure spaces of homogeneous type. Let (X,d,μ) be a metric measure space with a doubling measure μ with $\mu(X) < \infty$. We define a new geometric condition called the $A^*(\varepsilon,\delta)$-condition which in the Euclidean case is weaker than the $A(c)$-condition of Hajłasz and Martio [HM97, p. 225]. Using the method of Hajłasz and Martio we prove that if $\Omega \subset X$ is a domain which satisfies the $A^*(\varepsilon,\delta)$-condition and $\mu(\partial\Omega) = 0$, then there exists a bounded linear extension operator

$$E : M^{1,p}(\Omega,d,\mu) \to M^{1,p}(X,d,\mu)$$

such that $Eu|_\Omega = u$ for all $u \in M^{1,p}(\Omega,d,\mu)$. In the Euclidean case we may assume that $X = \mathbb{R}^n$ and we obtain the result of Hajłasz and Martio [HM97, Theorem 10, p. 242].

In the paper (C) we study geometric conditions on bounded domains $\Omega \subset \mathbb{R}^n$ which guarantee that there exist a constant $C > 0$ and an exponent $0 < \alpha \le 1$ such that all $u \in W^{1,p}(\Omega)$ satisfy the inequality

$$|u(x)-u(y)| \le C|x-y|^\alpha \big(M\nabla u(x) + M\nabla u(y)\big) \tag{2}$$

for almost all $x, y \in \Omega$. A variant of the inequality (2) has been studied by R. A. DeVore, R. C. Sarpley [DS84], Hajłasz [Haj95] and Martio [HM97]. We define a new class of bounded domains called (s,m)-uniform domains, $s \ge 1$ and $0 < m \le 1$. The special case $s = m = 1$ is the class of bounded uniform domains defined by Martio and J. Sarvas [MS79]. A class of bounded domains whose boundary is locally the graph of a λ-Hölder continuous function, $0 < \lambda \leq 1$, forms a proper subclass of $(\frac{1}{\lambda}, \lambda)$-uniform domains. We prove that if $\Omega \subset \mathbb{R}^n$ is a (s, m)-uniform domain, $1 \leq s < \frac{n}{n-1}$, then all $u \in W^{1,p}(\Omega)$ satisfy the inequality (2) with the exponent $\alpha = \frac{m}{s}\big(n - s(n-1)\big) < 1$. We calculate an upper bound for the exponent α in the class of (s, m)-uniform domains: if $1 \leq s < \frac{n}{n-1}$, then the exponent has to be less than or equal to $\frac{s(n-1)+1}{n}\big(n - s(n-1)\big) < 1$, and if $s \geq \frac{s}{s-1}$, then the inequality (2) does not hold with any exponent $\alpha > 0$ for all $1 < p < \infty$.

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