WEAKLY SELF-SIMILAR STATIONARY INCREMENT PROCESSES FROM THE SPACE $\mathrm{SSub}_{\varphi}(\Omega)$

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Abstract. We consider weakly self-similar processes with stationary increments that belong to the space $\mathrm{SSub}_{\varphi}(\Omega)$. We prove that all separable modifications of these processes are continuous with probability one on compacts. We provide estimates for the probabilities of large deviations and assumptions under which these processes belong to the weighted spaces $C(\mathbb{R}_+, c)$. The results hold true for the fractional Brownian motion with the choice $\varphi(x) = \frac{x^2}{2}$.

1. Introduction

We consider a centred square integrable process $Z_\alpha = (Z_\alpha(t) : t \geq 0)$ that has the covariance function

$$R_\alpha(t,s) \;=\; \frac{1}{2}\left(t^{2\alpha} + s^{2\alpha} - |t-s|^{2\alpha}\right)$$

and belongs to the space $\mathrm{SSub}_{\varphi}(\Omega)$ (to be defined later in Section 2). We shall assume that $\alpha \in (0,1)$ the other cases being either uninteresting or impossibe (cf. Beran [2]). For short, we shall say that Z_α is wsssi-$\mathrm{SSub}_{\varphi}(\Omega)$ (the acronym wsssi is explained below).

The motivation for the covariance function R_α is the following. Suppose that Z_α in self-similar with index α and has stationary increments (sssi, for short). By α-self-similarity we mean that

$$(Z_\alpha(t) : t \geq 0) \;\stackrel{d}{=}\; \left(x^{-\alpha} Z_\alpha(xt) : t \geq 0\right)$$

for all $x > 0$. Here d means equality in distributions. Assume further that the process Z_α is centred and square integrable. Then it is easy to see that Z_α has R_α as its covariance function. Note that the inverse is not true even in the case of stationary increments (for an example we refer to Benassi et al. [1]). Whence the name "weakly sssi" or "wsssi".

The parameter $\alpha \in (0,1)$ has the following role. If $\alpha \neq \frac{1}{2}$ then Z_α is a process with dependent increments. (However, there are α-self-similar processes with independent increments. These are processes with no variance, or course. For details of we refer to Samorodnitsky and Taqqu [12]) If $\alpha > \frac{1}{2}$ then the process Z_α exhibits the so-called long-range dependency property. The case $\alpha < \frac{1}{2}$ corresponds to short-range dependence. For details of long-range dependence see Beran [2].

In the Gaussian case the properties sssi and wsssi of course coincide. In this case Z_α is called the fractional Brownian motion. This process was originally defined and

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studied by Kolmogorov [4] within a Hilbert space framework where it was called a "Wiener helix". It was further studied by Yaglom [15]. The name "fractional Brownian motion" comes from Mandelbrot and Van Ness [9]. They defined it as a stochastic integral with respect to the standard Brownian motion. The self-similarity property of the fractional Brownian motion has made it a popular model in telecommunications [11] and in mathematical finance [10, 13, 14]. The fractional Brownian motion belongs to the space $\mathrm{SSub}_\varphi(\Omega)$ if $\varphi(x)=\frac{x^2}{2}$, i.e. it is (w)sssi-$\mathrm{SSub}_{\frac{x^2}{2}}$.

The paper is organised as follows. In Section 2 we recall some facts about the spaces $\mathrm{Sub}_\varphi(\Omega)$ and $\mathrm{SSub}_\varphi(\Omega)$ and the concept of metric entropy. In Section 3 we consider the process Z_α on compact sets. We show that their separable modifications are continuous with probability one and provide estimates for probabilities of large deviations. In Section 4 we provide assumption under which Z_α belongs to the weighted spaces $C([0,\infty),c)$ and provide an estimate for the supremum of $c(t)Z_\alpha(t)$.

2. Preliminaries

2.1. **Space** $\mathrm{SSub}_\varphi(\Omega)$**.** We recall briefly some basic facts about the generalised sub-Gaussian spaces $\mathrm{Sub}_\varphi(\Omega)$ and $\mathrm{SSub}_\varphi(\Omega)$. For details and proofs we refer to Buldygin and Kozachenko [3].

Definition 2.1. A continuous even convex function u is an Orlitcz N-function if it is increasing for $x>0$, $\frac{u(x)}{x}\to 0$ as $x\to 0$ and $\frac{u(x)}{x}\to\infty$ as $x\to\infty$.

For details of convex functions in Orlitcz spaces we refer to Krasnoselskii and Rutitskii [8].

Let $(\Omega,\mathscr{F},\mathbf{P})$ be a standard probability space.

Definition 2.2. Let φ be an Orlitcz N-function such that there exist some positive constants c and x_0 suct that $\varphi(x)=cx^2$ for all $|x|<x_0$. A zero mean random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if there exists a positive constant a such that the inequality

$$\mathbf{E}\exp(\lambda\xi) \leq \exp(\varphi(a\lambda))$$

holds for all $\lambda\in\mathbb{R}$.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm

$$\tau_\varphi(\xi)=\sup_{\lambda\neq 0}\frac{\varphi^{-1}(\ln\mathbf{E}\exp(\lambda\xi))}{|\lambda|}$$

and the inequalities

$$\begin{aligned}\mathbf{E}\exp(\lambda\xi) &\leq \exp(\varphi(\lambda\tau_\varphi(\xi))), \\ (\mathbf{E}\xi^2)^{\frac12} &\leq \tau_\varphi(\xi).\end{aligned}\tag{2.1}$$

hold for all $\lambda\in\mathbb{R}$.

If $\tau_\varphi(\xi)=(\mathbf{E}\xi^2)^{\frac12}$ then ξ is called strong $\mathrm{Sub}_\varphi(\Omega)$.

Definition 2.3. A family of random variables Δ from the space $\mathrm{Sub}_{\varphi}(\Omega)$ is called strong $\mathrm{Sub}_{\varphi}(\Omega)$ if the equality

$$\tau_{\varphi}\left(\sum_{i\in I}\lambda_i\xi_i\right) = \left(\mathbf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{\frac{1}{2}}$$

holds for all countable subsets $I \subset \Delta$.

If Δ is a strong $\mathrm{Sub}_{\varphi}(\Omega)$ family of random variables then the linear closure of Δ in $L^2(\Omega)$ is also a strong $\mathrm{Sub}_{\varphi}(\Omega)$ family. Linearly closed families of strong $\mathrm{Sub}_{\varphi}(\Omega)$ random variables form a space of strong $\mathrm{Sub}_{\varphi}(\Omega)$ random variables. This space is denoted by $\mathrm{SSub}_{\varphi}(\Omega)$.

Remark 2.4. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSub}_{\varphi}(\Omega)$ if $\varphi(x) = \frac{x^2}{2}$.

Definition 2.5. Let T be some parameter space. A process $X = (X(t), t \in T)$ belongs to the space $\mathrm{SSub}_{\varphi}(\Omega)$ if the corresponding family of random variables belongs to the space $\mathrm{SSub}_{\varphi}(\Omega)$.

The next examples follow from Kozachenko and Kovalchuk [5].

Example 2.6. Let φ be such an Orlitcz N-function that the function $\varphi(\sqrt{\cdot})$ is convex. Let

$$X(t) = \sum_{k=1}^{\infty}\xi_k\psi_k(t),$$

where the series converge in mean square for all $t \in T$ and the family $\{\xi_k : k = 1, 2, \ldots\}$ belongs to the space $\mathrm{SSub}_{\varphi}(\Omega)$ (e.g. the ξ_k's are independent strong $\mathrm{Sub}_{\varphi}(\Omega)$ random variables). Then X is a stochastic process from $\mathrm{SSub}_{\varphi}(\Omega)$.

Example 2.7. Let k be a deterministic kernel and suppose that $X = (X(t) : t \in T)$ is given by

$$X(t) = \int_T k(t,s)\,\mathrm{d}\xi(s),$$

where ξ is a random process from $\mathrm{SSub}_{\varphi}(\Omega)$ and the integral above is defined in the mean square sense. Then X is a stochastic process from $\mathrm{SSub}_{\varphi}(\Omega)$.

Let us now give an example of a process that is wsssi- SSub_{φ} but not the fractional Brownian motion. The example is rather implicit, admittedly.

Example 2.8. Let $Z_\alpha = (Z_\alpha(t) : t \in [0,T])$ be a fractional Brownian motion. Then Z_α may be presented in the form of a sum that is uniformly convergent in mean square, viz.

$$Z_\alpha(t) = \sum_{n=1}^{\infty}\lambda_n\xi_n\psi_n(t).$$

Here the ξ_n's are independent Gaussian random variables with $\mathbf{E}\xi_n = 0$ and $\mathbf{E}\xi_n^2 = 1$. The λ_n^2's are eigenvalues and the ψ_n's are the corresponding eigenfunctions of the integral equation

$$\psi(s) = \frac{1}{\lambda^2}\int_0^T R_\alpha(t,s)\psi(t)\,\mathrm{d}t.$$

Now, let η_n, $n = 1, 2, \dots$, be independent random variables such that $E\eta_n = 0$, $E\eta_n^2 = 1$ and $\eta_n \in \mathrm{SSub}_\varphi(\Omega)$, where the function φ satisfies the assumptions of Example 2.6. Then the process

$$\tilde{Z}_\alpha(t) = \sum_{n=1}^{\infty} \lambda_n \eta_n \psi_n(t)$$

is centered with the covariance function R_α. From Example 2.6 it follows that this process belongs to the space $\mathrm{SSub}_\varphi(\Omega)$, i.e. $\tilde{Z}_\alpha = (\tilde{Z}_\alpha(t), t \in [0,T])$ is wsssi-SSub_φ.

2.2. **Metric entropy.** Let us recall the concept of metric entropy. For details see e.g. [3].

Definition 2.9. Let (T, ρ) be a pseudometric space. The metric entropy is

$$H(u) \quad := \quad \ln N_{(T,\rho)}(u)$$

where $N_{(T,\rho)}(u)$ denotes the least number of closed ρ-balls whose diameter do not exceed $2u$ needed to cover T.

Remark 2.10. It should be noted that usually one does not define the metric entropy in terms of the pseudometric ρ but rather with respect to a pseudometric induced by the incremental variance of a stochastic process. In our case this pseudometric would be $\rho_\alpha(t,s) = \sigma(t-s) = |t-s|^\alpha$. We choose to use the underlying (pseudo)metric in our formulations, however.

Let us elaborate Remark 2.10 above.

Example 2.11. Let T be the interval $[a,b]$ equipped with the metric $\sigma(t) = t^\alpha$ induced by the process Z_α. Then

$$H(u) \quad = \quad H_0(\sigma^{-1}(u)) \quad = \quad H_0(u^{-\alpha}),$$

where H_0 is the Euclidian metric entropy of $[a,b]$.

Example 2.12. If T is the interval $[a,b]$ and ρ is the Euclidian distance then

$$\ln\left(\frac{b-a}{2u} \bigvee 1\right) \quad \le \quad H(u) \quad \le \quad \ln\left(\frac{b-a}{2u} + 1\right)$$

where $\vee$ denotes the maximum.

Remark 2.13. The metric entropy is used to provide the continuity of the wsssi-$\mathrm{SSub}_\varphi(\Omega)$ processes. It should be noted that for the sssi-$\mathrm{SSub}_\varphi(\Omega)$ case, e.g. the case of fractional Brownian motion, the continuity follows easily from the Kolmogorov criterion. Indeed, for any $n \ge 1$ we have

$$\mathbf{E}\big|Z_\alpha(t) - Z_\alpha(s)\big|^n \quad = \quad \mathbf{E}\big||t-s|^\alpha Z_\alpha(1)\big|^n \quad = \quad |t-s|^{n\alpha}\gamma_n,$$

where γ_n, the nth absolut moment of $Z_\alpha(1)$, exists by virtue of the inequality (2.1). So, a sssi-$\mathrm{SSub}_\varphi(\Omega)$ process has a version with β-Hölder continuous sample paths with any index $\beta < \alpha$. In the general case we cannot use the self-similarity property or the strict stationarity of the increments, however. Moreover, the metric entropy provides us estimates for the probabilities of large deviations. Nothing can be said about the Hölder continuity of the wsssi-$\mathrm{SSub}_\varphi(\Omega)$ processes, though.

3. WSSSI-SSub$_\varphi$ ON COMPACT SETS

Let us recall two lemmas. Lemma 3.1 is a modification of Lemma 3.1 of Kozachenko and Vasilik [6]. Lemma 3.2 is presented in [7] as Theorem 2.4. It may be proved like Theorem 3.5.1 of [3].

Lemma 3.1. *Let (T,ρ) be a metric separable compact space and let $X=(X(t),t\in T)$ is a separable process from the space $\mathrm{Sub}_\varphi(\Omega)$. Suppose that there exists a strictly increasing continuous function σ with $\sigma(0)=0$ such that*

$$\sup_{\rho(t,s)\le h}\tau_\varphi(X(t)-X(s)) \;\le\; \sigma(h) \tag{3.1}$$

and for some $\varepsilon>0$ we have

$$\int_0^\varepsilon \zeta(u)\,\mathrm{d}u \;<\; \infty \tag{3.2}$$

where

$$\zeta(u) \;:=\; \frac{H(\sigma^{-1}(u))}{\varphi^{-1}(H(\sigma^{-1}(u)))}$$

and H is the metric entropy on (T,ρ). Then for all $p\in(0,1)$ and $\lambda>0$ we have

$$\mathbf{E}\exp\left(\lambda\sup_{t\in T}X(t)\right) \;\le\; \Gamma(\lambda;p,\beta), \tag{3.3}$$

$$\mathbf{E}\exp\left(-\lambda\inf_{t\in T}X(t)\right) \;\le\; \Gamma(\lambda;p,\beta), \tag{3.4}$$

where

$$\Gamma(\lambda;p,\beta) \;=\; \exp\Bigg(\varphi\left(\frac{\lambda\gamma_0}{(1-p)}\right)(1-p)+\varphi\left(\frac{\lambda\beta}{(1-p)}\right)p+ \\ 2\lambda\left(\gamma_0\zeta(p\beta)+\frac{1}{(1-p)p}\int_0^{\beta p^2}\zeta(u)\,\mathrm{d}u\right)\Bigg).$$

Here β is any positive number such that $\beta\le\sigma(\inf_{s\in T}\sup_{t\in T}\rho(t,s))$ and $\gamma_0=\sup_{t\in T}\tau_\varphi(X(t))$.

Lemma 3.2. *The assumption (3.2) provides continuity with probability one of the process X on (T,ρ). Moreover, we have*

$$\mathbf{P}\Bigg(\lambda\sup_{\rho(t,s)<\varepsilon}|X(t)-X(s)|>\delta\Bigg) \;\le\; \exp\Bigg(-\lambda\delta+(1-p)\varphi\left(\frac{\lambda\sigma(\varepsilon)}{1-p}\right)+ \\ 2p\varphi\left(\frac{2\lambda\sigma(\varepsilon)}{p(1-p)}\right)+\frac{8\lambda}{p(1-p)}\int_0^{p\sigma(\varepsilon)}\zeta(u)\,\mathrm{d}u\Bigg)$$

for any $\delta>0$ and $\lambda\ge0$.

The following theorem is an application of the lemmas above.

Theorem 3.3. *Let $Z_\alpha = (Z_\alpha(t), t \in [a,b])$ be a separable* wsssi-$\mathrm{SSub}_\varphi(\Omega)$ *process. Then Z_α is continuous and bounded with probability one. Moreover,*

$$\mathbf{P}\Big(\sup_{t\in T} Z_\alpha(t) > \varepsilon\Big) \;\leq\; \Delta(\varepsilon;\beta,p,\lambda), \tag{3.5}$$

$$\mathbf{P}\Big(\inf_{t\in T} Z_\alpha(t) < -\varepsilon\Big) \;\leq\; \Delta(\varepsilon;\beta,p,\lambda), \tag{3.6}$$

$$\mathbf{P}\Big(\sup_{t\in T} |Z_\alpha(t)| > \varepsilon\Big) \;\leq\; 2\Delta(\varepsilon;\beta,p,\lambda), \tag{3.7}$$

for any $\varepsilon > 0$, $\beta \leq \left(\frac{b-a}{2}\right)^\alpha$, $p \in (0,1)$ and $\lambda > 0$, where

$$\Delta(\varepsilon;\beta,p,\lambda) \;=\; \exp\Bigg(-\lambda\varepsilon + \varphi\left(\frac{\lambda b^\alpha}{(1-p)}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{(1-p)}\right)p + 2\lambda\bigg(b^\alpha\zeta(p\beta) - \frac{1}{(1-p)p}\int_0^{\beta p^2} \zeta(u)\,\mathrm{d}u\bigg)\Bigg)$$

and

$$\zeta(u) = \frac{\ln\left(\frac{b-a}{2u^{1/\alpha}}+1\right)}{\varphi^{-1}\left(\ln\left(\frac{b-a}{2u^{1/\alpha}}+1\right)\right)}. \tag{3.8}$$

Proof. In our case the the condition (3.1) is satisfied with an equality with the choice $\sigma(u) = u^\alpha$. Now, it is easy to see that

$$H(\sigma^{-1}(u)) \;\leq\; \ln\left(\frac{b-a}{2u^{1/\alpha}}+1\right)$$

and for small enough u we have the inequality $H(\sigma^{-1}(u)) \leq C\ln\frac{1}{u}$, where $C > 0$ is some constant.

The function $v \mapsto \frac{v}{\varphi^{-1}(v)}$ is strictly increasing (cf. Lemma 2.2.3 of [3]). So, the integral (3.2) converges if the integral $\int_0^\varepsilon \zeta(C\ln\frac{1}{v})\,\mathrm{d}v$ converges. But if v is large, then $\zeta(v) \leq v$ and the integral (3.2) converges since $\int_0^\varepsilon \ln\frac{1}{u}\,\mathrm{d}u$ converges if ε is small enough. Therefore the assumption (3.2) of Lemma 3.1 is satisfied and the process Z_α is continuous and bounded with probability one by Lemma 3.2. The inequalities (3.5) – (3.6) follow from the inequalities (3.3) – (3.4) by using the Chebychev inequality

$$\mathbf{P}\Big(\sup_{t\in[a,b]} Z_\alpha(t) > \varepsilon\Big) \;\leq\; \exp(-\lambda\varepsilon)\mathbf{E}\exp\Big(\lambda\sup_{t\in[a,b]} Z_\alpha(t)\Big)$$

and noticing that $\gamma_0 = b^\alpha$. □

Let us denote by f^* the Young–Fenchel transform of the function f, i.e.

$$f^*(x) \;=\; \sup_{y>0}(xy - f(y)), \quad x > 0.$$

The next corollary is now evident.

Corollary 3.4. *Under the assumptions of Theorem 3.3 for any* $\varepsilon > D(p,\beta)$ *and* $p \in (0,1)$, *we have*

$$\begin{aligned} \mathbf{P}\Big(\sup_{t\in[a,b]} Z_\alpha(t) > \varepsilon\Big) &\le W(\varepsilon;\beta,p), \\ \mathbf{P}\Big(\inf_{t\in[a,b]} Z_\alpha(t) < -\varepsilon\Big) &\le W(\varepsilon;\beta,p), \\ \mathbf{P}\Big(\sup_{t\in[a,b]} |Z_\alpha(t)| > \varepsilon\Big) &\le 2W(\varepsilon;\beta,p), \end{aligned}$$

where

$$\begin{aligned} W(\varepsilon;\beta,p) &= \exp\big(-y^*(\varepsilon - D(p,\beta))\big), \\ y(\lambda) &= \varphi\left(\frac{\lambda b^\alpha}{(1-p)}\right)(1-p) - \varphi\left(\frac{\lambda\beta}{(1-p)}\right)p, \\ D(p,\beta) &= 2\left(b^\alpha\zeta(p\beta) + \frac{1}{(1-p)p}\int_0^{\beta p^2}\zeta(u)du\right) \end{aligned}$$

and ζ *is defined in (3.8).*

Corollary 3.5. *Under the assumptions of Theorem 3.3 we have for any* $\varepsilon > \tilde{D}(p)$ *and* $p \in (0,1)$ *the inequalities*

$$\begin{aligned} \mathbf{P}\Big(\sup_{t\in T} Z_\alpha(t) > \varepsilon\Big) &\le \tilde{W}(\varepsilon;p), \\ \mathbf{P}\Big(\inf_{t\in T} Z_\alpha(t) < -\varepsilon\Big) &\le \tilde{W}(\varepsilon;p), \\ \mathbf{P}\Big(\sup_{t\in T} |Z_\alpha(t)| > \varepsilon\Big) &\le 2\tilde{W}(\varepsilon;p), \end{aligned}$$

where

$$\begin{aligned} \tilde{W}(\varepsilon;p) &= \exp\left(-\varphi^*\left(\frac{1-p}{b^\alpha}(\varepsilon - \tilde{D}(p))\right)\right) \\ \tilde{D}(p) &= \frac{2b^\alpha\alpha}{(1-p)p}\int_{p^{-1/\alpha}}^{\infty}\frac{a(\ln(u+1))}{u^{\alpha+1}}\,\mathrm{d}u, \\ a(v) &= \frac{v}{\varphi^{-1}(v)}. \end{aligned}$$

Proof. Let $\beta = \left(\frac{b-a}{2}\right)^\alpha$. Since $\frac{\gamma_0}{\beta} = \left(\frac{2b}{b-a}\right)^\alpha > 1$ we have

$$\begin{aligned} D(p,\beta) &\leq \frac{2b^\alpha \alpha}{\beta(1-p)p} \int_0^{\beta p} \zeta(u)\,\mathrm{d}u \\ &= \frac{2b^\alpha \alpha}{\beta(1-p)p} \left(\frac{b-a}{2}\right)^\alpha \int_0^p a(\ln(\frac{1}{t^{\frac{1}{\alpha}}}+1))\,\mathrm{d}t \\ &= \frac{2b^\alpha \alpha}{(1-p)p} \int_{p^{\frac{-1}{\alpha}}}^\infty \frac{a(\ln(u+1))}{u^{\alpha+1}}\,\mathrm{d}u \\ &= \tilde{D}(p). \end{aligned}$$

Moreover, $\varphi\left(\frac{\lambda b^\alpha}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{p}\right)p \leq \varphi\left(\frac{\lambda b^\alpha}{1-p}\right)$. Therefore, the assertion of the corollary follows from (3.5) – (3.7). ☐

Corollary 3.6. *Let $u_0 \geq (e^{\varphi(1)}-1)^\alpha$ be such a number that for any $u \geq u_0$ we have $u \geq 1 + (\frac{1}{\alpha} + \ln(1+u^{1/\alpha}))$ Then, under the assumptions of Theorem 3.3 we have for any $\varepsilon > b^\alpha u_0$*

$$\begin{aligned} \mathbf{P}\bigg(\sup_{t\in T} Z_\alpha(t) > \varepsilon\bigg) &\leq W_0(\varepsilon), \\ \mathbf{P}\bigg(\inf_{t\in T} Z_\alpha(t) < -\varepsilon\bigg) &\leq W_0(\varepsilon), \\ \mathbf{P}\bigg(\sup_{t\in T} |Z_\alpha(t)| > \varepsilon\bigg) &\leq 2W_0(\varepsilon), \end{aligned}$$

where

$$W_0(\varepsilon) = \exp\bigg(-\varphi^*\bigg(\frac{\varepsilon}{b^\alpha} - 1 - \frac{2}{(1-\frac{b^\alpha}{\varepsilon})}\Big(\frac{1}{\alpha} + \ln\big(1+(\frac{\varepsilon}{b^\alpha})^{\frac{1}{\alpha}}\big)\Big)\bigg)\bigg).$$

Proof. If $\varphi^{-1}(\ln(p^{-\frac{1}{\alpha}}+1)) > 1$ then

$$\tilde{D}(p) \leq \frac{2b^\alpha\alpha}{(1-p)p}\int_{p^{\frac{-1}{\alpha}}}^\infty \frac{\ln(u+1)}{u^{\alpha+1}}\,\mathrm{d}u \leq \frac{2b^\alpha}{(1-p)}\left(\frac{1}{\alpha} + \ln(p^{-\frac{1}{\alpha}}+1)\right) = \hat{D}(p)$$

Now, since $\varepsilon > \hat{D}(p)$, it follows from Corollary 3.5 that

$$\tilde{W}(p,\varepsilon) \leq \exp\left(-\varphi^*\left(\frac{\varepsilon}{b^\alpha} - \frac{p\varepsilon}{b^\alpha} - \frac{\hat{D}(p)}{b^\alpha}\right)\right)$$

So, the claim follows from this by setting $p = \frac{b^\alpha}{\varepsilon}$. ☐

Example 3.7. If $\varphi(x)=\frac{x^2}{2}$ then $\varphi^*(x)=\frac{x^2}{2}$ and

$$\begin{aligned} W_0(\varepsilon) &= \exp\Bigg(-\frac{1}{2}\Big(\frac{\varepsilon}{b^\alpha}-1-\frac{2}{(1-\frac{b^\alpha}{\varepsilon})}\Big(\frac{1}{\alpha}+\ln\big(1+(\frac{\varepsilon}{b^\alpha})^{\frac{1}{\alpha}}\big)\Big)\Big)\Bigg) \\ &\sim \exp\Bigg(-\frac{1}{2}\frac{\varepsilon^2}{b^{2\alpha}}+O\Big(\ln\Big(1+\frac{\varepsilon}{b^\alpha}\Big)\Big)\Bigg). \end{aligned}$$

Of course, this is also an estimate for the fractional Brownian motion.

Example 3.8. If $\varphi(x)=\frac{x^p}{p}$, $p>1$, for $x>x_0>0$, then for large enough ε we have

$$W_0(\varepsilon) = \exp\Bigg(-\frac{1}{q}\Big(\frac{\varepsilon}{b^\alpha}-1-\frac{2}{(1-\frac{b^\alpha}{\varepsilon})}\Big(\frac{1}{\alpha}+\ln\big(1+(\frac{\varepsilon}{b^\alpha})^{\frac{1}{\alpha}}\big)\Big)\Big)^q\Bigg),$$

where q is such number that $\frac{1}{p}+\frac{1}{q}=1$. □

Remark 3.9. The estimates of this section can be improved by using of other theorems from the book [3]. Also, if Z_α is *strictly* sssi then the estimates can be improved by using the obvious equality

$$\mathbf{P}\Big(\sup_{t\in[0,b]} Z_\alpha(t)>\varepsilon\Big) = \mathbf{P}\Big(\sup_{t\in[0,1]} Z_\alpha(t)>\varepsilon b^{-\alpha}\Big).$$

4. Wsssi-SSub$_\varphi$ on $\mathbb{R}_+$.

In this section we suppose that the process Z_α is defined on $\mathbb{R}_+$ and consider its weighted supremum.

Definition 4.1. Let c be a strictly positive continuous function on $\mathbb{R}_+$. Then $C(\mathbb{R}_+,c)$ is the space of continuous functions f over $\mathbb{R}_+$ such that we have $\sup_{t\geq 0} c(t)|f(t)|<\infty$.

Space $C_0(\mathbb{R}_+,c)$ is the space of continuous functions f over $\mathbb{R}_+$ such that we have $\sup_{t\geq 0} c(t)f(t)\to\infty$ as $t\to\infty$.

Remark 4.2. Let us note here that for a wsssi-SSub$_\varphi(\Omega)$ process Z_α to have sample paths in $C(\mathbb{R}_+,c)$ it is necessary that $c(t)\to 0$ as $t\to\infty$ since $\mathbf{Var}\, Z_\alpha(t)=t^{2\alpha}\to\infty$ as $t\to\infty$.

We shall use the next lemma from Kozachenko and Vasilik [6].

Lemma 4.3. *Let (T,ρ) be a pseudometric separable space and $X=(X(t):t\in T)$ be a separable random process from the space* Sub$_\varphi(\Omega)$. *Suppose there exists a strictly increasing continuous function σ such that $\sigma(h)>0$ as $h>0$, $\sigma(h)\to 0$ as $h\to 0$ and*

$$\sup_{\rho(t,s)\leq h} \tau_\varphi\left(X(t)-X(s)\right) \leq \sigma(h).$$

Let $T = \cup_{\ell=1}^{\infty} B_\ell$, where the B_ℓ's are compact sets and let c be a continuous function such that $|c(t)| \le 1$. Denote

$$\begin{aligned} \delta_\ell &= \sup_{t\in B_\ell} |c(t)|, \\ \alpha_\ell &= \sigma(\inf_{t\in B_\ell} \sup_{s\in B_\ell} \rho(t,s)), \\ \gamma_\ell &= \sup_{t\in B_\ell} \tau_\varphi(X(t)), \\ \beta &= \inf_{\ell\ge 1} \frac{\alpha_\ell}{\gamma_\ell}. \end{aligned}$$

Suppose that the following conditions hold

$$d = \sum_{\ell=1}^{\infty} \delta_\ell \gamma_\ell < \infty, \tag{4.1}$$

$$\int_0^\beta \zeta_\ell(u)\,\mathrm{d}u < \infty, \tag{4.2}$$

where $\zeta_\ell(u) = \frac{H_\ell(\sigma^{-1}(u))}{\varphi^{-1}(H_\ell(\sigma^{-1}(u)))}$ and H_ℓ is the metric entropy of the set B_ℓ, and for any $p \in (0,1)$

$$\sum_{\ell=1}^{\infty} \left(\delta_\ell \gamma_\ell \zeta_\ell(p\beta\gamma_\ell) + \frac{\delta_\ell}{(1-p)p} \int_0^{\beta\gamma_\ell p^2} \zeta_\ell(u) du \right) < \infty. \tag{4.3}$$

Then for all $\lambda > 0$ and $p \in (0,1)$ we have

$$\mathbf{E} \exp\left(\lambda \sup_{t\in T} |c(t)x(t)| \right) \le \Phi(\lambda, p),$$

where

$$\begin{aligned} \Phi(\lambda,p) &= 2\exp\Bigg(\varphi\left(\frac{\lambda d}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda d\beta}{1-p}\right)p + \\ &\qquad 2\lambda\Bigg(\sum_{\ell=1}^{\infty} \left(\delta_\ell\gamma_\ell\zeta_\ell(p\beta\gamma_\ell) + \frac{\delta_\ell}{(1-p)p}\int_0^{\beta\gamma_\ell p^2} \zeta_\ell(u) du \right)\Bigg)\Bigg). \end{aligned}$$

The following theorem is an application of Lemma 4.3 above.

Theorem 4.4. *Let $Z_\alpha = (Z_\alpha(t), t \ge 0)$ be a separable* wsssi-SSub$_\varphi(\Omega)$ *process. Suppose there exists a sequence $(x_\ell : \ell = 1, 2, \ldots)$ increasing to infinity such that*

$x_1 = 0$ and

$$C = \sup_{\ell\geq 1} \frac{x_\ell}{x_{\ell+1}} < 1, \tag{4.4}$$

$$\sum_{\ell=1}^{\infty} c(x_\ell) x_\ell^\alpha < \infty. \tag{4.5}$$

Then for any $\varepsilon > 0$ and $p \in (0,1)$ we have

$$\mathbf{P}\left(\sup_{t\in T} |c(t) Z_\alpha(t)| > \varepsilon\right) \leq 2\exp\bigl(-A^*(\varepsilon - L(p))\bigr), \tag{4.6}$$

where

$$\begin{aligned}
A(y) &= \varphi\left(\frac{yd}{1-p}\right)(1-p) + \varphi\left(\frac{yd\beta}{1-p}\right)p, \\
L(p) &= \frac{2^{1-\alpha}}{\beta(1-p)p} d_1 I(p), \\
I(p) &= \int_0^p a\left(\ln t^{-\frac{1}{\alpha}} + 1\right) \mathrm{d}t, \\
a(v) &= \frac{v}{\varphi^{-1}(v)}, \\
d_1 &= \sum_{\ell=1}^{\infty} c(x_\ell)(x_{\ell+1} - x_\ell)^\alpha.
\end{aligned}$$

In particular, the process Z_α has sample paths in $C(\mathbb{R}_+, c)$ almost surely.

Proof. Set $B_\ell = [x_\ell, x_{\ell+1}]$, $\ell = 1, 2, \ldots$. Since in our case $\sigma(h) = h^\alpha$ we have $\gamma_\ell = x_{\ell+1}^\alpha$, $\alpha_\ell = (\frac{x_{\ell+1} - x_\ell}{2})^\alpha$, $\frac{\alpha_\ell}{\gamma_\ell} = \frac{1}{2^\alpha}\left(1 - \frac{x_\ell}{x_{\ell+1}}\right)^\alpha$ and $\beta = \frac{1}{2^\alpha}(1-C)^\alpha$. From the assumption (4.5) follows that $d = c(x_\ell)x_{\ell+1}^\alpha$ is finite.

Since

$$\gamma_\ell \zeta_\ell(\beta p \gamma_\ell) \leq \frac{1}{\beta p(1-p)} \int_{\beta\gamma_\ell p^2}^{\beta\gamma_\ell p} \zeta_\ell(u)\,\mathrm{d}u$$

and $\beta < 1$ we have

$$\sum_{\ell=1}^{\infty} \delta_\ell \gamma_\ell \zeta_\ell(p\beta\gamma_l) + \frac{\delta_\ell}{(1-p)p} \int_0^{\beta\gamma_\ell p^2} \zeta_\ell(u)\,\mathrm{d}u \leq \frac{1}{\beta p(1-p)} \sum_{\ell=1}^{\infty} \delta_\ell \int_0^{\beta\gamma_\ell p} \zeta_\ell(u)\,\mathrm{d}u.$$

As for the metric entropy of $[x_\ell, x_{\ell+1}]$ we have

$$H_\ell(\sigma^{-1}(u)) \leq \ln\left(\frac{x_{\ell+1} - x_\ell}{2\sigma^{-1}(u)} + 1\right) = \ln\left(\frac{x_{\ell+1} - x_\ell}{2u^{1/\alpha}} + 1\right).$$

Set $a(v) = \frac{v}{\varphi^{-1}(v)}$. So, we have

$$\begin{aligned}\int_0^{\beta\gamma_\ell p} \zeta_\ell(u)\,\mathrm{d}u &= \int_0^{\beta\gamma_\ell p} a\left(\ln\left(\frac{x_{\ell+1}-x_\ell}{2u^{1/\alpha}}+1\right)\right)\,\mathrm{d}u \\ &= \frac{(x_{\ell+1}-x_\ell)^\alpha}{2^\alpha}\int_0^{\frac{\beta\gamma_\ell p 2^\alpha}{(x_{\ell+1}-x_\ell)^\alpha}} a\left(\ln\left(\frac{1}{t^{1/\alpha}}+1\right)\right)\,\mathrm{d}t.\end{aligned}$$

Since

$$\begin{aligned}\frac{\beta\gamma_\ell p 2^\alpha}{(x_{\ell+1}-x_\ell)^\alpha} &= \beta\frac{x_{\ell+1}^\alpha 2^\alpha}{(x_{\ell+1}-x_\ell)^\alpha} \\ &= \beta 2^\alpha\frac{1}{(1-\frac{x_\ell}{x_{\ell+1}})^\alpha} \\ &\le \beta 2^\alpha\frac{1}{(1-C)^\alpha} \\ &= 1\end{aligned}$$

we have

$$\int_0^{\beta\gamma_\ell p} \zeta_\ell(u)\,\mathrm{d}u \le \frac{(x_{\ell+1}-x_\ell)^\alpha}{2^\alpha}\int_0^p a\left(\ln\left(\frac{1}{t^{1/\alpha}}+1\right)\right)\,\mathrm{d}t.$$

Combining the bounds above we obtain

$$\begin{aligned}&\sum_{\ell=1}^\infty \delta_\ell\gamma_\ell\zeta_\ell(p\beta\gamma_\ell) + \frac{\delta_\ell}{(1-p)p}\int_0^{\beta\gamma_\ell p^2}\zeta_\ell(u)\,\mathrm{d}u \\ &\le \frac{1}{2^\alpha\beta p(1-p)}\left(\sum_{\ell=1}^\infty c(x_\ell)(x_{\ell+1}-x_\ell)^\alpha\right)\int_0^p a\left(\ln\left(\frac{1}{t^{1/\alpha}}+1\right)\right)\,\mathrm{d}t \qquad (4.7)\\ &\le \frac{1}{2^\alpha\beta p(1-p)}\int_0^p a\left(\ln\left(\frac{1}{t^{1/\alpha}}+1\right)\right)\,\mathrm{d}t\sum_{\ell=1}^\infty c(x_\ell)x_{\ell+1}^\alpha \\ &< \infty.\end{aligned}$$

So, we proved that the conditions (4.1) – (4.3) of Lemma 4.3 hold. Therefore, Z_α belongs to the space $C(\mathbb{R}_+, c)$ almost surely and for all $\lambda > 0$ we have, by (4.7),

$$\mathbf{E}\exp\left(\lambda\sup_{t\in T}|c(t)x(t)|\right) \le \tilde{\Phi}(\lambda, p) \qquad (4.8)$$

where

$$\tilde{\Phi}(\lambda, p) = 2\exp\left(\varphi\left(\frac{\lambda d}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda d\beta}{1-p}\right)p + 2\lambda\frac{1}{2^\alpha \beta p(1-p)}\int_0^p a\left(\ln\left(\frac{1}{t^{1/\alpha}}+1\right)\right)\mathrm{d}t d_1\right)$$

and

$$d = \sum_{\ell=1}^{\infty} c(x_\ell) x_{\ell+1}^\alpha, \qquad d_1 = \sum_{\ell=1}^{\infty} c(x_\ell)(x_{\ell+1} - x_\ell)^\alpha$$

Finally, the inequality (4.6) follows from (4.8) and the Chebyshev inequality. □

Remark 4.5. It is easy to obtain from Theorem 4.4 the same corollaries as from Theorem 3.3.

Example 4.6. Let $x_\ell = e^\ell$. Obviously $(x_\ell : \ell = 1, 2, \ldots)$ grows fast enough to infinity to satisfy the condition (4.4). Now, it is easy to see that the condition (4.5) is satified for the following functions

$$\begin{aligned} c(t) &= t^{-(\alpha+\varepsilon)}, \qquad \varepsilon > 0, t > 1, \\ c(t) &= t^{-\alpha}(\ln t)^{-(1+\varepsilon)}, \qquad \varepsilon > 0, t > e, \\ c(t) &= t^{-\alpha}(\ln t)^{-1}(\ln\ln t)^{-(1+\varepsilon)}, \qquad \varepsilon > 0, t > e^e. \end{aligned}$$

Remark 4.7. If a process belongs to the space $C(\mathbb{R}_+, c)$ then it belongs to the space $C_0(\mathbb{R}_+, cg)$, where $g = (g(t), t \in \mathbb{R}_+)$ is a function satisfying $g(t) \geq 0$ and $g(t) \to 0$ as $t \to \infty$. Therefore, the process Z_α belongs to the space $C_0(\mathbb{R}_+, c)$, where c is a function from Example 4.6 above.

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