Locally bilipschitz maps of roughly dense sets

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Abstract

If the set $A \subset \mathsf{R}^n$ is δ-dense in R^n and if $f\colon A \to \mathsf{R}^n$ is L-bilipschitz in sufficiently large balls, then f is L-bilipschitz in A. A converse result is also given.

1 Introduction

We start with some terminology. Let X and Y be metric spaces, where the distance between points a and b is written as $|a-b|$. For $L \geq 0$, a map $f\colon X \to Y$ is *L-Lipschitz* if $|fx - fy| \leq L|x-y|$ for all $x, y \in X$, and f is *L-bilipschitz*, $L \geq 1$, if $|x-y|/L \leq |fx - fy| \leq L|x-y|$ for all $x, y \in X$. If each point $a \in X$ has a neighborhood U such that the restriction $f|U$ is L-Lipschitz or L-bilipschitz, then f is said to be *locally L-Lipschitz* or *locally L-bilipschitz*, respectively. Finally, if this condition holds for the balls $U = \bar{B}(a, R)$ with R independent of a, we say that f is *R-locally L-Lipschitz* or *R-locally L-bilipschitz*, respectively.

If A is a convex subset of a normed space, it is easy to show that a locally L-Lipschitz map $f\colon A \to Y$ is L-Lipschitz. The bilipschitz version of this result is not true. However, it follows from a result of F. John [Jo1, Th. III, p. 93] that if E and F are Banach spaces, then every locally L-bilipschitz open map $f\colon E \to F$ is an L-bilipschitz map onto F.

The purpose of this paper is to give a rough variation of this result of John. We say that a subset A of a metric space X is *roughly dense* in a set $D \subset X$ if there is a number $\delta \geq 0$ such that for each $x \in D$ there is $a \in A$ with $|a - x| \leq \delta$. We do not assume that $A \subset D$. We also say that A is *δ-dense* in D if this holds with a given δ. For example, the set Z^n is δ-dense in the euclidean n-space R^n with $\delta = \sqrt{n}/2$.

We can now formulate the main result of this paper.

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1.1. Theorem. *Suppose that $L \geq 1$, that $K \geq 16L^3(L+1)$, that $A \subset \mathsf{R}^n$ is δ-dense in R^n, and that $f\colon A \to \mathsf{R}^n$ is a $K\delta$-locally L-bilipschitz map. Then f is L'-bilipschitz with*

$$L' = L'(L,K) = \frac{1 + 30L^2/K}{1 - 8L^2/K} L \to L$$

as $K \to \infty$.

The proof of the theorem is given in Section 3 after some auxiliary considerations in Section 2. It can be regarded as a rough version of the proof of John [Jo1, Th. III]. In Section 4 we give a converse of Theorem 1.1. In Section 5 we consider possible generalizations.

2 Auxiliary results

2.1. *Notation.* In a metric space X, we let $B(a,r)$ and $\bar{B}(a,r)$ denote the open and the closed ball with center a and radius r. Let $S(a,r)$ be the sphere $\bar{B}(a,r) \setminus B(a,r)$. In the case of a normed space, we use the abbreviations $B(r) = B(0,r)$ and $\bar{B}(r) = \bar{B}(0,r)$. The distance between sets $A, B \subset X$ is written as $d(A,B)$, and the diameter of A is $d(A)$. Expressions like abc/xyz mean $(abc)/(xyz)$.

We start with an improvement of [Vä, Th. 3.17]. It can be considered as a rough version of the invariance of domain, and it is the only nonelementary result needed in this paper. Since the proof in [Vä] is somewhat sketchy and hidden in the quasisymmetric case, we give a detailed proof.

2.2. Lemma. *Let $L \geq 1$ and $0 < \delta \leq \delta_0 = 1/16L^3(L+1)$. Suppose that $0 \in A \subset \bar{B}(1) \subset \mathsf{R}^n$, that A is δ-dense in $\bar{B}(1-\delta)$, and that $f\colon A \to \mathsf{R}^n$ is an L-bilipschitz map with $f(0) = 0$. Then fA is $L\delta$-dense in $\bar{B}(1/2L)$.*

Proof. By the theorem of Kirszbraun [Fe, 2.10.43], there is an L-Lipschitz extension $g\colon \bar{B}(1) \to \mathsf{R}^n$ of f. We shall use degree theory to show that $g\bar{B}(1-\delta)$ contains the ball $\bar{B}(1/2L)$. The lemma will then follow easily. Indeed, if $y \in \bar{B}(1/2L)$, we can find first $x \in \bar{B}(1-\delta)$ with $gx = y$, and then $a \in A$ with $|a - x| \leq \delta$. Then $|fa - y| = |ga - gx| \leq L|a - x| \leq L\delta$, and the lemma is proved.

If $x, y \in \bar{B}(1-\delta)$, then there are $a, b \in A$ with $|x - a| \leq \delta$, $|y - b| \leq \delta$, and thus $|gx - gy| \geq |ga - gb| - 2L\delta \geq (|x-y| - 2\delta)/L - 2L\delta$. It follows that

$$|gx - gy| \geq |x - y|/L - 2(L+1)\delta \tag{2.3}$$

for all $x, y \in \bar{B}(1-\delta)$.

Set $S = S(1-\delta)$, $Y = gS$. For $x \in S$ we obtain by (2.3)

$$|gx| \geq |x|/L - 2(L+1)\delta \geq 1/L - (2L+3)\delta_0 > 1/2L.$$

Hence

$$Y \cap \bar{B}(1/2L) = \varnothing. \tag{2.4}$$

Set $t_0 = 1/4L$, $t_1 = t_0/2 = 1/8L$ and $Y_1 = Y + \bar{B}(t_1)$. For $y \in Y_1$ we write $Q(y) = S \cap g^{-1}[Y \cap B(y, t_0)]$. If $x, x' \in Q(y)$, then (2.3) gives

$$|x - x'| + \delta \leq L|gx - gx'| + 2L(L+1)\delta_0 + \delta_0 < 2Lt_0 + 3L(L+1)\delta_0 < 1,$$

and thus $d(Q(y)) < 1 - \delta$ for all $y \in Y_1$. It follows that we can choose an open hemisphere $H(y)$ of S containing $Q(y)$.

Since Y_1 is compact, we can choose a finite set $\{y_1, \ldots, y_m\} \subset Y$ such that the balls $B(y_j, t_0)$ cover Y_1. If $Y \subset B(y_j, t_0)$ for some j, then $Q(y_j) = S$, which is impossible, because $d(Q(y_j)) < 1 - \delta$. Hence the sets $F_j = Y_1 \setminus B(y_j, t_0)$ are nonempty. Choosing points $x_j \in S \cap g^{-1}\{y_j\}$ and writing

$$hy = \sum_{j=1}^{m} d(y, F_j) x_j$$

we get a continuous map $h \colon Y_1 \to \mathsf{R}^n$.

Let $y \in Y_1$. Since the balls $B(y_j, t_0)$ cover Y_1, there is j with $d(y, F_j) > 0$. For each such j, we have $x_j \in Q(y) \subset H(y)$. Consequently, $hy \neq 0$ for each $y \in Y_1$. Let $p \colon \mathsf{R}^n \setminus \{0\} \to S$ be the central projection $px = (1-\delta)x/|x|$. Define $u \colon Y_1 \to S$ by $uy = phy$. Then $uy \in H(y)$.

If $x \in S$, then trivially $x \in Q(gx) \subset H(gx)$. It follows that

$$|ugx - x| < 2(1-\delta) \tag{2.5}$$

for all $x \in S$. Let $g_1 \colon S \to Y_1$ be the map defined by g. Then the map $ug_1 \colon S \to S$ is homotopic to the identity. Hence the induced homomorphism $g_1^* \colon H^{n-1}(Y_1) \to H^{n-1}(S)$ is not the zero map. By [Vä, 2.1], there is a component U of $\mathsf{R}^n \setminus Y_1$ such that the topological degree $\mu(y, g, B(1-\delta))$ is nonzero for $y \in U$. We show that $\mu(0, g, B(1-\delta)) \neq 0$. By (2.4) this will imply that $\bar{B}(1/2L) \subset g\bar{B}(1-\delta)$ and prove the lemma.

Fix a point $x_0 \in B(1-\delta)$ with $gx_0 \in U$. We may assume that $x_0 \neq 0$, and it suffices to show that $Y \cap gJ = \varnothing$ for $J = [0, x_0]$. Set $x_1 = px_0$ and $s = |x_1 - x_0|$. Then

$$1/8L = t_1 < |gx_1 - gx_0| \leq L|x_1 - x_0| = Ls,$$

and thus $s > 1/8L^2$. If $a \in J, b \in S$, then (2.3) implies that

$$\begin{aligned} |ga - gb| &\geq |a-b|/L - 2(L+1)\delta \geq s/L - 2(L+1)\delta_0 \\ &> 1/8L^3 - 2(L+1)\delta_0 = 0, \end{aligned}$$

and hence $Y \cap gJ = \varnothing$. □

The following result follows easily from 2.2 by means of auxiliary similarity maps:

2.6. Lemma. *Suppose that $a \in A \subset \bar{B}(a, r) \subset \mathsf{R}^n$ and that that $f\colon A \to \mathsf{R}^n$ is L-bilipschitz. Suppose also that A is δ-dense in $\bar{B}(a, r-\delta)$, where $\delta \leq \delta_0 r$ and δ_0 is as in 2.2. Then fA is $L\delta$-dense in $\bar{B}(fa, r/2L)$.* □

In the rest of this section, we give some elementary results on roughly dense sets and roughly Lipschitz maps. The following result is obvious:

2.7. Lemma. *If A is δ-dense in D, then $A \cap \bar{B}(x, r)$ is δ-dense in $D \cap \bar{B}(x, r-\delta)$ for each $x \in E$ and $r > \delta$.* □

2.8. *Terminology.* Let $L \geq 0$ and $C \geq 0$. A map $f\colon X \to Y$ (not necessarily continuous) between metric spaces is said to be *C-roughly L-Lipschitz* if

$$|fx - fy| \leq L|x-y| + C \tag{2.9}$$

for all $x, y \in X$. If $R > 0$ and if this inequality holds in each ball $\bar{B}(a, R)$, $a \in X$, we say that f is *R-locally C-roughly L-Lipschitz.*

We next give a rough version of the uniqueness of path lifting for local homeomorphisms.

2.10. Lemma. *Let X and Y be metric spaces and let $f\colon X \to Y$ be an R-locally L-bilipschitz map. Suppose that $\Delta \subset \mathsf{R}$ is an interval (possibly unbounded) and that $\alpha, \beta\colon \Delta \to X$ are C-roughly M-Lipschitz maps such that $|f\alpha(t) - f\beta(t)| \leq \lambda$ for all $t \in \Delta$, where $C + L\lambda < R$. If $|\alpha(t_0) - \beta(t_0)| \leq L\lambda$ for some $t_0 \in \Delta$, then $|\alpha(t) - \beta(t)| \leq L\lambda$ for all $t \in \Delta$.*

Proof. Assume that $s, t \in \Delta$ are such that $|\alpha(s) - \beta(s)| \leq L\lambda$ and $|s-t| \leq (R - L\lambda - C)/M$. It suffices to show that $|\alpha(t) - \beta(t)| \leq L\lambda$. We have

$$\begin{aligned} |\alpha(t) - \alpha(s)| &\leq M|s-t| + C \leq R - L\lambda \leq R, \\ |\beta(t) - \alpha(s)| &\leq |\beta(t) - \beta(s)| + |\beta(s) - \alpha(s)| \\ &\leq M|s-t| + C + L\lambda \leq R. \end{aligned}$$

Since f is L-bilipschitz in $A \cap \bar{B}(\alpha(s), R)$, we obtain

$$|\alpha(t) - \beta(t)| \leq L|f\alpha(t) - f\beta(t)| \leq L\lambda. \quad \square$$

2.11. *Remark.* The constant M plays no role in 2.10. In fact, the condition on α and β could be replaced by *C-rough continuity*, which means that $\limsup_{s\to t} |\alpha(s) - \alpha(t)| \le C$ for each $t \in \Delta$ and similarly for β.

In the rest of this section, we let E and F denote (real) normed spaces.

2.12. Theorem. *Suppose that $Q \subset E$ is convex and that $f\colon Q \to F$ is R-locally C-roughly L-Lipschitz. Then f is C-roughly L_0-Lipschitz with $L_0 = L + C/2R$.*

Proof. Let $x, y \in Q$. If $|x - y| \le 2R$, then $x, y \in \bar{B}((x + y)/2, R)$, and (2.9) is true. Assume that $|x - y| > 2R$. There is an integer $m \ge 2$ such that $2(m-1)R \le |x - y| < 2mR$. We divide the line segment $[x, y]$ into segments $[x_{k-1}, x_k]$ of length $|x - y|/m < 2R$ by points $x = x_0, \dots, x_m = y$. Then

$$\begin{aligned} |fx - fy| &\le \sum_{k=1}^{m} |fx_{k-1} - fx_k| \le \sum_{k=1}^{m} (L|x_{k-1} - x_k| + C) = L|x - y| + mC \\ &\le L|x - y| + (1 + |x - y|/2R)C = L_0|x - y| + C. \quad \square \end{aligned}$$

2.13. Theorem. *Suppose that $Q \subset E$ is convex, that $A \subset Q$ is δ-dense in Q, and that $f\colon A \to F$ is R-locally C-roughly L-Lipschitz with $R > 2\delta$. Then f extends to an R_1-locally C_1-roughly L-Lipschitz map $f^*\colon Q \to F$ with $R_1 = R - 2\delta$ and $C_1 = C + 2L\delta$. Moreover, the maps f and f^* are C_1-roughly L_1-Lipschitz with $L_1 = L + (C + 2L\delta)/2(R - 2\delta)$.*

Proof. Since A is δ-dense in Q, there is a map $\varrho\colon Q \to A$ such that $\varrho|A = \mathrm{id}$ and $|\varrho x - x| \le \delta$ for all $x \in Q$. We show that $f^* = f\varrho\colon Q \to F$ is the desired map.

Let $z \in Q$ and let $x, y \in \bar{B}(z, R - 2\delta)$. Then ϱx and ϱy lie in $\bar{B}(\varrho z, R)$, and hence

$$|f^*x - f^*y| \le L|\varrho x - \varrho y| + C \le L(|x - y| + 2\delta) + C = L|x - y| + C_1.$$

The last part of the theorem follows from 2.12. $\square$

2.14. Theorem. *Suppose that $Q \subset E$ is convex, that $A \subset Q$ is δ-dense in Q and that $f\colon A \to F$ is R-locally L-Lipschitz with $R > 2\delta$. Then f is L_2-Lipschitz with $L_2 = L(R + \delta)/(R - 2\delta)$.*

Proof. Applying 2.13 with $C = 0$ we see that f is $2L\delta$-roughly L_1-Lipschitz with $L_1 = L + L\delta/(R - 2\delta)$. Assume that $a, b \in A$ and that $|a - b| \ge R$. Then

$$|fa - fb| \le L_1|a - b| + 2L\delta \le (L_1 + 2L\delta/R)|a - b| \le L_2|a - b|. \quad \square$$

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. The proof can be considered as a rough version of the proof of John for [Jo1, Th. III]. Roughly speaking, the main idea of the proof is to construct a roughly Lipschitz map $h\colon \mathsf{R}^n \to A$ that is a rough inverse of f.

3.1. *Basic assumptions.* Throughout this section we assume that we have the situation of Theorem 1.1. Thus A is a given δ-dense subset of R^n, and $f\colon A \to \mathsf{R}^n$ is a given R-locally L-bilipschitz map, where $R = K\delta$ and $K \geq 16L^3(L+1)$. We shall several times use the estimates

$$K \geq 32L^3, \quad \delta \leq R/32L^3. \tag{3.2}$$

We normalize the situation so that $0 \in A$ and $f(0) = 0$.

The map $f\colon A \to \mathsf{R}^n$ has a unique extension to a map $\bar{f}\colon \bar{A} \to \mathsf{R}^n$, which is R'-locally L-bilipschitz for all $R' < R$. Hence we may assume that A is closed in R^n.

¿From 2.14 we readily obtain:

3.3. Lemma. *The map f is L_2-Lipschitz with $L_2 = L(K+1)/(K-2)$.* $\square$

Since $L_2 < L'$, it remains to prove that $|a-b| \leq L'|fa - fb|$ for all $a, b \in A$. The proof will be completed in 3.16.

3.4. *Notation.* For $a \in A$ we set $A(a) = A \cap \bar{B}(a, R)$. Then $f_a = f|A(a)$ is L-bilipschitz. By 2.6 and 2.7, the set $A'(a) = fA(a)$ is $L\delta$-dense in $\bar{B}(fa, R/2L)$. We let $g_a\colon A'(a) \to A(a)$ denote the inverse of f_a. Then g_a is L-bilipschitz.

3.5. *The map h.* We construct a rough inverse $h\colon \mathsf{R}^n \to A$ of f. Let $J \subset \mathsf{R}^n$ be a closed ray emanating from the origin. To define $h_J = h|J$, we set $r = R/10L$ and divide J into segments $J_k = [y_{k-1}, y_k]$, $k \in \mathsf{N} = \{1, 2, \dots\}$ by points $y_j \in J$ with $|y_j| = jr$. Since $A'(0)$ is $L\delta$-dense in $\bar{B}(R/2L)$, there is a map $\varrho_1\colon J_1 \to A'(0)$ such that $\varrho_1(0) = 0$ and $|\varrho_1 y - y| \leq L\delta$ for $y \in J_1$. We define $h_1\colon J_1 \to A$ by $h_1 = g_0\varrho_1$, where $g_0\colon A'(0) \to A$ was defined in 3.4. Then $h_1(0) = 0$. Since $fg_0 = \mathrm{id}$, we have $|fh_1 y - y| \leq L\delta$ for $y \in J_1$. Set $a_1 = 0$.

Proceeding inductively, we assume that $p \in \mathsf{N}$ and that for $1 \leq k \leq p$ we have defined points $a_k \in A$ and maps $\varrho_k\colon J_k \to A'(a_k)$ such that setting $h_k = g_{a_k}\varrho_k\colon J_k \to A$ we have

(1) $a_k = h_{k-1}y_{k-1} = h_k y_{k-1}$ for $2 \leq k \leq p$,

(2) $|\varrho_k y - y| \leq L\delta$ for $y \in J_k$.

Set $a_{p+1} = h_p y_p$. Then $fa_{p+1} = \varrho_p y_p$. Since (3.2) implies that $L\delta + r \leq R/32L + R/10L < R/2L$, the segment J_{p+1} lies in $\bar{B}(fa_{p+1}, R/2L)$. Hence there is a map $\varrho_{p+1}\colon J_{p+1} \to A'(a_{p+1})$ with $|\varrho_{p+1}y - y| \le L\delta$ for all $y \in J_{p+1}$, and then (2) is valid for $k = p+1$. Since $\varrho_p y_p = f a_{p+1} \in A'(a_{p+1})$, we can choose the map ϱ_{p+1} so that $\varrho_{p+1}y_p = \varrho_p y_p$. Setting $h_{p+1} = g_{a_{p+1}}\varrho_{p+1}$ we get $h_{p+1}y_p = g_{a_{p+1}} f a_{p+1} = a_{p+1} = h_p y_p$, and also (1) holds for $k = p+1$.

If $y, z \in J_k$ for some k, then

$$|h_k y - h_k z| = |g_{a_k}\varrho_k y - g_{a_k}\varrho_k z| \le L|y-z| + 2L^2\delta. \tag{3.6}$$

In particular,

$$|a_{k+1} - a_k| = |h_k y_k - h_k y_{k-1}| \le Lr + 2L^2\delta \le R/10 + R/16 < R/2. \tag{3.7}$$

By (1), the maps $h_k\colon J_k \to A$ define a map $h_J\colon J \to A$ with $h(0) = 0$, and these maps define a map $h\colon \mathsf{R}^n \to A$. We have

$$|fhy - y| \le L\delta \tag{3.8}$$

for all $y \in \mathsf{R}^n$.

We next show that each $h|J$ satisfies the rough Lipschitz condition

$$|hy - hz| \le (L + 20L^3/K)|y-z| + 4L^2\delta \tag{3.9}$$

for $y, z \in J$. By (3.6) we may assume that $y \in J_k$ and $z \in J_{k+m}$ for some $k, m \in \mathsf{N}$. With the aid of (3.6) we get

$$\begin{aligned}
&|hy - hz| \le |hy - hy_k| + \sum_{i=k+1}^{k+m-1} |hy_{i-1} - hy_i| + |hy_{k+m-1} - hz| \\
&\le L|y - y_k| + 2L^2\delta + \sum_{i=k+1}^{k+m-1} (L|y_{i-1} - y_i| + 2L^2\delta) + L|y_{k+m-1} - z| + 2L^2\delta \\
&= L|y-z| + 2L^2\delta(m+1).
\end{aligned}$$

Since $(m-1)r \le |y-z|$, we have

$$m + 1 \le |y-z|/r + 2 = 10L|y-z|/K\delta + 2,$$

and (3.9) follows. $\square$

3.10. Lemma. *The map $h\colon \mathsf{R}^n \to A$ is C_1-roughly L_1-Lipschitz with $L_1 = L + 30L^3/K$, $C_1 = 6L^2\delta$.*

Proof. By 2.12, it suffices to show that h is r-locally C_1-roughly L-Lipschitz. Let $z_0 \in \mathsf{R}^n$. We must show that

$$|hy - hz| \le L|y-z| + 6L^2\delta \tag{3.11}$$

for all $y, z \in \bar{B}(z_0, r)$. We assume that $|z_0| > r = R/10L$; the proof in the case $|z_0| \le r$ is similar but much easier. Let J be the ray from 0 through z_0, subdivided into segments $J_k = [y_{k-1}, y_k]$ as in 3.5. Let J' be a ray from 0 meeting $\bar{B}(z_0, r)$, subdivided into segments $J'_k = [y'_{k-1}, y'_k]$ in the corresponding way. Let m be an integer for which $(m-1)r < |z_0| \le mr$. Then $m \ge 2$, $z_0 \in J_m$, and $J' \cap \bar{B}(z_0, r)$ is contained in the segment $I' = [y'_{m-2}, y'_{m+1}] = J'_{m-1} \cup J'_m \cup J'_{m+1}$. Let α be the angle between J and J'. Then $\alpha < \pi/2$, and hence

$$(3.12) \qquad |y_{m-2} - y'_{m-2}| = d(y_{m-2}, J')/\cos(\alpha/2) \le r\sqrt{2}.$$

We construct a map $h' \colon [0, y'_{m+1}] \to A$, following the idea of 3.5 with two modifications: (i) We use the subdivision of $[0, y'_{m+1}]$ into the segments $J'_1, \ldots, J'_{m-2}, I'$. (ii) We use the points $0 = a_1, \ldots, a_{m-1}$, chosen in the construction of $h|J$.

More precisely, if $m \ge 3$, we start with a map $\varrho'_1 \colon J'_1 \to A'(0)$ such that $\varrho'_1(0) = 0$ and $|\varrho'_1 y - y| \le L\delta$ for $y \in J'_1$. Set $h'_1 = g_0 \varrho'_1 \colon J'_1 \to A$. Assume that $1 \le p \le m-3$ and that for $1 \le k \le p$ we have found maps $\varrho'_k \colon J'_k \to A'(a_k)$ such that setting $h'_k = g_{a_k} \varrho'_k \colon J'_k \to A$ we have

(1) $h'_{k-1} y'_{k-1} = h'_k y'_{k-1}$ for $2 \le k \le p$,

(2) $|\varrho'_k y - y| \le L\delta$ for $y \in J'_k$.

Since $|f a_{p+1} - y_p| \le L\delta \le R/32L$ and since (3.12) implies that $|y_p - y'_p| \le r\sqrt{2}$, we have $|f a_{p+1} - y'_p| + r \le R/32L + r\sqrt{2} + r < R/2L$. Thus $J'_{p+1} \subset \bar{B}(f a_{p+1}, R/2L)$. Hence there is a map $\varrho'_{p+1} \colon J'_{p+1} \to A'(a_{p+1})$ such that (2) is true for $k = p+1$.

To obtain (1), we first show that $w := \varrho'_p y'_p \in A'(a_{p+1}) = fA(a_{p+1})$. As $w \in A'(a_p)$, we can write $w = fx$ for some $x \in A(a_p)$. Since $f|A(a_p)$ is L-bilipschitz, we get

$$\begin{aligned} |x - a_p| &\le L|fx - fa_p| = L|w - \varrho_p y_{p-1}| \le L|y'_p - y_{p-1}| + 2L^2\delta \\ &\le L(r\sqrt{2} + r + 2L\delta) < R/2. \end{aligned}$$

By (3.7) this yields $|x - a_{p+1}| < R$, and hence $w \in A'(a_{p+1})$. We can thus choose $\varrho'_{p+1} y'_p = w$. Furthermore, since $a_{p+1} = g_{a_p} \varrho_p y_p$ and $|\varrho_p y_p - y_p| \le L\delta$, we get by (3.2)

$$|g_{a_p} w - a_{p+1}| \le L|w - \varrho_p y_p| \le L|y'_p - y_p| + 2L^2\delta \le Lr\sqrt{2} + 2L^2 R/32L^2 < R.$$

Since $f|A(a_{p+1})$ is injective, this implies that $h'_{p+1} y'_p = g_{a_{p+1}} \varrho'_{p+1} y'_p = g_{a_{p+1}} w = g_{a_p} w = h'_p y'_p$, and thus (1) holds for $k = p+1$.

We have obtained a map $h' \colon [0, y'_{m-2}] \to A$. If $m = 2$, we are still at $y'_{m-2} = 0$.

The extension of h' to I' is carried out similarly, choosing a map $\varrho'\colon I' \to A'(a_{m-1})$ such that $|\varrho' y - y| \le L\delta$ for $y \in I'$ and such that $\varrho' y'_{m-2} = \varrho'_{m-2} y'_{m-2}$. Then $h'|I' = g_{a_{m-1}}\varrho'\colon I' \to A$. The details are easy modifications of the previous case. For example, we have

$$|y'_{m-2} - f a_{m-1}| + 3r \le r\sqrt{2} + L\delta + 3r \le (\sqrt{2}+3)R/10L + R/32L < R/2L,$$

and hence I' lies in $\bar{B}(f a_{m-1}, R/2L)$.

The map $h'\colon [0, y'_{m+1}] \to A$ satisfies the inequalities

$$|fh'y - y| \le L\delta,\ \ |h'y - h'z| \le (L + 20L^3/K)|y - z| + 4L^2\delta \tag{3.13}$$

for all $y, z \in [0, y'_{m+1}]$. The second estimate is proved as in (3.9).

Let $e' \in J'$ be the unique unit vector. Define $\alpha, \beta\colon [0, (m+1)r] \to A$ by $\alpha(t) = h(te')$, $\beta(t) = h'(te')$. Since $\alpha(0) = \beta(0) = 0$, it follows from (3.8), (3.9) and (3.13) that we can apply Lemma 2.10 to α and β with $C = 4L^2\delta$, $\lambda = 2L\delta$ provided that $C + L\lambda < R$. By (3.2) we obtain $C + L\lambda = 6L^2\delta < 6R/32L < R$, and hence Lemma 2.10 yields

$$|hy - h'y| \le 2L^2\delta \tag{3.14}$$

for all $y \in [0, y'_{m+1}]$.

The ball $\bar{B}(z_0, r)$ is contained in the union V of the segments $[0, y'_{m+1}]$ over all rays J' meeting $\bar{B}(z_0, r)$. We have defined a map $h'\colon V \to A$, and (3.14) holds for all $y \in V$. Setting $g = g_{a_{m-1}}$ we can write $h'y = g\varrho' y$ for $y \in \bar{B}(z_0, r)$, where $|\varrho' y - y| \le L\delta$. Since g is L-Lipschitz, we obtain for all $y, z \in \bar{B}(z_0, r)$

$$|hy - hz| \le |h'y - h'z| + 4L^2\delta \le L|\varrho' y - \varrho' z| + 4L^2\delta \le L|y - z| + 6L^2\delta,$$

and thus (3.11) is true. □

3.15. Lemma. $|hfx - x| \le L^2\delta$ *for all* $x \in A$.

Proof. Let $x \in A$. Since $hf(0) = 0$, we may assume that $x \ne 0$. Set $e = x/|x|$ and $\Delta = [0, |x|]$. Since A is δ-dense in R^n, there is a map $\alpha\colon \Delta \to A$ such that $\alpha(0) = 0$, $\alpha(|x|) = x$, and $|\alpha(t) - te| \le \delta$ for all t. For $s, t \in \Delta$ we have $|\alpha(s) - \alpha(t)| \le |s - t| + 2\delta$.

Set $\beta = hf\alpha\colon \Delta \to A$. By 3.10 and 3.3 we obtain

$$\begin{aligned}|\beta(s) - \beta(t)| &\le L_1|f\alpha(s) - f\alpha(t)| + C_1 \le L_1L_2|\alpha(s) - \alpha(t)| + C_1\\ &\le L_1L_2|s - t| + 2L_1L_2\delta + C_1\end{aligned}$$

for all $s, t \in \Delta$. Here $L_2 = L(K+1)/(K-2) \le 11L/10$ by (3.2). Furthermore, (3.8) yields

$$|f\alpha(t) - f\beta(t)| = |f\alpha(t) - fhf\alpha(t)| \le L\delta$$

for all $t \in \Delta$. Since $\alpha(0) = \beta(0) = 0$ and since

$$2L_1L_2\delta + C_1 + L^2\delta = 2\delta(L + 30L/32)11L/10 + 6L^2\delta + L^2\delta < 12L^2\delta < R,$$

we may apply Lemma 2.10, which gives $|\alpha(t) - \beta(t)| \le L^2\delta$ for all $t \in \Delta$. For $t = |x|$ this gives the lemma. $\square$

3.16. *Proof of Theorem* 1.1. Let $a, b \in A$. We must show that $|a - b| \le L'|fa - fb|$ where $L' = L(1 + 30L^2/K)/(1 - 8L^2/K)$. The case $|a - b| \le R$ is trivial, and we assume that $|a - b| > R$. By 3.15 and 3.10 we obtain

$$|a - b| \le |hfa - hfb| + 2L^2\delta \le L_1|fa - fb| + C_1 + 2L^2\delta.$$

Here

$$L_1 = L + 30L^3/K,\ C_1 + 2L^2\delta = 8L^2\delta < 8L^2\delta|a - b|/R = 8L^2|a - b|/K,$$

and we obtain the desired estimate. $\square$

4 A converse

In this section we prove the following converse for Theorem 1.1, which shows that the property of Theorem 1.1 gives a characterization for roughly dense sets in R^n.

4.1. Theorem. *Suppose that $A \subset \mathsf{R}^n$ is closed and unbounded, that $R > 0$, and that every R-locally 25-bilipschitz map $f: A \to \mathsf{R}^n$ is injective. Then A is $8R$-dense in R^n.*

The proof of the theorem will be completed in 4.5. Remarks on the numerical constants are given in 4.6, and in 4.7 we give some characterizations for roughly dense sets.

4.2. *Auxiliary maps.* Suppose that $b \in \mathsf{R}^n$ and that $e \in \mathsf{R}^n$ is a unit vector. Let T be the line $\{b + te : t \in \mathsf{R}\}$. For $t > 0$ and $M \ge 0$ we define the map $g = g_{b,e,t,M}: \mathsf{R}^n \to \mathsf{R}^n$ by

$$gx = \begin{cases} x + M(t - d(x,T))e & \text{if } d(x,T) \le t, \\ x & \text{if } d(x,T) \ge t. \end{cases}$$

Then g is bijective with $g^{-1} = g_{b,-e,t,M}$. The restriction $g|T$ is the translation $gx = x + Mte$. Since

$$|gx - gy| \le |x - y| + M|d(x,T) - d(y,T)| \le (M+1)|x-y|,$$

and since a similar inequality holds for g^{-1}, the map g is $(M+1)$-bilipschitz.

4.3. *Notation.* For nonzero vectors $x, y \in \mathsf{R}^n$, we let $\operatorname{ang}(x,y)$ denote the angle between x and y, $0 \le \operatorname{ang}(x,y) \le \pi$.

4.4. Lemma. *Suppose that $A \subset \mathsf{R}^n$ is closed and unbounded. Let $0 < \delta < \sup\{d(x,A)\colon x \in \mathsf{R}^n\}$. Then there are points $x_0 \in \mathsf{R}^n$ and $a_1, a_2 \in A$ such that*

(1) *$d(x_0, A) = \lambda \ge \delta$,*
(2) *$|x_0 - a_1| \le 3\lambda$, $|x_0 - a_2| \le 3\lambda$.*
(3) *$\operatorname{ang}(a_1 - x_0, a_2 - x_0) \ge \pi/6$.*

Proof. Set $\delta_0 = \sup\{d(x,A)\colon x \in \mathsf{R}^n\}$. We consider two cases.

Case 1. $\delta_0 < \infty$. Choose a point $x_0 \in \mathsf{R}^n$ such that $d(x_0,A) = \lambda \ge \delta$ and $\lambda > 4\delta_0/5$. Then (1) is true and $B(x_0,\lambda) \cap A = \varnothing$, $Q = S(x_0,\lambda) \cap A \ne \varnothing$. Choose any point $a_1 \in Q$. We show that there is a point $a_2 \in A$ such that the conditions (2) and (3) hold, even in the stronger forms

(2′) $|x_0 - a_1| = \lambda$, $|x_0 - a_2| \le 2\lambda$.
(3′) $\operatorname{ang}(a_1 - x_0, a_2 - x_0) \ge \pi/3$.

Assume that there is no such a_2. Then A does not meet the sector $\{x\colon |x - x_0| \le 2\lambda,\ \operatorname{ang}(x - x_0, a_1 - x_0) > \pi/3\}$. Set $x_1 = x_0 + \frac{1}{2}(x_0 - a_1)$. By elementary geometry we get the contradiction

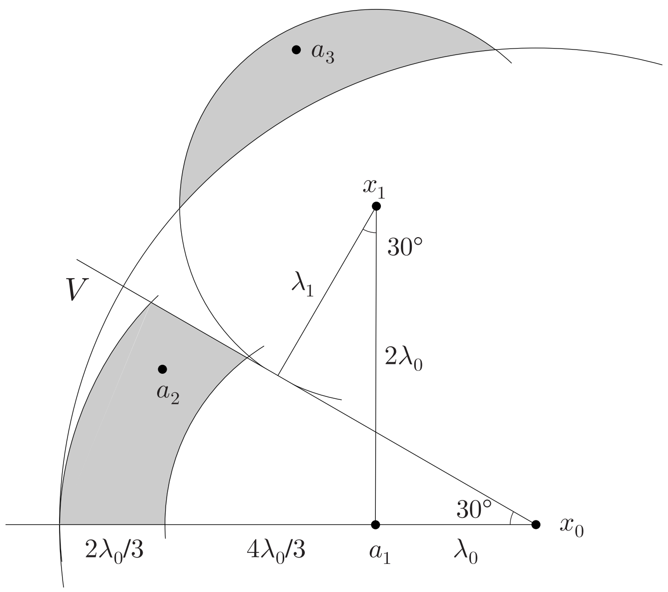
$$\delta_0 \ge d(x_1, A) \ge \lambda\sqrt{7}/2 > 5\lambda/4 > \delta_0,$$

and Case 1 is proved.

Case 2. $\delta_0 = \infty$. Assume that the lemma is not true for some δ.

Fact 1. If $a \in A$ and $r \ge 2\delta$, then $B(a,3r) \setminus B(a,2r)$ meets A.

Assume that Fact 1 is not true. Choose $b_1 \in A$ with $|b_1 - a| = d(a, A \setminus B(a,3r))$. Then $|b_1 - a| \ge 3r$. There is a point $y_0 \in [a, b_1]$ such that $B(y_0, |y_0 - b_1|) \cap A = \varnothing$ and the radius $s = |y_0 - b_1|$ is as large as possible. Then $\delta \le r/2 \le s \le |b_1 - a|/2$ and $|y_0| \ge |b_1 - a|/2 \ge 3r/2$. There is a point $b_2 \in S(y_0, s) \cap \bar{B}(a, 2r) \cap A$. Clearly $\operatorname{ang}(b_1 - y_0, b_2 - y_0) \ge \pi/6$. Since the lemma was assumed to be false, we obtain a contradiction, and Fact 1 is proved.



Since $d(x, A)$ is unbounded, we can choose a point $x_0 \in \mathsf{R}^n$ with $d(x_0, A) = \lambda_0 \geq 3\delta$. Choose $a_1 \in A$ with $|a_0 - a_1| = \lambda_0$. By Fact 1, there is a point $a_2 \in A$ such that $4\lambda_0/3 \leq |a_2 - a_1| = 2\lambda_0$. Then $|a_2 - x_0| \leq 3\lambda_0$. Since the lemma was assumed to be false, $A \cap \bar{B}(x_0, 3\lambda_0)$ is contained in the cone

$$V = \{x\colon \operatorname{ang}(x - x_0, a_1 - x_0) \leq \pi/6\}.$$

Thus $a_2 \in V$. Let $x_1 \in \mathsf{R}^n$ be a point such that $|x_1 - a_1| = 2\lambda_0$, the vector $x_1 - a_1$ is perpendicular to $x_0 - a_1$, and such that the points x_1, a_2 lie in a half plane containing x_0 and a_1 as boundary points. Then $\lambda_1 = d(x_1, V) = (\sqrt{3} - 1/2)\lambda_0$ and $|x_1 - a_2| \leq 2\lambda_0\sqrt{2} < 3\lambda_1$. Moreover, $\operatorname{ang}(a_1 - x_1, a_2 - x_1) \geq \pi/6$. If $d(x_1, A) \geq \lambda_1$, then the points x_1, a_1, a_2 satisfy the lemma. Hence $d(x_1, A) < \lambda_1$. Choose $a_3 \in A$ with $|x_1 - a_3| = d(x_1, A)$. Then $|a_3 - x_0| \geq 3\lambda_0$, and thus $|x_1 - a_3| \geq 3\lambda_0 - \lambda_0\sqrt{5} > 2\lambda_0/3$, which implies that $|a_1 - x_1| = 2\lambda_0 < 3|x_1 - a_3|$. Clearly $\operatorname{ang}(a_1 - x_1, a_3 - x_1) \geq \pi/6$, and thus the points x_1, a_1, a_3 satisfy the lemma. □

4.5. *Proof of Theorem* 4.1. Assume that A is not $8R$-dense in R^n. Then $\sup\{d(x, A)\colon x \in \mathsf{R}^n\} > 8R$. Let x_0, a_1, a_2 be the points given by Lemma 4.4 for $\delta = 8R$. To simplify notation, we assume that $x_0 = 0$. For $\lambda = d(0, A)$ we have $\delta \leq \lambda \leq |a_k| \leq 3\lambda$ for $k = 1, 2$. Set $t = 1/8$. Let J_k be the ray from the origin through a_k, and set $U_k = \{x\colon |x| \geq \lambda,\ d(x, J_k) < t\lambda\}$. Since $\operatorname{ang}(a_1, a_2) \geq \pi/6$, it follows by elementary trigonometry that $d(U_1, U_2) \geq \lambda/4$. For a more precise estimate, see 4.6. Set $e_k = -a_k/|a_k|$ and

$g_k = g_{a_k, e_k, t\lambda, |a_k|/t\lambda}$, where g is defined in 4.2. Define $f\colon A \to \mathsf{R}^n$ by

$$fx = \begin{cases} g_k x & \text{if } x \in U_k \cap A,\ k = 1, 2, \\ x & \text{if } x \in A \setminus (U_1 \cup U_2). \end{cases}$$

Since $d(U_1, U_2) \geq \lambda/4$, no ball of radius $\lambda/8$ meets both U_1 and U_2. As $\lambda/8 \geq \delta/8 = R$ and $8|a_k| \leq 24\lambda$, it follows from 4.2 that f is R-locally 25-bilipschitz. Furthermore, $fa_1 = fa_2 = 0$, and thus f is not injective. This contradiction proves the theorem. □

4.6. *Remarks.* The numerical estimates in Theorem 4.1 are presumably far from the best possible. For a general t, the proof works as long as $d(U_1, U_2) > 0$. We have $d(U_1, U_2) \geq s(t)\lambda$, where

$$s(t) = (\sqrt{1 - t^2} - t(2 + \sqrt{3}))\sqrt{2 - \sqrt{3}} > 0$$

if $t > t_0 = \frac{1}{2}\sqrt{2 - \sqrt{3}} = 0.25881\ldots$. For these t, Theorem 4.1 is valid if the constant 25 is replaced by $L(t) = 1 + 3/t$ and if $8R$ is replaced by $\beta(t)R$ where $\beta(t) = 2/s(t)$. Then

$$\begin{aligned} &L(t) \to 1 + 3/t_0 = 12.591\ldots,\ \beta(t) \to \infty && \text{as } t \to t_0, \\ &L(t) \to \infty,\ \beta(t) \to 1/t_0 = 3.8637\ldots && \text{as } t \to 0. \end{aligned}$$

For example, $L(1/4) = 13$, $\beta(1/4) = 109.66\ldots$.

On the other hand, the constant 25 cannot be replaced by any $L < 2^{1/4}$. In fact, John [Jo2, Th. A, p. 266] has proved that for such L, every locally L-bilipschitz map of an open n-ball into R^n is injective. This implies the corresponding result for an open half space, and the case of a closed half space is obtained by a simple auxiliary map with bilipschitz constant close to 1. Consequently, a closed half space satisfies the hypotheses of Theorem 4.1 for locally L-bilipschitz maps but it is not roughly dense in R^n.

Together with Theorem 1.1, these considerations yield the following characterizations for roughly dense sets.

4.7. Theorem. *For an unbounded closed set $A \subset \mathsf{R}^n$, the following conditions are equivalent:*

(1) *A is roughly dense in R^n,*

(2) *There is $R > 0$ such that every R-locally* 13*-bilipschitz map $f\colon A \to \mathsf{R}^n$ is injective,*

(3) *There is $R > 0$ such that every R-locally* 13*-bilipschitz map $f\colon A \to \mathsf{R}^n$ is* 14*-bilipschitz.* □

5 Generalizations

We consider without detailed proofs the possibilities to generalize Theorem 1.1 to four directions:

(1) other norms of the n-space,
(2) infinite-dimensional spaces,
(3) maps of a ball,
(4) quasisymmetric maps.

5.1. *Other norms.* Suppose that E and F are n-dimensional normed spaces. It is natural to ask whether there is a version of Theorem 1.1 for maps $f\colon A \to F$, where A is δ-dense in E.

By a result of John (see [To, 9.12]), there is a linear $\sqrt{n}$-bilipschitz isomorphism $\varphi\colon E \to \mathsf{R}^n$, where R^n has the euclidean norm. By such maps we can transfer Theorem 1.1 to normed spaces, but the result is not satisfactory, because the resulting estimate $L'(L, K, n)$ does not converge to L as $K \to \infty$.

We get a better result by rewriting the whole proof of 1.1 in the ne w setting. The proof of 2.2 makes use of Kirszbraun's theorem, which is not valid in normed spaces. However, we can extend f to an nL-Lipschitz map g of $\bar{B}(1)$. The proof of inequality (2.5) is valid in strictly convex spaces, but in an arbitrary space it must be replaced by a somewhat longer argument. As a result we obtain that fA is $nL\delta$-dense in $\bar{B}(1/2L)$ if δ is sufficiently small.

The proof for inequality (3.12) made use of angles. In a normed space, the estimate is true with $r\sqrt{2}$ replaced by $2r$. Therefore we now choose $r = R/11L$. With some minor modifications, the proof gives the following result:

5.2. Theorem. *Let E and F be normed spaces with* $\dim E = \dim F = n < \infty$*. Suppose that $L \geq 1$, that $K \geq 16nL^3(nL+1)$, that $A \subset E$ is δ-dense in E, and that $f\colon A \to F$ is a $K\delta$-locally L-bilipschitz map. Then f is L'-bilipschitz with*

$$L' = L'(L, K, n) = \frac{1 + 33nL^2/K}{1 - 8nL^2/K} L \to L$$

as $K \to \infty$.

5.3. *Question.* Is 5.2 true with a constant L' independent of n?

5.4. *Infinite-dimensional spaces.* Easy examples show that Theorem 1.1 is false for maps $f\colon A \to F$, where E and F are infinite-dimensional Hilbert spaces and A is δ-dense in E. The classical result of John is true for locally L-bilipschitz *open* maps $f\colon E \to F$. In the present situation, we could replace openness by the requirement that f has the property given by Lemma 2.2 in every ball, but the result does not look interesting.

5.5. *Maps of a ball.* It follows from the theorem [Jo1, Th. III] of John that a locally L-bilipschitz map of a ball $B(R_0) \subset \mathsf{R}^n$ into R^n is L-bilipschitz in the ball $B(R_0/L^2)$. This result has the following rough version:

5.6. Theorem. *Suppose that $L \ge 1$, that $K \ge 16L^3(L+1)$, that $0 < R = K\delta < R_0/2$, that $0 \in A \subset B(R_0) \subset \mathsf{R}^n$, that A is δ-dense in $B(R_0)$, and that $f\colon A \to \mathsf{R}^n$ is R-locally L-bilipschitz. Then $f|A \cap B(R_1)$ is L'-bilipschitz, where $L' = L'(L,K)$ as in Theorem* 1.1 *and*

$$R_1 = R_1(L,K,R_0,R) = \frac{(K-2)(R_0-2R)}{L^2(K+1)(1+20L^2/K)}.$$

The proof of 5.6 is similar to that of 1.1 but somewhat more complicated. The main difference is that the rough inverse h of f is not defined in the whole space but only in the ball $B(R')$ where $R' = (R_0 - 2R)/L(1 + 20L^2/K)$.

Since $R_1 \to \infty$ as $R_0 \to \infty$, Theorem 1.1 follows from 5.6. We show that also the result of John for locally L-bilipschitz maps follows from 5.6.

Suppose that $f\colon B(R_0) \to \mathsf{R}^n$ is locally L-bilipschitz. Choose a number s with $0 < s < R_0$. Then there is $R > 0$ such that $R < s/2$ and $f|B(s)$ is R-locally L-bilipschitz. Let $K > 16L^3(L+1)$. It follows from 5.6 that $f|R_1$ is L'-bilipschitz, where $L' = L'(L,K)$ and $R_1 = R_1(L,K,s,R)$. Letting $K \to \infty$ and $R \to 0$ we see that f is L-bilipschitz in the ball $B(s/L^2)$ for each $s < R_0$, and hence in $B(R_0/L^2)$.

5.7. *Quasisymmetric maps.* Let $\eta\colon [0,\infty[\to [0,\infty[$ be a homeomorphism. We recall that a map $f\colon X \to Y$ between metric spaces is η-quasisymmetric if $|fa - fx| \le \varphi(t)|fb - fx|$ whenever $|a - x| \le t|b - x|$. A homeomorphism $f\colon \mathsf{R}^n \to \mathsf{R}^n$, $n \ge 2$, is K-quasiconformal if and only if f is η-quasisymmetric, where K and η depend only on each other and on n.

By a theorem of V.A. Zorich [Zo], a locally K-quasiconformal map $f\colon \mathsf{R}^n \to \mathsf{R}^n$ is K-quasiconformal provided that $n \ge 3$. In view of this result and Theorem 1.1, it is natural to ask whether there is a quasisymmetric version of Theorem 1.1. For example, given an integer $n \ge 3$, a number $\delta > 0$ and a homeomorphism $\eta\colon [0,\infty[\to [0,\infty[$, does there exist a number r_0 such that if $A \subset \mathsf{R}^n$ is δ-dense in R^n, if $R \ge r_0$ and if a map $f\colon A \to \mathsf{R}^n$ is η-quasisymmetric in $A \cap B(a,R)$ for each $a \in A$, then f is η'-quasisymmetric, where η' depends only on η, δ, R, n?

I do not know the answer. A quasisymmetric version of Lemma 2.2 is given in [Vä, 3.1].

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