Look–down equivalence without BPI equivalence

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Abstract

We prove that there exist BPI spaces F and F' with a weak (1,1)-Poincaré inequality that are David–Semmes -regular images of each other but do not contain a set of positive measure that could be mapped to the other space by a bilipschitz mapping. In particular, they are look–down equivalent but not BPI equivalent.

0. Introduction

In [DS] a general class of self–similar spaces, so called *BPI spaces* was introduced. Let us begin with definitions.

Definition 0.1 Let X be a complete metric space. We say that X is *Ahlfors regular* with dimension d if there exists a Borel regular measure m in X and a constant C such that for any ball $B(x,r)$ we have $r^d/C \leq m(B(x,r)) \leq Cr^d$, when $r \leq \mathrm{diam}(X)$. Measure m as above we call also Ahlfors regular. We use also shorter expression Ahlfors d–regular for space or measure to indicate the dimension.

It is easy to see that any Ahlfors regular measure is comparable to the Hausdorff measure of the same dimension. We sometimes shorten the expression Hausdorff d-measure to d-measure and use also symbol H^d.

Definition 0.2 We say that an Ahlfors d–regular space is a *BPI space* if for any two balls $B(x,r)$ and $B(y,s)$ there exists $A \subset B(x,r)$ with $H^d(A) \geq r^d/C$ that one can bilipschitz embed to $B(y,s)$ after scaling the metric in $B(y,s)$ such that it becomes same–sized as $B(x,r)$. The constant C and bilipschitz constant should be independent of the balls.

Definition 0.3 Two BPI spaces of dimension d are *BPI equivalent* if there exists a set of positive d-measure that one can bilipschitz embed from one space to the other. They are *look–down equivalent* if both of them contain sets of positive d-measure that can be mapped to the other space with a Lipschitz map such that the image has positive d-measure.

In [DS] it was asked whether look–down equivalence of BPI–spaces implies BPI equivalence. We show in this paper that it does not, not even in spaces with Poincaré inequality. One of the spaces we use comes directly from [L1] and the other is a modification of the construction presented there. These two spaces are look–down equivalent and even David–Semmes -regular images of each other, but still not BPI equivalent. Recall that David–Semmes regular mappings are natural finite–to–one analogs of bilipschitz mappings.

Definition 0.4. A mapping $f : X \to Y$ between metric spaces is called *David–Semmes regular with data* (L, M, N) if f is L–Lipschitz and if for each closed ball B in Y the preimage $f^{-1}(B)$ can be covered by M closed balls in X each of diameter at most N times the diameter of B. We abbreviate this to *DS–regular.* In [DS] they used the term *regular mapping* for the same thing.

It was proved in [L1] that spaces constructed there do not admit bilipschitz embeddings to any R^n. There will be a sequel of this paper showing that there also exist two simple tree-like fractals in the Euclidean plane that are look–down equivalent but not BPI equivalent. The constructions and proofs are easier in the sequel than here, so for readers that are not interested in Poincaré inequalities in this connection it might be good to read that paper instead of this one.

1. Construction

Construction. Let $I = [0, 1]$ be the unit interval. Denote by K the usual Cantor set in the real line with scale factor 5, i.e. $K = K/5 \cup (K/5 + 4/5)$. Set $K_0 = K/5$ and $K_1 = K/5 + 4/5$. Denote by K_{00} and K_{01} the similar parts of K_0, i.e. $5^{-2}K$ and $(K/5 + 4/5)/5$ respectively. Continue this naming inductively. For K_a, $|a|$ denotes the length of the binary string a and $a1$ denotes concatenation of the strings a and "1".

Now $K \times I$ consists of 10 natural pieces that are isometric to the whole after scaling with factor 5. Call them first order similarity parts. Their analogous parts are then second order similarity parts and so on. Natural similarity mappings to nth order similarity parts we call nth order similarity mappings. Let h be the natural projection $K \times I \to I$ and call it *height.* **Warning!** In [L1] we talked about $I \times K$ instead. The height denoted the projection to I there as well and this caused some confusion. We changed the order of symbols to $K \times I$ here in order to make the name height to be as natural as possible and to avoid confusion.

Identify each boundary of a first order similarity part in $K_0 \times I$ to the corresponding boundary in the same height in $K_1 \times I$ such that any point (x, y) in the boundaries in $K_0 \times I$ will be identified to $(x + 4/5, y)$. We take the boundaries with respect to $K \times I$, so we get no identifications at heights 0 and 1. Let the natural self–similarities of $K \times I$ carry the identifications to the interior of the similarity parts of all orders. In other words, we identify also (u, z) and (v, z) if they are images of some (x, y) and $(x + 4/5, y)$ as above under some nth order similarity mapping for some n. Call the resulting space F. Note that the boundaries of similarity parts consist of similarity images of the space at heights 0 and 1. Because we make no identifications at heights 0 and 1, identifications corresponding to different order similarity mappings will not happen at the same heights.

Since we only identified points at the same heights, the height stays well defined in F, too. Restriction of F to a height where boundaries of nth order similarity parts are identified we call *an nth order identification level*. In F we use the metric that is the infimum of lengths of paths connecting given points, where the length is measured with respect to change of height. Denote by s_F the projection $K \times I \to F$ that we used in the construction of F. We call s–images of nth order similarity parts of $K \times I$ by the same name, i.e. nth order similarity parts (of F). The corresponding similarity mappings carry to F, too, remaining isometries after scalings. Since F is symmetric with respect to height, i.e. there is an isometry that maps height 0 to height 1 and preserves the Cantor set direction, we can use also similarity mappings that are compositions of the natural similarity mappings with this height reversing isometry. We call them height direction reversing similarity mappings (of order n). Similarly the usual ones we call height direction preserving. There are many isometries of F, much more than in $K \times I$, and hence there are also many similarity mappings. We need only abovementioned natural similarity mappings. (The expression nth order similarity mapping refers to exactly those natural similarity mappings.)

We create a new space F' such that it has the same family of self similarities, but there is a modification in the overlapping of the boundaries of the similarity parts. To obtain F' we remove from F the lowest first order identification level. We also remove all images of that level under the following similarity mappings. For the lowest two first order similarity parts that are on the same height we use the height direction preserving similarity, and for the two parts above them use the height direction reversing similarity. We continue similarly upwards with changing between reversing and preserving in each step. So, in the two highest parts we have again height preserving isometry. We remove also all identifications that are images of the lowest first order level under some finite composition of the similarity mappings we used to the first order similarity parts. Then the result will be self–similar.

In the second order similarity parts also every other part in the height direction has a height preserving isometry and the rest have height reversing isometry to the whole, beginning with height preserving in the bottom. Similarly in the smaller parts.

Ahlfors–regularity of F and F' can be deduced as follows from the Ahlfors regularity of the push–forward measure coming through projections s_F and $s_{F'}$. After multiplication by a constant either one of the push-forwards measures gives measure 10^{-n} to any similarity part of order n. Note that such a similarity part has diameter 5^{-n}. It is easy to see that every ball contains a similarity part of comparable size and is contained in the union of four similarity parts of comparable diameter. Thus the measure of a ball $B(x, r)$ is comparable to $r^{1+ln(2)/ln(5)}$. Previous reasoning also shows together with the Ahlfors regularity that F and F' are BPI spaces, and even self–similar in a stronger way since they consist of 10 pieces isometric to the whole after scaling with factor 5.

Theorem 1.1. The BPI spaces F and F' are not BPI equivalent but still look–down equivalent; they are even DS–regular images of each other. Moreover, both admit a weak $(1,1)$–Poincaré inequality.

Two last chapters will be dedicated to proof of 1.1.

2. DS–regular mappings and Poincaré inequalities

Let us first prove that $s_F : K \times I \to F$ is DS–regular. Same argument works for $s_{F'} : K \times I \to F'$.

To prove the Lipschitz condition, let x and y be two points in $K \times I$. There exists K_a such that $\{x, y\} \subset K_a \times I$ and $\mathrm{diam}(K_a) \leq C|x-y|$. Now all the identification levels needed to pass from any point of K_a to any other are within the distance $5^{-|a|}2$ which is proportional to $\mathrm{diam}(K_a)$. A path that goes upwards from $s(x)$ of height $5^{-|a|}2$ using all necessary levels to change side and then continues solely upwards or solely downwards to $s(y)$ has length not more than $C|x-y|$ in F.

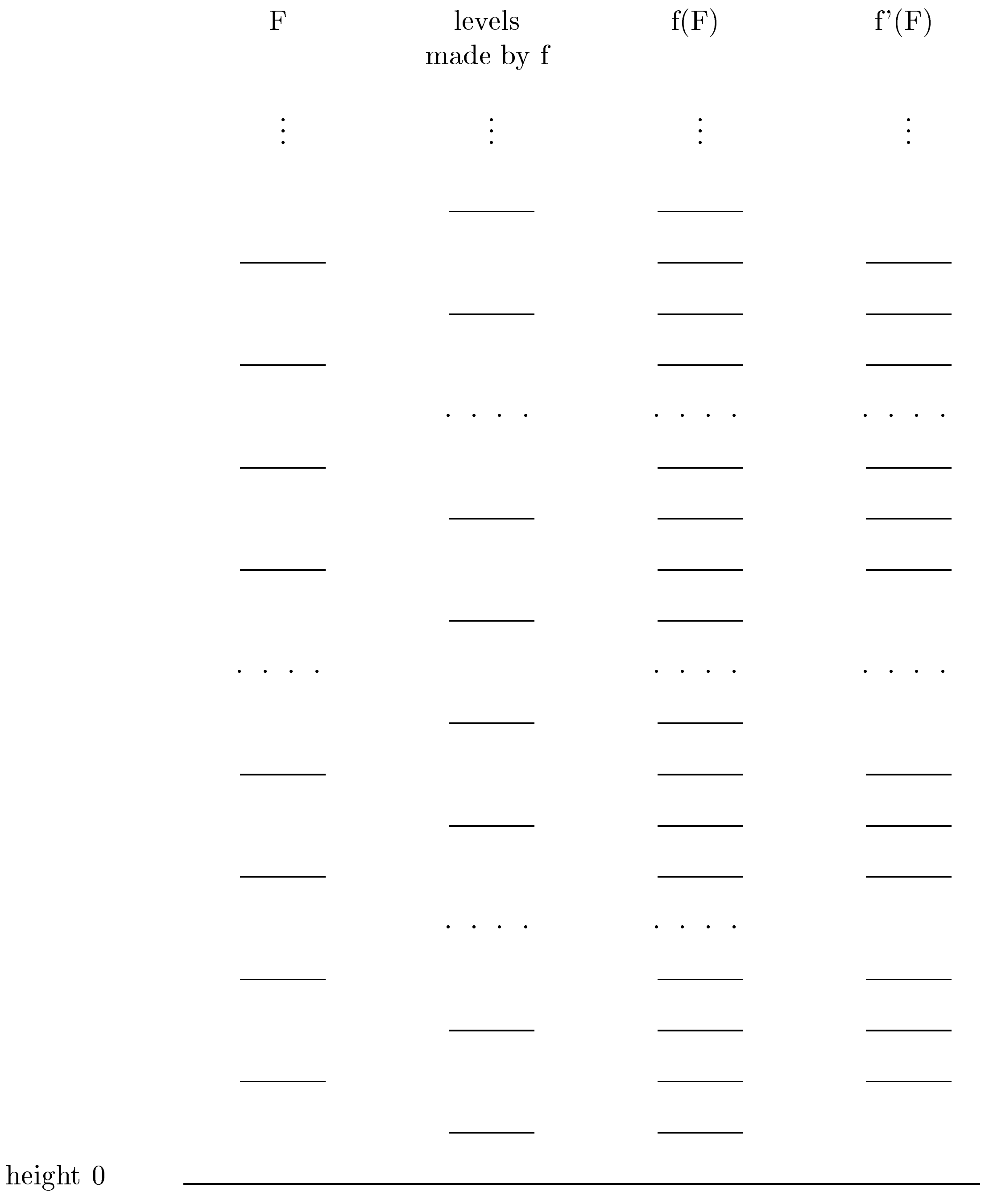
Next consider $s^{-1}(B(x,r))$. We can assume $r \leq \mathrm{diam}(F)$. Let m be such that $5^{-m} > 2r \geq 5^{-m-1}$. The minimal distance between identification levels of orders from 1 to $m+1$ is 5^{-m}, so $B(x,r)$ can reach only one identification level of those orders. On the other hand, identification levels of greater orders can only decrease distances of amount $\sum_{k=m+1}^{\infty} 5^{-k}$, because passing twice through identification levels of the same order does not help at all. The preimage $s^{-1}(B(x,r))$ is therefore contained in at most two balls of radius $r + 5^{-m-1}/4 \leq Cr$.

It is quite clear that we have a DS–regular mapping m from F' onto F. This mapping only puts back the removed identifications, and is distance non-increasing. This fact together with DS-regularity of the projection $s_F : K \times I \to F$ and Lipschitzness of $s_{F'}$ implies that also m is DS-regular.

The other direction is treated as follows. Consider first a projection mapping f that identifies some distinct points of F such that $f(F)$ has in addition to the identification levels of F an another set of identification levels that is similar to the original one, but shifted upwards of a height $1/2$. We understand this upward shift in a modular way. This will say that if there is an original level at height q, then there exists in $f(F)$ a new level of the same order at height $[q+1/2]_{Mod\ 1}$. See Picture 3. We naturally use similar path metric in $f(F)$ as in F. Note that the shift $1/2$ can be written in a form $5^{-k}/2 + \sum_{j=1}^{k} 2/5^j$. Because the $k+1$th order identification level pattern repeates itself with period 5^{-k}, only the last term matters for $k+1$th order level pattern. This remaining shift is also half of the period for each k. So the Picture 3 covers the pattern for every k. Thus f induces to the image a similar identification level pattern as F has with double density with respect to height. (See Picture 3.) (This shows in particular that new and old identification levels cannot overlap.) In $f(F)$ every other cycle of identification levels of order k has the lowest level induced by f, and the rest have the highest level induced by f. This pattern begins for all k such that the lowest period has the lowest level induced by f. See Picture 3. A mapping f' that makes the same new identification levels as f except those just mentioned induces to $f'(F)$ a similar identification level pattern as F'

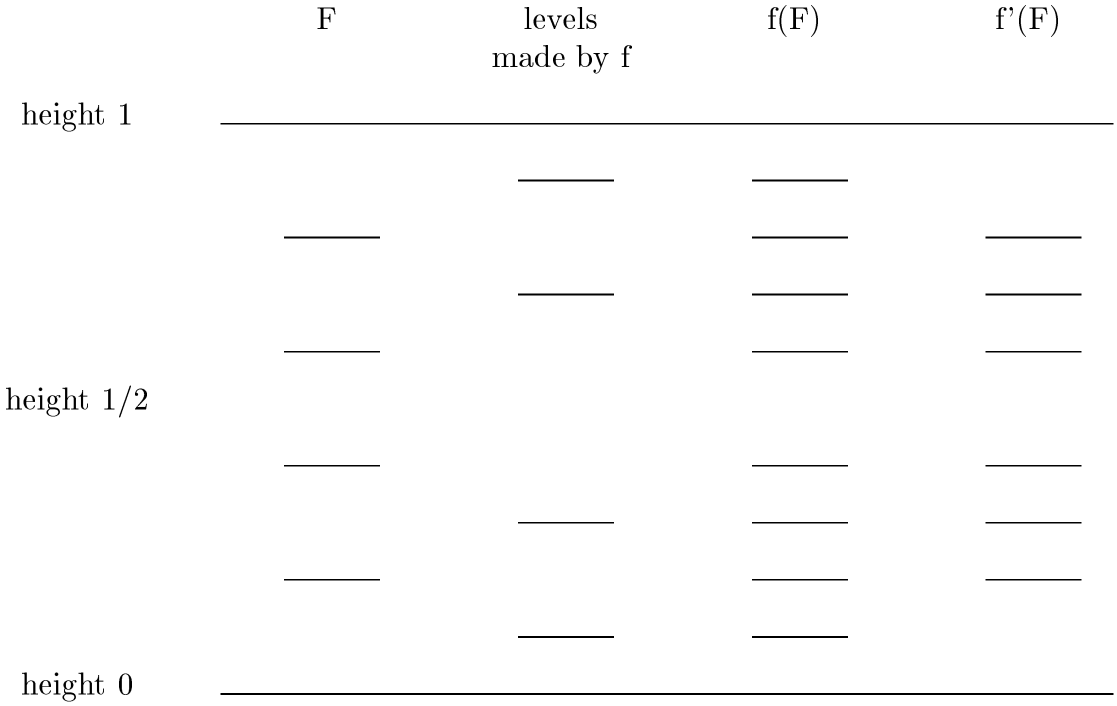
has. See Picture 3. (Observe for the picture that also for levels of order less than k the pattern of F' omits the first level of $f(F)$)

Picture 3: Identification levels of order k
(Horizontal dotted lines mark identification levels of some order less than k.)



To see that the projection f' is DS-regular, observe that it is distance non-increasing and that the other part of DS–regularity can be proved just as for s_F. Now the double density of the level system of $f'(F)$ makes every similarity part of F into two similarity parts of F'. "Similarity part" meaning here that the parts are isometric to F' after scaling. Every other part of fixed order admits height direction preserving and the rest admit height direction reversing similarity mapping to F'. We understand "every other" naturally with respect to height. Especially, the entire space becomes two similarity parts. See picture 4. So a certain mapping that maps points at height t to height $2t$ in height interval $[0,1/2]$ and points at height t to height $2-2t$ in the height interval $[1/2,1]$ will map $f'(F)$ DS-regularly onto F'. To give a precise mapping, take first a mapping w of $I \times K$ to itself such that the height direction is mapped as above and the K direction is the identity. Any pair of points of $I \times K$ that is identified by $f' \circ s_F$ is also identified by $s_{F'} \circ w$, as we see from the matching of the identification levels. Hence $s_{F'} \circ w$ carries to a well defined mapping $w' : f'(F) \to F'$. From the previous discussion we get also that w' is also piecewise bilipschitz with two aforementioned pieces. Thus there also exists an onto DS-regular map $w' \circ f' : F \to F'$.

Picture 4: First order identification levels



Poincaré inequalities. The space called F here corresponds to the space called F in [L1] with parameter $t = 1/5$ and sequence $j_k = 5$ for all k. There we proved that F admits a weak (1,1)–Poincaré inequality in the sense of [HK]. Also F' can be treated similarly. To see this we recall the main properties that were used in proving the desired properties. In proving the Poincaré inequality we used Ahlfors regularity of F, DS-regularity and height–preserving of the projection s from $K \times I$, and also the fact that the maximal distance of identification levels of order k is at most $C5^{-k}$ which is true for F' too. In addition to this we used the fact that a geodesic has a height at least half of its length. In other words, the height range of a geodesic is at least half as long as the geodesic itself. [L, Proposition 1.2.] In proving the last fact we do not need to know the exact locations of identification levels, not even their density. We need only the fact that identification levels depend only on height and identify points of the same height. In other words, at a height of an identification level of order k we identify $K_{u0} \times \{h_0\}$ to $K_{u1} \times \{h_0\}$ for *any* u with $|u| = k-1$; everywhere at that height there is a possibility of "switching sides" of some K_u with $|u| = k-1$.

3. F and F' are not BPI equivalent

Definition 3.1. Let (X, d) be a compact metric space. We define the Hausdorff-metric for compact subsets of X as follows: If A is in ϵ-neighborhood of B and B is in ϵ-neighborhood of A, then their Hausdorff distance is less than or equal to ϵ. The precise distance is naturally the infimum over all such ϵ.

Definition 3.2. Let (X, d) and (Y, e) be compact metric spaces and let $A_n \subset X$ and let $\{A_n\}$ converge to $A \subset X$ in the Hausdorff-metric. Let $f_n : A_n \to Y$ be mappings. We say that $\{f_n\}$ converges to $f : A \to Y$ if for all sequences $\{x_n\}$ converging to some $x \in A$ such that $x_n \in A_n$ we have that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

The following lemma is well known and not very difficult to prove.

Lemma 3.3. Let (X, d) and (Y, e) be compact metric spaces and let $A_n \subset X$ and let $f_n : A_n \to Y$ be L-Lipschitz mappings for any positive integer n. Now for some subsequence n_k we have that $\{A_{n_k}\}$ converges to some A and $\{f_{n_k}\}$ converges to f that is also L-Lipschitz.

In the following we use bold font for the essence of the proof. The reader is adviced to read only the bold text and skip the non-bold, unless something remains unclear. The ordinary font gives more explanations. Also formulas that appear in non-bold sentences can be skipped. The large scale structure is as follows: First we prove that BPI-equivalence of F and F' implies the existence of a height preserving isometry from F' into F and deduce a contradiction from this by a direct geometric argument. To obtain the isometry we bring the infinitesimal behaviour of the original mapping back to the original spaces using self-similarity and iterate this a couple of times.

Assume that there exists $A \subset F'$ with $H^d(A) > 0$ and an L–bilipschitz $g : A \to F$. Here d denotes the common Hausdorff dimension of F and F'. Take a density point p of A. For $n = 1, 2, 3, \ldots$ let $\{u_n\}$ be a sequence converging to infinity. For each positive integer n choose a similarity part Σ_n containing p such that $u_n$diam(Σ_n) is comparable to 1 with the constants independent of n. Let σ_n be the corresponding similarity mapping $\sigma_n : F' \to \Sigma_n$. Since g is bilipschitz, $g(\Sigma_n \cap A)$ is contained in at most four similarity parts of F with diameters comparable to diam(Σ_n). More precisely, it is either contained in one

such similarity part or in four similarity parts that all meet at an identification level. In the first case we use the inverse δ_n of the natural similarity mapping, ie δ_n maps the similarity part onto F. In the second case we use the four natural similarity mappings to obtain mappings from the four similarity parts consistently onto suitable four second order similarity parts of F, that meet in the same second order identification level. We then denote by δ_n the piecewise defined mapping coming from these. We choose naturally height direction preserving mappings. We obtain a uniformly bilipschitz mapping $g_n : \sigma_n^{-1}(A) \to F$ defined as $g_n = \delta_n \circ g \circ \sigma_n$. Uniformly means that the bilipschitz constant does not depend on n. **Now because p was a density point, $\sigma_n^{-1}(A)$ converges to F' in the Hausdorff-metric. For a suitable subsequence also g_n converges and we obtain a bilipschitz $g' : F' \to F$ defined everywhere on F'.** To obtain the similarity parts containing $g(\Sigma_n \cap A)$, recall that the distance of successive identification levels of order at most n is 5^{-n}. So we choose biggest n such that 5^{-n} is more than the diameter of $g(\Sigma_n \cap A)$. If we take one similarity part of order n that overlaps with $g(\Sigma_n \cap A)$, former either contains the latter, or the latter is contained in the union of the similarity part and some of its neighbors such that the distance of their interiors is less than the diameter of $g(\Sigma_n \cap A)$. Since the boundary of a similarity part of order n is at two identification levels of order at most n, the containing similarity parts have to be four similarity parts of order n that meet in a single identification level of order at most n. Then there are four cases that also appear among the second order similarity parts. If the similarity parts are called S_1, S_2, S_3 and S_4 such that S_1 and S_2 are above S_3 and S_4, then it might be that S_1 and S_2 meet at not only one but at two successive identification levels of order at most n. The same possibility occurs also for S_3 and S_4. So we choose four second order similarity parts that match the behaviour of S_i and then use the similarity mappigs to map S_i onto those.

Consider next $\phi = h_F \circ g' \circ s_{F'}$ that is a real valued Lipschitz function in $K \times I$. When restricted to $\{k\} \times I$, it is differentiable H^1-almost everywhere in the ordinary sense by basic real analysis, and the differential is a linear mapping.

Fubini theorem in $K \times I$ implies that at almost every height r we have for H^{d-1}–almost every point $\{k\} \times \{r\} \in K \times \{r\}$ the differentiability of $\phi|_{\{k\}\times I}$ at $\{k\}\times\{r\}$. Let h_0 be such a height. By Egorovs theorem we can take a set $B \subset K$ of positive H^{d-1}–measure such that the convergence of the difference quotient is uniform in $B \times \{h_0\}$. In other words, for any $k \in B$ we have $\phi(\{k\} \times \{h_0 + t\}) = \phi(\{k\} \times \{h_0\}) + \beta_k t + r_k(t)$ and $|r_k(t)| < r(t)$ where $\lim_{t\to 0} r(t)/t = 0$.

Let z be a density point of $B \times \{h_0\}$ (with respect to H^{d-1}–measure). **Let $\{u_n\}$ be a sequence of scalings converging to infinity. As before we blow up around z and bring things back to the unit scale with self similarities, and pass to the limit. We obtain a limit $\phi' = h_F \circ g'' \circ s_{F'}$ for some bilipschitz $g'' : F' \to F$.** For any n take a similarity part T_n of $K \times I$ containing z with diameter comparable to u_n^{-1}. Let τ_n be the corresponding similarity mapping from $K \times I$ to T_n. Now $s_{F'}(T_n)$ is a similarity part of F' with comparable diameter. Let v_n be the corresponding similarity mapping from F' onto $s_{F'}(T_n)$. We have $\phi \circ \tau_n = h_F \circ g' \circ v_n \circ s_{F'}$. Since g' is bilipschitz, as before we obtain that $g'(s_{F'}(T_n))$ is contained in at most four similarity parts of F with diameters comparable to $\mathrm{diam}(T_n)$. As before we use the natural self-similarities to obtain a mapping q_n that maps $g'(s_{F'}(T_n))$ so that the diameter of its image will be comparable to 1 in F. So q_n is now a mapping that increases the scale by a factor comparable to u_n, but otherwise it preserves the metric. Thus $q_n \circ g' \circ v_n$ is bilipschitz with a constant independent of n. For a suitable subsequence n_m the previous sequence converges to a bilipschitz mapping. Then also $\phi_n = h_F \circ q_n \circ g' \circ v_n \circ s_{F'}$ converges to a Lipschitz mapping for the same subsequence.

Since z was a density point of $B \times \{h_0\}$, we get that the limit of the similarity images of $B \times I$ is entire $K \times I$. Moreover, every line in the latter is a limit of lines of the former from smaller and smaller neighborhoods of z. Since the convergence of the differentials of ϕ was uniform at $B \times \{h_0\}$, we get that the limit mapping ϕ' is also linear in every line $\{k\} \times I$. Say it is $a_k + b_k t$ in $\{k\} \times \{t\}$. For following notation see the previous paragraph. Since z was a density point of $B \times \{h_0\}$ and hence a density point of $B \times I$, we get that $\tau_n^{-1}(T_n \cap (B \times I))$ converges to $K \times I$ and that every line of $K \times I$ is a Hausdorff limit of τ_n^{-1}-images of lines of $B \times I$. In other words, every $\{k\} \times I \subset K \times I$ is a Hausdorff limit of $\{k_n\} \times I \subset \tau_n^{-1}(T_n \cap (B \times I))$. Denote by $\bar{k}_n$ the images of k_n under τ_n. For following notation see the paragraph before previous one. Since $\bar{k}_n \in B$ we have $\phi(\{\bar{k}_n\} \times \{h_0 + t\}) = \phi(\{\bar{k}_n\} \times \{h_0\}) + \beta_{\bar{k}_n} t + r_{\bar{k}_n}(t)$. Now τ_n maps I-component linearly, say $h(\tau_n(x)) = c_n h(x) + d_n$. For the definition of v_n see previous paragraph. Now ϕ_n that is by definition $h_F \circ q_n \circ g' \circ v_n \circ s_{F'}$ is equal to $\theta_n \circ \phi \circ \tau_n$, where θ_n is a linear map from R^1 onto R^1, because q_n changes the height linearly. The last clause means that $h(q_n(x)) = \lambda_n h(x) + \eta_n$. So $\theta_n(t)$ is of the linear form $\lambda_n t + \eta_n$ with $c_n \lambda_n$ comparable to 1. Thus

$$\phi_n(\{k_n\} \times \{t\}) = \lambda_n(\ \phi(\{\bar{k}_n\} \times \{h_0\}) + \beta_{\bar{k}_n}(c_n t + d_n - h_0) + r_{\bar{k}_n}(c_n t + d_n - h_0)\ ) + \eta_n.$$

For some constant α_{k_n} we can write the above as $\alpha_{k_n} + c_n \lambda_n \beta_{\bar{k}_n} t + \lambda_n r_{\bar{k}_n}(c_n t + d_n - h_0)$, where the last term does not affect to the limit, because any r_k is bounded by r that satisfies $\lim_{s \to 0} r(s)/s = 0$. To see that the previous limit property implies that the last term goes to zero, observe that $c_n t + d_n$ maps the unit interval to an interval containing h_0 and hence $|c_n t + d_n - h_0| \leq c_n$. Moreover, c_n converges to 0 with n and $c_n \lambda_n$ is comparable to 1. Because the limit of linear maps is linear, we get that $\phi'(\{k\} \times \{t\}) = a_k + b_k t$ for some a_k and b_k.

We proceed to show that a_k and b_k are independent of k. If $U \subset F$ is a set of form $s_F(K_u \times Y)$, we call $s_F(K_{u0} \times Y)$ and $s_F(K_{u1} \times Y)$ *sides* of U. Similarly for F'. Naturally we also say that $K_{u0} \times Y$ and $K_{u1} \times Y$ are sides of $K_u \times Y$. Let $k \in K$. At all first order identification levels the projection $s_{F'}$ identifies points of $\{k\} \times I$ to points of some $\{k'\} \times I$ that is on the other side of $K \times I$. Naturally ϕ' takes the same values in both lines at such points. Since $\{k\} \times I$ and $\{k'\} \times I$ are identified to each other at more than one height, the linear mapping has to be the same in both lines, ie $a_k = a_{k'}$ and $b_k = b_{k'}$. At the second order identification levels the two aforementioned lines are identified to two other lines at other sides of $K_0 \times I$ and $K_1 \times I$. Thus all four share the constants. At the third order identification levels we get four more lines so that alltogether

the lines pass through all third order similarity parts. Continuing this way implies that in a dense set of lines the constants are the same. Lipschitzness then implies that the constants are the same in every line.

Since $s_{F'}$ is height preserving, the previous implies that g'' maps the height linearly. That will say: The height in the image depends linearly on the height in the domain. **It is also bilipschitz. Thus $g''(F')$ has a density point w. As before we lift the mapping from smaller and smaller similarity parts containing the preimage of w to uniformly bilipschitz mappings from F' to F. In the limit we obtain a mapping $g''' : F' \to F$ such that the image $g'''(F')$ contains a ball B and g''' still maps the height linearly.** Bilipschitzness implies that the g''-image of a ball contains a ball of $g''(F')$ with a comparable radius. Because w is a point of density, the ball of $g''(F')$ and the ball of F centered at w with radii r have Hausdorff distance $\psi(r)$ for some ψ with $\lim_{r->0} \psi(r)/r = 0$. Similar fact is true for balls in $B(w,r)$ with radius comparable to r (with a ψ' that naturally depends on the comparability constant). Therefore the similarity images of such balls of $g''(F')$ with smaller and smaller r converge to a ball of F in the Hausdorff metric. Indeed, by passing to a subsequence we can assure that the centers converge; radii have a lower bound by the way the balls were obtained.

After renormalising the height linearly in F (which amounts to scaling the metric) we get that g''' is height preserving. This implies that in a sufficiently small ball that is mapped inside the ball B we obtain that g''' is an isometry. Recall that the metric in F and F' is the infimum of change of the height along paths. **By self similarity we obtain a height-preserving isometry ζ from F' into F after renormalising the height of F again linearly.**

Next we show that the mapping ζ we obtained is an impossibility. By lines of F' or F we mean the $s_{F'}$- or s_F-images of lines of $K \times I$. Half-open interval is denoted by $]a,b]$ or $[c,d[$. Outwards pointing bracket denotes the open end.

Take two lines l_i from F' such that l_1 and l_2 meet at all identification levels of order k, where k is at least 2. Let $p_i,\ i = 1,2$ be the images of two successive points where l_1 and l_2 meet. Suppose $h(p_2) > h(p_1)$. Let $w_1 \in \zeta(l_1)$ and $w_2 \in \zeta(l_2)$ be the images of the points halfway p_1 and p_2. They are then at the same height also in F and they have to be at different sides of some $s_F(K_q \times I)$, even such that its sides are not identified at that height, since w_1 and w_2 have positive distance. The image $A = \zeta(l_1 \cup l_2)$ is a geodesic subset, because ζ is an isometry. It has to meet an identification level joining the sides of $s_F(K_q \times I)$ in the height interval $]h(w_i), h(p_2)]$ since there are paths beginning from w_1 and w_2 that meet in that interval. An another level of the same order has to be in the height interval $[h(p_1), h(w_i)[$, too. Therefore the minimal distance of identification levels of order $|q|$ is not more than $|h(p_1) - h(p_2)|$. On the other hand, the maximal distance of successive identification levels of order $|q| + 1$ is 2/5 times the minimal distance of identification levels of order $|q|$. Therefore identification levels of any order more than $|q|$ appear in the height interval $[h(w_i), h(p_2) - \frac{1}{10}(h(p_2) - h(p_1))]$. If for some $\epsilon > 0$ the interval $[h(w_i), h(p_2) - \epsilon]$ contained also an identification level of order $|q|$, there would exists a shorter path joining w_1 and w_2 than the restriction of A to the height

interval $[h(w_i), p_2]$. Since A is a geodesic subset, the $|q|$th order level has to be at the height $h(p_2)$. For the same reasons there has to be a level of order $|q|$ at $h(p_1)$. Moreover, these levels are successive. It follows that all meeting points of the images of l_1 and l_2 are at successive identification levels of the same order, $|q|$. Now, the maximal distance of successive levels of a fixed order in F is twice the minimal distance of levels of that fixed order. But the similar ratio is four in F'. This is a contradiction. Theorem 1.1 is proved.

Remark. We used in the proof of Theorem 1.1. something that can be seen as a version of differentiability of Lipschitz functions. Nik Weaver has developed very general theory of differentiability of Lipschitz functions in metric measure spaces that can also be used to show with little extra effort that real valued Lipschitz functions in F or F' or at their tangents cones are infinitesimally linear functions of height almost everywhere. See [W]. Jeff Cheeger has also developed in [C] a very general theory of differentiability of Lipschitz functions. It leads to more detailed conclusions than Weavers theory, but there Poincaré inequalities and a doubling condition are needed. Note that Ahlfors–regularity implies the doubling condition. Where defined, Cheegers differential is equal to Weavers one, even though their methods are entirely different. Weavers differential is sometimes identically zero, so the mere generality of his construction does not tell much. Also Cheegers theorems can be used to verify, that real valued Lipschitz functions are infinitesimally linear functions of the height in our spaces. It needs some extra arguments in addition to Cheegers theorems, though. Here a direct proof was given, without using Poincaré inequalities.

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