ON BOUNDARY BEHAVIOR OF Q-HOMEOMORPHISMS

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Abstract. The known Gehring–Martio results (1985) on the boundary behavior of quasiconformal mappings are extended to a wide class of the so–called Q-homeomorphisms and, in particular, to BMO–quasiconformal mappings. Our proofs are based on extremal length methods and properties of BMO functions.

1. Introduction

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [1, \infty]$. We say that a homeomorphism $f : D \to \mathbb{R}^n$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int_D Q(x)\rho^n(x)dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

Here we use only open paths $\gamma : (a, b) \to \mathbb{R}^n$. Recall that, given a family of paths Γ in a domain D, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in adm\Gamma$, if

$$\int_\gamma \rho(x)|dx| \geq 1 \tag{1.2}$$

for each $\gamma \in \Gamma$. The *(conformal) modulus* $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in adm\Gamma} \int_D \rho^n(x)dm(x). \tag{1.3}$$

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An example of $Q(x)$–homeomorphisms is provided by a class of homeomorphisms $f \in \mathrm{W}^{1,n}_{loc}(D)$ whose dilatation Q is in $L^{n-1}_{loc}(D)$, see [MRSY]. Given a function $Q : D \to [1,\infty)$, we say that a sense-preserving homeomorphism $f : D \to \mathbb{R}^n$ is $Q(x)$-*quasiconformal*, abbr. $Q(x)$-*qc*, if $f \in \mathrm{W}^{1,n}_{loc}(D)$ and

$$K(x,f) \le Q(x) \quad \text{a.e.} \tag{1.4}$$

We also say that $f : D \to \mathbb{R}^n$ is *BMO-quasiconformal*, abbr. *BMO-qc*, if f is $Q(x)$-qc for some BMO (*bounded mean oscillation*) function $Q : D \to [1,\infty)$. In particular, for $Q \in L^\infty(D)$, we obtain usual quasiconformal mappings.

We use the standard notation for the *maximal dilatation*, or in short the *dilatation*, of f at x

$$K(x,f) = \max(K_O(x,f), K_I(x,f)) \tag{1.5}$$

where , for $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$ and $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$,

$$K_O(x,f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{1.6}$$

is the *outer dilatation* of f at x and

$$K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{1.7}$$

is the *inner dilatation* of f at x, cf. [MRV] and [Re].

For the investigations of the boundary behavior, some regularity of the boundary is needed and, for this purpose, we introduce a number of classes of domains below.

We say that a proper subdomain D of $\mathbb{R}^n$ is an L^1-*BMO domain* if $u \in L^1(D)$ whenever $u \in$BMO(D). By [Sta], p. 106–107, cf. [Go$_1$], p. 69, D is an L^1-BMO domain if and only if $k_D(\cdot, x_0) \in \mathrm{L}^1(\mathrm{D})$ where k_D is *quasihyperbolic metric* on D,

$$k_D(x, x_0) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d(y, \partial D)} \tag{1.8}$$

where ds denotes the Euclidean length element, $d(y, \partial D)$ the Euclidean distance from $y \in D$ to ∂D, and the infimum is taken over all rectifiable paths $\gamma \in D$ joining x to x_0. L^1-BMO domains are not invariant under quasiconformal mappings of $\mathbb{R}^n$, however, they are invariant under quasi–isometries, see [Sta], p. 119 and 112.

In particular, every John domain is an L^1-BMO domain, see Theorem 3.14 in [Sta], p. 115. A domain $D \subset \mathbb{R}^n$ is called *John domain* if there exist $0 < \alpha \le \beta < \infty$ and a point $x_0 \in D$ such that, for every $x \in D$, there is a rectifiable path $\gamma : [0, l] \to D$, parametrized by arclength such that $\gamma(0) = x$, $\gamma(l) = x_0$, $l \le \beta$ and

$$d(\gamma(t), \partial D) \ge \frac{\alpha}{l} \cdot t \tag{1.9}$$

for all $t \in [0, l]$. Note that John domains are invariant under qc mappings of $\mathbb{R}^n$, see [MS], p. 388. More generally, the Hölder domains, see [SS], are also L^1-BMO domains.

We say that a domain D in $\mathbb{R}^n$ is a *BMO extension domain* if every $u \in$BMO(D) has an extension to $\mathbb{R}^n$ which belongs to BMO($\mathbb{R}^n$). Evidently, D is an L^1-BMO domain, if D is bounded and a BMO extension domain. It was shown in [GO] and [Jo] that a domain D is a BMO extension domain if and only if D is a *uniform domain*, i.e., for some a and $b > 0$, each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ such that

$$l(\gamma) \le a \cdot |x_1 - x_2| \tag{1.10}$$

and, for all $x \in \gamma$,

$$\min_{i=1,2} l(\gamma(x_i, x)) \le b \cdot d(x, \partial D) \tag{1.11}$$

where $l(\gamma)$ is the Euclidean length of γ, $\gamma(x_i, x)$ is the part of γ between x_i and x.

The uniform domains have been introduced in [MS] and their various characterizations can be found in [Ge], [Ma], [Mar], [Vä$_2$] and [Vu]. It has been shown in [MS], p. 387, that the notion of uniform domains is invariant under quasiconformal mappings of $\mathbb{R}^n$. In particular, every domain which is bounded by a quasisphere, i.e., the image of $\partial\mathbb{B}^n$ under qc automorphism of $\mathbb{R}^n$, is a uniform domain and, consequently, a BMO extension domain. We also write $u \in$BMO$(\overline{D})$ if u has a BMO extension to an open set $U \subset \mathbb{R}^n$ such that $\overline{D} \subset U$. Domains D in $\mathbb{R}^n$ for which every $u \in$BMO(D) admits such extension can be characterized as relatively uniform domains, see e.g. [Go$_2$].

A domain $D \subset \mathbb{R}^n$ is called a *quasiextremal distance domain* or a *QED domain* if there is $K \geq 1$ such that, for each pair of disjoint continua E and F in D,

$$M(\Gamma(E,F;\mathbb{R}^n)) \leq K \cdot M(\Gamma(E,F;D)) \tag{1.12}$$

where $\Gamma(E,F;D)$ denotes the family of all paths joining E and F in D. Here we say that an open path $\gamma : (a,b) \to \mathbb{R}^n$ joins E and F in D if $\gamma((a,b)) \subset D$ and γ is the restriction of a closed path $\overline{\gamma} : [a,b] \to \mathbb{R}^n$ to (a,b) such that $\overline{\gamma}(a) \in E$ and $\overline{\gamma}(b) \in F$. It is known that every uniform domain D is a QED domain and there exist QED domains which are not uniform, see [GM], p. 189 and 194. Every QED domain $D \subset \mathbb{R}^n$ is *quasiconvex*, i.e., (1.10) holds for all x_1 and $x_2 \in D$, see Lemma 2.7 in [GM], p. 184. Hence D is *locally connected* at ∂D, i.e., every point $x \in \partial D$ has an arbitrarily small neighborhood U such that $U \cap D$ is connected, cf. also Lemma 2.11 in [GM], p. 187, and [HK$_1$], p. 190. Note that every Jordan domain D in $\mathbb{R}^n$ is locally connected at ∂D, see [Wi], p. 66.

Below we denote by $B^n(a,r)$ the Euclidean ball $|x-a| < r$ in $\mathbb{R}^n$ with center a and radius r, $S^n(a,r) = \partial B^n(a,r)$. We also let $B^n(r) = B^n(0,r)$ and $\mathbb{B}^n = B^n(1)$, $S^n = \partial\mathbb{B}^n$. A history survey of closely ralated investigations can be found in our previous preprint [MRSY].

2. On weakly accessible and flat boundaries

We say that ∂D is *weakly accessible* if, for nondegenerate continua E and F in $\overline{D}$,

$$M(\Gamma(E, F; D)) > 0, \tag{2.1}$$

and that ∂D is *weakly flat* if, for nondegenerate continua E and F in $\overline{D}$ with $E \cap F \neq \emptyset$,

$$M(\Gamma(E, F; D)) = \infty. \tag{2.2}$$

These properties are clearly invariant under qc mappings of $\mathbb{R}^n$ and they are closely related to properties P_1 and P_2 by Väisälä in [Vä$_1$, 17.5] as well as to the notions of quasiconformal flatness and quasiconformal accessibility by Näkki in [Nä] and by Herron–Koskela in [HK$_1$].

We shall show below that the weak flatness implies the weak accessibility. The converse conclusion does not hold as an example of the unit disk with a cut shows. Note that the conditions (2.1) and (2.2) automatically hold if E and F are inside of D, see e.g. Lemma 1.15 in [Nä], p. 16, and [Vä$_1$, 10.12], but not if E and F are in ∂D. It is known that in a QED domain the inequality (1.12) holds for each pair of disjoint continua E and F in $\overline{D}$, see Theorem 2.8 in [HK$_2$], p. 173, cf. Lemma 6.11 in [MV], p. 35. Hence QED domains and, consequently, uniform domains have weakly flat boundaries, cf. Lemma 3.1 in [HK$_1$], p. 196.

2.3. Lemma. *If a boundary of a domain D in $\mathbb{R}^n$, $n \geq 2$, is weakly flat, then it is weakly accessible.*

Proof. Let E and F be nondegenerate continua in $\overline{D}$. Without loss of generality we may assume that $E \cap F = \emptyset$, ∞ lies outside of $E \cup F$ and $E \subset \partial D$, see Lemma 1.15 in [Nä], p. 16. Take $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(E, F)$. Then $E_\varepsilon \cap F_\varepsilon = \emptyset$ where $E_\varepsilon = \{x \in D : \mathrm{dist}(x, E) < \varepsilon\}$ and $F_\varepsilon = \{x \in D : \mathrm{dist}(x, F) < \varepsilon\}$. Since each path in $\Gamma(E, E; D)$ contains a subpath which belongs to $\Gamma(E, E_\varepsilon; D)$, it follows that $M(\Gamma(E, E; D)) \leq M(\Gamma(E, E_\varepsilon; D))$. In view of the weak flatness of ∂D, $M(\Gamma(E, E; D)) = \infty$. Therefore $M(\Gamma(E, E_\varepsilon; D)) = \infty$. By the

Lindelöf principle, see e.g. [Ku], p. 54, the open subsets E_ε and F_ε of D can be covered by countable collections of open and, consequently, closed balls B inside of E_ε and F_ε, respectively. Thus, by countable subadditivity of the modulus we can find a couple of closed balls, $B_0 \subset E_\varepsilon$ and $B_0^* \subset F_\varepsilon$, $B_0 \cap B_0^* = \emptyset$, such that

$$M(\Gamma(E, B_0; D)) > 0 \tag{2.4}$$

and

$$M(\Gamma(F, B_0^*; D)) > 0, \tag{2.5}$$

see Theorem 1 in [Fu], p. 176, as well as

$$M(\Gamma(B_0, B_0^*; D)) > 0, \tag{2.6}$$

see e.g. Lemma 1.15 in [Nä].

Now, we use the idea of Näkki in the proof of Theorem 1.16 in [Nä]. More precisely, let $\rho \in \text{adm } \Gamma(E, B_0^*; D)$. If

$$\int_\gamma \rho ds \ge \frac{1}{3} \quad \text{or} \quad \int_{\gamma'} \rho ds \ge \frac{1}{3} \tag{2.7}$$

for all rectifiable paths $\gamma \in \Gamma(E, B_0; D)$ and $\gamma' \in \Gamma(B_0, B_0^*; D)$, respectively, then $3\rho \in \text{adm } \Gamma(E, B_0; D)$ or $3\rho \in \text{adm } \Gamma(B_0, B_0^*; D)$, and hence by (2.4) and (2.6)

$$\int_D \rho^n dm \ge 3^{-n} \min(M(\Gamma(E, B_0; D)), M(\Gamma(B_0, B_0^*; D))) > 0. \tag{2.8}$$

If (2.7) does not hold for a couple of such paths γ and γ', then

$$\int_\alpha \rho ds > \frac{1}{3} \tag{2.9}$$

for every rectifiable path $\alpha \in \Gamma(\gamma, \gamma'; R_0)$ where R_0 is a ring $r_0 < |x - c_0| < r_0'$, c_0 and r_0 is the center and the radius of B_0, respectively, and $r_0' = r_0 + \text{dist}(B_0, B_0^* \cup \partial D)$, i.e., $3\rho \in \text{adm } \Gamma(\gamma, \gamma'; R_0)$, and hence

$$\int_D \rho^n dm \ge 3^{-n} c_n \log \frac{r_0'}{r_0} > 0, \tag{2.10}$$

see [Vä$_1$, 10.12]. Thus, by (2.8) and (2.10)

$$M(\Gamma(E, B_0^*; D)) > 0. \tag{2.11}$$

If $I = \mathrm{Int}B_0^* \cap F \neq \emptyset$, then by Lemma 1.15 in [Nä] $M(\Gamma(B_0, \overline{I}; D)) > 0$, and arguing as above (take $\overline{I} = B_0^* \cap F$ instead of B_0^*) we obtain $M(\Gamma(E, \overline{I}; D)) > 0$ and hence by monotonicity of the modulus

$$M(\Gamma(E, F; D)) > 0. \tag{2.12}$$

If $B_0^* \cap F = \partial B_0^* \cap F \neq \emptyset$, then by subadditivity of the modulus we obtain (2.5), (2.6) and (2.11) for another ball $B_0' \subset \mathrm{Int}B_0^*$ with $B_0' \cap F = \emptyset$. Finally, if $B_0^* \cap F = \emptyset$, then repeating the above arguments (as in the proof of (2.11), replace B_0 and B_0^* by B_0^* and F, respectively) we again obtain (2.12) by (2.5) and (2.11).

3. Boundary extension

Here $Q \in BMO(\overline{D})$ means that $Q \in BMO(U)$ for some open set U which contains $\overline{D}$.

3.1. Lemma. *Let $D \subset \mathbb{R}^n, n \geq 2$, be locally connected at ∂D and let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in BMO(\overline{D})$. If $\partial D'$ is weakly accessible, then f has a continuous extension $\tilde{f} : \overline{D} \to \overline{D}'$.*

Proof. Let $x_0 \in \partial D$. As BMO functions and Q-homeomorphisms are Möbius invariant, we may assume that $x_0 = 0$ and that $\partial\mathbb{B}^n \cap D \neq \emptyset$. We will show that the cluster set $E = C(0, f)$ of f at 0 is a point, which will prove that $f(x)$ has a limit at 0.

Since D is locally connected at $0, E$ is a continuum. Suppose that E is nondegenerate. For $t \in (0, 1/e)$, let D_t denote the component of $B^n(t) \cap D$ for which $0 \in \overline{D}_t$. Note, that D_t is well defined, since D is locally connected at 0, and that $D_t \subset D_{t'}$ for $t < t'$. For $t \in (0, e^{-2}]$, let Γ_t denote the family of path joining D_t and the set $S = D \cap \partial B^n(1/e)$ in

$D_{1/e}\setminus\overline{D}_t$. As in Lemma 2.3 in [MRSY], we let $A(t)$ denote the spherical ring $t<|x|<1/e$. Then the function $\rho(x)$ given by

$$\rho(x)=\frac{1}{(\log\log 1/t)|x|\log 1/|x|} \tag{3.2}$$

for $x\in A(t)$ and $\rho(x)=0$ otherwise, is admissible for Γ_t, and hence

$$M(f\Gamma_t)\leq\int\limits_{D_t}Q(x)\rho^n(x)dm(x) \tag{3.3}$$

For $Q\in \mathrm{BMO}(\overline{D})$, we may apply (3.3) in [RSY] for $n=2$ and Lemma 2.3 in [MRSY] for $n\geq 3$, and get

$$\int\limits_{D_t}Q(x)\rho^n(x)dm(x)\leq\int\limits_{A(t)}Q(x)\rho^n(x)dm(x)\to 0 \tag{3.4}$$

as $t\to 0$. On the other hand

$$M(f\Gamma_t)\geq M(\Gamma(\overline{f(S)}),E;D'). \tag{3.5}$$

Now, $\partial D'$ is weakly accessible, $f(S)$ contains a nondegenerate continuum and E is nondegenerate. Therefore, the right hand side in (3.5) is positive contradicting (3.3) and (3.4). This shows that the cluster set of f at every point of ∂D is degenerate and thus f has a continuous extension on $\overline{D}$.

3.6. Lemma. *Let D be a domain in $\mathbb{R}^n$ and f a Q-homeomorphism of D onto a domain D' in $\mathbb{R}^n$ with $Q\in L^1(D)$. Suppose that D is locally connected at ∂D. If $\partial D'$ is weakly flat, then $C(x_1,f)\cap C(x_2,f)=\emptyset$ for every two distinct points x_1 and x_2 in ∂D.*

Proof. Without loss of generality we may assume that the domain D is bounded. For $i=1,2$, let E_i denote the cluster sets $C(x_i,f)$ and suppose that $E_1\cap E_2\neq\emptyset$. Write $d=|x_1-x_2|$. Since D is locally connected in ∂D, there are neighborhoods U_i of x_i, such that $W_i=D\cap U_i$ is connected and $U_i\subset B^n(x_i,d/3), i=1,2$. Then the function

$\rho(x) = 3/d$ if $x \in D \cap B^n((x_1 + x_2)/2, d)$ and $\rho(x) = 0$ elsewhere is admissible for the family $\Gamma = \Gamma(\overline{W}_1, \overline{W}_2; D)$. Thus,

$$M(f\Gamma) \leq \int_D Q(x)\rho^n(x)dm(x) \leq \frac{3^n}{d^n} \int_D Q(x)dm(x) < \infty. \tag{3.7}$$

On the other hand

$$M = M(\Gamma(E_1, E_2; D)) \leq M(f\Gamma). \tag{3.8}$$

But as $\partial D'$ is weakly flat, and $E_i, i = 1, 2$, are nondegenerate continua in $\overline{D'}$ with nonempty intersection, it follows that $M = \infty$, contradicting (3.7). The assertion follows.

3.9. Corollary. *Let D, D', f and Q be as in Lemma 3.6. Then f^{-1} has a continuous extension to $\overline{D'}$.*

3.10. Corollary. *Let E be a nondegenerate continuum in $\mathbb{B}^n$ and $Q \in L^1(\mathbb{B}^n \setminus E)$. Then there exists no Q-homeomorphism of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$. If $Q \in L^{n-1}(\mathbb{B}^n \setminus E)$, then there exists no $Q(x)$-qc mapping of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$.*

3.11. Corollary. *Let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in BMO(\overline{D})$. If D locally connected at ∂D and $\partial D'$ is weakly flat, then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

3.12. Theorem. *Let $f : D \to D'$ be a Q-homeomorphism between QED domains D and D' with $Q \in BMO(\overline{D})$. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

This and the next theorem extend the Gehring–Martio results, see [GM], p. 196, and [MV], p. 36, from qc mappings to Q–homeomorphisms with $Q \in BMO(\overline{D})$ and to BMO–qc mappings, respectively.

3.13. Theorem. *Let $f : D \to D'$ be a BMO–qc mapping between uniform domains D and D'. Then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

3.14. Corollary. *If D is a domain in $\mathbb{R}^n$ which is locally connected at ∂D but not a Jordan domain, then D cannot be mapped onto $\mathbb{B}^n$ by a Q-homeomorphism with $Q \in BMO(\overline{D})$.*

3.15. Corollary. *If a domain D in $\mathbb{R}^n$ is uniform but not Jordan, then there is no BMO–qc mapping of D onto $\mathbb{B}^n$.*

The following example shows that, for every $n \geq 3$, there is a proper subdomain of $\mathbb{R}^n$ which is a topological ball and not a BMO–quasiball. Here we say that a domain D in $\mathbb{R}^n, n \geq 2$, is a *BMO-quasiball* if there exists a homeomorphism of D onto $\mathbb{B}^n$ which is BMO-qc.

3.16. Example. Let $B = \mathbb{B}^n \setminus C^n(\varepsilon)$ where $C^n(\varepsilon)$ is a cone with its vertex v at the point of $S^{n-1} = \partial\mathbb{B}^n$ in the hyperplane $x_n = 1$ and with the $(n-1)$–ball $B^{n-1}(\varepsilon)$, $0 < \varepsilon < 1$, in the hyperplane $x_n = 0$ as its base. For $n \geq 3$, the domain B is uniform. Evidently B is a topological ball. However, the boundary of B is not homeomorphic to the sphere S^{n-1} because the point v splits ∂B into two components. Thus, B is not a BMO-quasiball by Corollary 3.15.

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