Relative distance and boundary properties of nonparametric surfaces with finite area

O. Martio, V.M. Miklyukov, and M. Vuorinen

File: qcont9.tex, March 12, 2002

Abstract

Some boundary properties of nonparametric surfaces with finite area are proved.

1 Main Theorem

Let $D \subset \mathbf{R}^2$ be a simply connected domain with a Jordan boundary ∂D and $O \in D$ be a fixed point. Let $U \subset D$ be an open set. We denote by $\overline{U}$ the closure in the Euclidean topology of $\mathbf{R}^2$, by $[U] = \overline{U} \setminus \partial D$, and $\partial' U = [U] \setminus U$.

1.1. Definition. Let $f : D \to \mathbf{R}^m$ be a continuous vector function. We will say that $a \in \partial D$ is the *point of ∂'-continuity* of f, if there exists a sequence of subdomains $\{D_k\}_{k=1}^{\infty} \subset D$ with the following properties:
(α) each $\partial' D_k$ separates the point a from the point O;
(β) $\cap_k [D_k] = \emptyset$; $\operatorname{diam} \partial' D_k \to 0$ as $k \to \infty$;
(γ) $\liminf_{k\to\infty} \operatorname{osc}(f, \partial' D_k) = 0$.

Here

$$\operatorname{osc}(f, E) = \sup_{x,y\in E} |f(y) - f(x)|$$

is the *oscillation* of $f : D \to \mathbf{R}^m$ in the set $E \subset D$.

The rest of the points of ∂D are called *∂'-jump points* of the function f. These concepts are independent of the point O.

We set

$$\delta(a, f, O) = \inf_{\gamma} \max\{\operatorname{osc}(f, \gamma), \operatorname{diam}(\gamma)\},$$

where infimum is taken over all arcs $\gamma \subset D$, $\overline{\gamma} \cap \partial D \neq \emptyset$, separating points a and O on D. It is easy to see that a point $a \in \partial D$ is the point of ∂'-continuity of f if and only if $\delta(a, f, O) = 0$.

A vector function $f : D \to \mathbf{R}^m$ is called *monotone*, if for each subdomain $\Delta \subset\subset D$,

$$\operatorname{osc}(f, \Delta) \leq \operatorname{osc}(f, \partial\Delta).$$

A vector function $f : D \to \mathbf{R}^m$ belongs to $\operatorname{Lip} D$ if for some $L < \infty$,

$$|f(y) - f(x)| \leq L\,|y - x| \quad \text{for all} \quad x,\, y \in D.$$

A vector function is called *locally Lipschitz* if $f \in \mathrm{Lip}\,\Delta$ for each subdomain $\Delta \subset\subset D$.

Let $\mathcal{S}$ be a 2-dimensional surface in $\mathbf{R}^m$, $m \geq 3$, given by the vector function

$$\chi = \chi(\xi) = (\xi_1, \xi_2, g_1(\xi), \dots, g_{m-2}(\xi)) : D \to \mathbf{R}^m\,, \tag{1.2}$$

where $\xi = (\xi_1, \xi_2)$ and g_i, $i = 1, \dots, m-2$, are locally Lipschitz functions in D.

The system (1.2) describes a two–dimensional nonparametric surface $\mathcal{S} = \chi(D)$.

The following statement contains the main result of this article.

1.3. Theorem. *Let $D \subset \mathbf{R}^2$ be a bounded Jordan domain with $O \in D$. Let $\mathcal{S}$ be a nonparametric surface given by a monotone vector function (1.2).*

If $\mathrm{area}\,\mathcal{S} < \infty$, then $\chi(\xi)$ is ∂'-continuous at all points $a \in \partial D$ except possibly on a countable set. Moreover, if $a_1, a_2, \dots$ are ∂'-jump points of $\chi(\xi)$, then

$$\begin{aligned}&\textstyle\sum_{i=1}^{3} \ \max\left\{\pi - \frac{K^2}{\pi\delta^2(a_i,\chi,O)}, 0\right\} + \\ &\qquad +2\ \textstyle\sum_{i=4}^{\infty} \arcsin\left(\frac{1}{2}\exp\left\{-\frac{K^2}{\delta^2(a_i,\chi,O)}\right\}\right) \leq \pi\,,\end{aligned} \tag{1.4}$$

where

$$K = 2\sqrt{\pi}\,(\mathrm{area}\,\mathcal{S})^{1/2}\ .$$

2 Relative Distance in $\mathcal{S}$

Let D be a subdomain of $\mathbf{R}^2$ and let $\mathcal{S}$ be a nonparametric locally Lipschitz surface (1.2). The line element on $\mathcal{S}$ is defined in the usual way

$$\begin{aligned}ds^2_{\mathcal{S}} &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + \ldots + (dx_m)^2 = \\ &= (d\xi_1)^2 + (d\xi_2)^2 + (\textstyle\sum_{i=1}^{2} g'_{1,\xi_i})^2 + \ldots + (\textstyle\sum_{i=1}^{2} g'_{m-2,\xi_i})^2 = \\ &= \sum_{i,j=1}^{2}\left(\delta_{ij} + \sum_{k=1}^{m-2}\frac{\partial g_k}{\partial \xi_i}\frac{\partial g_k}{\partial \xi_j}\right)\,d\xi_i\,d\xi_j\end{aligned} \tag{2.1}$$

and this generates a Riemannian metric on $\mathcal{S}$.

The mapping (1.2) is one-to-one. We denote by $j : \mathcal{S} \to D$ its inverse mapping. It is a projection of $\mathcal{S}$ into D.

Suppose that the domain D is simply connected and fix a point $O \in D$. Let $O' = \chi(O)$ be the image of O on $\mathcal{S}$.

For points $p, q \in \mathcal{S} \setminus \{O'\}$ we set

$$\rho(p, q; O', \mathcal{S}) = \min\{\rho_1(p,q),\ \rho_2(p,q)\}\,, \tag{2.2}$$

where ρ_1 is infimum of diameters in the metric of $\mathbf{R}^m$ of arcs $\gamma \subset \mathcal{S} \setminus \{O'\}$ connecting p and q; ρ_2 is infimum of diameters of open Jordan arcs γ lying in $\mathcal{S} \setminus \{O'\}$ and separating p, q from O' on $\mathcal{S}$. The quantity ρ is called the *relative distance* between points $p,\, q \in \mathcal{S} \setminus \{O'\}$. The relative distance between a point $p \in \mathcal{S} \setminus \{O'\}$ and the point O' is

$$\rho(p, O';\ O', \mathcal{S}) =: \lim_{\substack{q \to O' \\ q \neq O'}} \rho(p, q;\ O', \mathcal{S})\,.$$

2.3. Lemma. *The function ρ defines a distance in $\mathcal{S}$.*

Proof. Only the triangle inequality requires a proof. Moreover it suffices to prove the triangle inequality in the case the three points p, q, r belong to $\mathcal{S} \setminus \{O'\}$. We consider three cases.

In the *first case* we assume that

$$\rho(p,\, r; O', \mathcal{S}) = \rho_1(p,\, r), \quad \rho(r,\, q; O', \mathcal{S}) = \rho_1(r,\, q)\,.$$

Let $\epsilon > 0$. We choose arcs γ_1 connecting p to r and γ_2 connecting r to q such that

$$\mathrm{diam}\,(\gamma_1) \leq \rho_1(p,\, r) + \frac{\epsilon}{2}, \quad \mathrm{diam}\,(\gamma_2) \leq \rho_1(r,\, q) + \frac{\epsilon}{2}\,.$$

Because the arc $\gamma_1 \cup \gamma_2$ joins p and q, we have

$$\rho_1(p,\, q) \leq \mathrm{diam}\,(\gamma_1) + \mathrm{diam}\,(\gamma_2) \leq \rho_1(p,\, r) + \rho_1(r,\, q) + \epsilon\,,$$

and hence

$$\begin{aligned} \rho(p,\, q; O', \mathcal{S}) \ &\leq \rho_1(p,\, q) \leq \rho_1(p,\, r) + \rho_1(r,\, q) + \epsilon \leq \\ &\leq \rho(p,\, r; O', \mathcal{S}) + \rho(r,\, q; O', \mathcal{S}) + \epsilon\,. \end{aligned}$$

Setting $\epsilon \to 0$ we obtain the triangle inequality.

In the *second case* let

$$\rho(p,\, r; O', \mathcal{S}) = \rho_2(p,\, r)\,, \quad \rho(r,\, q; O', \mathcal{S}) = \rho_2(r,\, q)\,.$$

Choose arcs γ_1, γ_2 separating p, r and r, q from O' on $\mathcal{S}$, respectively, such that for a given $\epsilon > 0$,

$$\mathrm{diam}\,(\gamma_1) \leq \rho_2(p,\, r) + \frac{\epsilon}{2}\,, \quad \mathrm{diam}\,(\gamma_2) \leq \rho_2(r,\, q) + \frac{\epsilon}{2}\,,$$

and that $q \notin \gamma_1$, $p \notin \gamma_2$.

The surface $\mathcal{S}$ is simply connected and, if $\gamma_1 \cup \gamma_2$ is connected, then it separates points p and q from O'. Hence,

$$\begin{aligned} \rho_2(p,\,q) &\le \operatorname{diam}(\gamma_1) + \operatorname{diam}(\gamma_2) \le \\ &\le \rho_2(p,\,r) + \rho_2(r,\,q) + \epsilon. \end{aligned}$$

If the set $\gamma_1 \cup \gamma_2$ is not connected, then one of arcs γ_1 or γ_2 separates p and q from O'. Hence we have

$$\begin{aligned} \rho_2(p,\,q) &\le \max\{\operatorname{diam}(\gamma_1), \operatorname{diam}(\gamma_2)\} \le \\ &\le \rho_2(p,\,r) + \rho_2(r,\,q) + \epsilon. \end{aligned}$$

Setting now $\epsilon \to 0$ we conclude in both situations that

$$\rho(p,\,q; O', \mathcal{S}) \le \rho_2(p,\,q) \le \rho(p,\,r; O', \mathcal{S}) + \rho(r,\,q; O', \mathcal{S})\,.$$

By symmetry it suffices to consider the last *third* case

$$\rho(p,\,r; O', \mathcal{S}) = \rho_1(p,\,r)\,, \quad \rho(r,\,q; O', \mathcal{S}) = \rho_2(r,\,q)\,.$$

Choose an arc γ_1 joining points p and r on $\mathcal{S}$ such that

$$\operatorname{diam}(\gamma_1) \le \rho_1(p,\,r) + \frac{\epsilon}{2},$$

and an arc γ_2 separating points r and q from O' on $\mathcal{S}$ with the property

$$\operatorname{diam}(\gamma_2) \le \rho_2(r,\,q) + \frac{\epsilon}{2}.$$

We can also assume that $q \notin \gamma_1$ and $p \notin \gamma_2$. If γ_2 separates p from O', then

$$\begin{aligned} \rho(p,\,q; O', \mathcal{S}) &\le \rho_2(p, q) \le \operatorname{diam}(\gamma_2) \\ &\le \rho_2(r,\,q) + \tfrac{\epsilon}{2} \le \rho(p,\,r; O', F) + \rho(r,\,q; O', F) + \tfrac{\epsilon}{2}\,. \end{aligned}$$

If γ_2 does not separate p from O', then $\gamma_1 \cup \gamma_2$ is connected, contains γ and separated q from O'. Now it is easy to construct an arc γ_3 such that γ_3 separates q and r from O' and

$$\operatorname{diam}(\gamma_3) \le \operatorname{diam}(\gamma_1 \cup \gamma_2) + \epsilon\,.$$

Hence we obtain

$$\begin{aligned}\rho(p,\,q;O',\mathcal{S}) &\leq \operatorname{diam}(\gamma_3) \leq \\ &\leq \operatorname{diam}(\gamma_1)+\operatorname{diam}(\gamma_2)+\epsilon \leq \\ &\leq \rho_1(p,\,r)+\rho_2(r,\,q)+2\epsilon \leq \\ &\leq \rho(p,\,r;O',\mathcal{S})+\rho(r,\,q;O',\mathcal{S})+2\epsilon.\end{aligned}$$

Setting again $\epsilon \to 0$ we get the triangle inequality in the third case. This completes the proof of the lemma. □

By the symbol $\partial_\rho \mathcal{S}$ below we denote the *boundary* of $\mathcal{S}$ in the metric ρ, that is, up to the standard equivalence relation, the set of all Cauchy sequences (p_n) of points on $\mathcal{S}$ fundamental in the relative metric ρ and not having a condensation point on $\mathcal{S}$.

In the plane domain D we also employ the relative distance obtained from the above definition with $m = 2$. Then the relative boundary $\partial_\rho \mathcal{S}$ of a plane domain $\mathcal{S} = D$ coincides with the Caratheodory boundary.

For a discussion of the relative distance on planar domains see [3] and [6, Part I, §4].

Obviously the projection $j : \mathcal{S} \to D$ does not increase diameters of arcs and hence we have

2.4. Lemma. *Let $\mathcal{S}$ be a locally Lipschitz nonparametric surface (1.2) given over a simply connected domain $D \subset \mathbf{R}^2$, and $j(O') = O$.*

Then for an arbitrary pair of points $p, q \in \mathcal{S}$,

$$\rho(j(p),\, j(q); O, D) \leq \rho(p,\, q; O', \mathcal{S}). \tag{2.5}$$

3 Estimates for Relative Distance

Here we obtain some estimates of the relative distance under conformal mappings of a plane domain onto a nonparametric surface.

Let A be an ellipse on the (ξ_1, ξ_2)-plane. *Characteristics of the ellipse A* are called a ratio $p \geq 1$ of its axes and, if $p > 1$, an angle θ, $0 \leq \theta < \pi$, between its greater axis and the $\overrightarrow{0\xi_1}$-axis.

These characteristics give A up to transformations of a displacement and a similarity.

We assume that A is an ellipse with center at origin, the smaller semiaxis $h > 0$ and characteristics (p, θ). We have

$$\gamma\,(\xi_1)^2 - 2\,\beta\,\xi_1\,\xi_2 + \alpha\,(\xi_2)^2 = p\,h^2, \tag{3.1}$$

where

$$
\begin{cases} \alpha = p\,\cos^2\theta + \frac{1}{p}\,\sin^2\theta, \\ \beta = \left(p - \frac{1}{p}\right)\,\cos\theta\,\sin\theta, \\ \gamma = p\,\sin^2\theta + \frac{1}{p}\,\cos^2\theta. \end{cases} \tag{3.2}
$$

In fact, setting

$$
\begin{aligned} \tilde{\xi}_1 &= \xi_1\,\cos\theta + \xi_2\,\sin\theta, \\ \tilde{\xi}_2 &= -\xi_1\,\sin\theta + \xi_2\,\cos\theta, \end{aligned}
$$

we find

$$
\frac{(\tilde{\xi}_1)^2}{(p\,h)^2} + \frac{(\tilde{\xi}_2)^2}{h^2} = 1.
$$

This relation implies (3.2).

The quantities α, β and γ are connected by equations

$$
\alpha\,\gamma - \beta^2 = 1.
$$

If an ellipse is given by the equation

$$
A\,(\xi_1)^2 + 2\,B\,\xi_1\,\xi_2 + C\,(\xi_2)^2 = D, \quad AC - B^2 > 0, \;\; A > 0, \tag{3.3}
$$

then its characteristics are defined by relations

$$
\frac{\gamma}{A} = -\frac{\beta}{B} = \frac{\alpha}{C} = \frac{p\,h^2}{D} = \frac{1}{\sqrt{AC - B^2}}, \tag{3.4}
$$

which follow by comparison of (3.1) with (3.3).

From (3.2) we find

$$
p + \tfrac{1}{p} = \alpha + \gamma,
$$

$$
\tan 2\theta = \tfrac{2\,\beta}{\alpha - \gamma},
$$

$$
\beta\,\tan\theta \geq 0.
$$

Thus,

$$
\begin{cases} p = \frac{\alpha+\gamma}{2} + \sqrt{\left(\frac{\alpha+\gamma}{2}\right)^2 - 1} = \frac{\alpha+\gamma}{2} + \sqrt{\left(\frac{\alpha-\gamma}{2}\right)^2 + \beta^2}, \\ \tan\theta = \frac{\gamma - \alpha + \sqrt{(\gamma-\alpha)^2 + 4\beta^2}}{2\,\beta}. \end{cases} \tag{3.5}
$$

Let $\mathcal{S}$ be a locally Lipschitz nonparametric surface given by a vector function (1.2). We fix arbitrarily a point $a \in D$, where the vector function $\chi = \chi(\xi)$ is differentiable. By the Stepanoff theorem this vector function is differentiable almost everywhere.

The quadratic form (2.1) is positive definite and gives a distribution of ellipses with characteristics (p, θ) almost everywhere on D. Using (3.4) and (3.5) we compute

$$
p(a) = \frac{A + C}{2\sqrt{AC - B^2}} + \left(\left(\frac{A + C}{2\sqrt{AC - B^2}} \right)^2 - 1 \right)^{1/2},
$$

where

$$A = 1 + \textstyle\sum_{k=1}^{m-2} \left(\frac{\partial g_k}{\partial \xi_1}\right)^2 ,$$

$$B = \textstyle\sum_{k=1}^{m-2} \frac{\partial g_k}{\partial \xi_1}\frac{\partial g_k}{\partial \xi_2} ,$$

$$C = 1 + \textstyle\sum_{k=1}^{m-2} \left(\frac{\partial g_k}{\partial \xi_2}\right)^2 .$$

Next let $u = u(\xi)$ be a quasiconformal mapping from the domain D into the plane of variables $u = (u_1, u_2)$ with characteristics which are equal almost everywhere on D to $p(\xi)$ and $\theta(\xi)$.

We have

$$AC - B^2 \geq 1 + \sum_{k=1}^{m-2} \left(\frac{\partial g_k}{\partial \xi_1}\right)^2 + \sum_{k=1}^{m-2} \left(\frac{\partial g_k}{\partial \xi_2}\right)^2 \geq 1$$

and therefore

$$\operatorname*{ess\,sup}_{D'} p(\xi) \leq \operatorname*{ess\,sup}_{D'} \left[\frac{A+C}{2} + \left(\left(\frac{A+C}{2}\right)^2 - 1 \right)^{1/2} \right] < \infty$$

for each subdomain $D' \subset\subset D$. Now by the well-known measurable Riemann mapping theorem [1, theorem 9] there exists a locally quasiconformal mapping $u = u(\xi)$ of D into $\mathbf{R}^2$ with characteristics which are almost everywhere equal to $p(\xi)$ and $\theta(\xi)$ in D. This mapping is unique up to conformal transformations on the image domain.

The following properties of the mapping $u : D \to \mathbf{R}^2$ are well-known:

i) The mapping $u = u(\xi)$ has a differential almost everywhere on D and at each point of differentiability it transforms infinitesimal ellipses with characteristics $p(\xi)$, $\theta(\xi)$ onto infinitesimal circles (see [1, §1]).

ii) The mapping $u = u(\xi)$ has generalized locally square integrable derivatives, that is, it belongs to the class $ACL^2_{\mathrm{loc}}(D)$ (see [1, §4]).

iii) Let $G = u(D)$ and let $\xi = \xi(u) = (\xi_1(u), \xi_2(u))$ be the inverse mapping of u. The characteristic $p(u)$ of the inverse mapping $\xi : G \to D$ is locally bounded on G (see [1, §1]) and, by virtue of the property *ii*), the mapping $\xi = \xi(u)$ also belongs to the class $ACL^2_{\mathrm{loc}}(G)$.

Consider the composed mapping

$$(3.6)\quad \chi^*(u) = \chi \circ \xi(u) = (x_1(u), x_2(u), \ldots, x_m(u)) : G \to \mathcal{S}\,, \quad u = (u_1, u_2)\,.$$

Since the functions $g_1, \ldots, g_{m-2}$ are locally Lipschitz on D, then by virtue of *iii*), the functions $x_k(u) \in ACL^2_{\mathrm{loc}}(G)$, $k = 3, \ldots, m$.

Suppose that the vector function (1.2) is monotone. Then the vector function (3.6) is monotone too. By the Stepanoff theorem $\chi^*(u)$ is differentiable almost everywhere on G.

The vector function $\chi^* : G \to \mathcal{S}$ defines a one-to-one mapping from the domain $G = u(D)$ onto the surface $\mathcal{S}$. The mapping $\chi^* : G \to \mathcal{S}$ is a composition of mappings $\xi(u)$ and j^{-1}, and so transforms infinitesimal circles, firstly, by property $i)$ onto infinitesimal ellipses on the $\xi-$plane and, secondly, by the definition of the characteristic $p(x)$ onto infinitesimal circles on the surface $\mathcal{S}$.

Thus, $\chi^* : G \to \mathcal{S}$ is conformal almost everywhere and almost everywhere on G it has the following properties:

$$\left|\frac{\partial \chi^*}{\partial u_1}\right| = \left|\frac{\partial \chi^*}{\partial u_2}\right|, \qquad \left\langle \frac{\partial \chi^*}{\partial u_1}, \frac{\partial \chi^*}{\partial u_2} \right\rangle = 0. \tag{3.7}$$

Here the symbol $\langle .,. \rangle$ means the scalar product on $\mathbf{R}^m$.

The variables (u_1, u_2) are called *isothermal coordinates* on $\mathcal{S}$.

Suppose that the domain D is simply connected and its image $G = u(D)$ is different from the entire plane $\mathbf{R}^2$. Because relations (3.7) are invariant under conformal mappings in the plane of variables $u = (u_1, u_2)$, without loss of generality we can assume that the domain G is bounded.

We consider the conformal mapping χ^*. We let $O'' = (\chi^*)^{-1}(O')$ and denote by $r(G)$ the Euclidean distance from O'' to ∂G.

We will need an estimate of relative distance under conformal mappings from plane domains onto nonparametric surfaces in $\mathbf{R}^m$. For quasiconformal mappings between plane domains such a result has be obtained in [6, Chapter I, §8] and for conformal mappings from domains $D \subset \mathbf{R}^2$ onto graphs of functions $x_3 = f(x_1, x_2)$ in [5].

3.8. Theorem. *Let $\mathcal{S}$ be a nonparametric surface in $\mathbf{R}^m$ given by a locally Lipschitz vector function (1.2). If the vector function $\chi^* : D \to \mathbf{R}^m$ is monotone and* area $(\mathcal{S}) < \infty$, *then for each pair of points $p, q \in G$ satisfying the condition*

$$\rho(p, q; O'', G) < \min\{1, \frac{1}{16}\, r^4(G)\}\,, \tag{3.9}$$

the following estimate of the relative distance holds

$$\rho(\chi^*(p), \chi^*(q); O', \mathcal{S}) \le K \log^{-1/2} \frac{1}{\rho(p, q; O'', G)}\,. \tag{3.10}$$

Here $r(G) = \inf_{u \in \partial G} |u - O''|$ and K is the constant of Theorem 1.3.

Proof. We fix arbitrary points $p, q \in G \setminus \{O''\}$ with (3.9). Let $\gamma \subset G \setminus \{O''\}$ be an arc such that

$$\operatorname{diam}(\gamma) < \rho(p,\, q; O'', G) + \epsilon < \min\{1,\, \frac{1}{16}\, r^4(G)\}\,,$$

where $\epsilon > 0$ is a sufficiently small number.

We choose a point $a \in \gamma$ and consider a family of circles

$$L_\tau = \{u \in \mathbf{R}^2 : |u - a| = \tau\}, \quad r < \tau < R, \tag{3.11}$$

where

$$r = \rho(p,\, q; O'', G) + \epsilon, \quad R = \sqrt{\rho(p,\, q; O'', G)}\,.$$

We have

$$\rho(p,\, q; O'', G) < \sqrt{\rho(p,\, q; O'', G)} < 1,$$

and so that this family is not empty for all sufficiently small $\epsilon > 0$. Each of the circles L_τ contains inside an arc γ.

Next we consider two cases.

The *first case.* Suppose that

$$\operatorname{dist}(a, \partial G) \le \sqrt{\operatorname{diam}(\gamma)}\,. \tag{3.12}$$

Here $\operatorname{dist}(w, B)$ stands for the Euclidean distance from the point w to the set B. Now

$$\sqrt{\rho(p,\, q; O'', G)} < |a - O''|, \tag{3.13}$$

that is, each of L_τ separates p, q from O'.

To see this suppose the contrary. Let

$$|a - O''| \le \sqrt{\rho(p,\, q; O'', G)}.$$

For every point $x \in \partial G$ such that $|x - a| = \operatorname{dist}(a, \partial G)$, we find

$$|x - O''| \le |x - a| + |a - O''|.$$

Thus by virtue of (3.12),

$$\begin{aligned} r(G) \ &\le |x - O''| \le \\ &\le \sqrt{\operatorname{diam}(\gamma)} + \sqrt{\rho(p,\, q; O'', G)} \le \\ &\le \sqrt{\rho(p,\, q; O'', G) + \epsilon} + \sqrt{\rho(p,\, q; O'', G)}\,. \end{aligned}$$

The relation (3.11) between r and R now implies

$$\begin{aligned} r(G) \ &\le \rho^{1/4}(p,\, q; O'', G) + \rho^{1/2}(p,\, q; O'', G) < \\ &< 2\, \rho^{1/4}(p,\, q; O'', G)\,, \end{aligned}$$

or

$$\frac{1}{16} r^4(G) \le \rho(p,\, q; O'', G)\,.$$

This contradicts with (3.9), and the inequality (3.13) follows.

For an arbitrary $\tau \in [r, R]$ let C_τ be a connected component of the set $L_\tau \cap G$ separating points p, q from O''. The existence of this component follows from (3.13).

Let $\Delta_{r,R}$ be a subdomain of the domain G lying between C_r and C_R.

The mapping $\chi^* : G \to \mathbf{R}^m$ belongs to $ACL^2_{\text{loc}}(G)$ and so it is absolutely continuous along almost all C_τ, $r < \tau < R$.

Let $u(s)$, $0 \le s \le l(C_\tau)$, be the natural arc length parametrization of the arc C_τ, where $l(C_\tau)$ is the length of C_τ. Then for almost all $\tau \in [r, R]$,

$$l(\chi^*(C_\tau)) = \int\limits_{C_\tau} \left| \frac{d\chi^*}{ds}(u(s)) \right| \, ds \le \int\limits_{C_\tau} \sqrt{\sum_{i=1}^m |\nabla x_i(u)|^2} \, |du| \, .$$

By the Cauchy inequality,

$$l^2(\chi^*(C_\tau)) \le l^2(C_\tau) \int\limits_{C_\tau} \sum_{i=1}^m |\nabla x_i(u)|^2 \, |du| \, .$$

It is not difficult to show that the function $l(\chi^*(C_\tau))$ is measurable as a function of the parameter $\tau \in [r, R]$. Integrating we obtain

$$\int\limits_r^R \frac{l^2\,(\chi^*(C_\tau))}{\tau} \, d\tau \le 2\pi \int\limits_{\Delta_{r,R}} \sum_{i=1}^m |\nabla x_i(u)|^2 \, du_1 \, du_2 \, . \tag{3.14}$$

Consider the conformal mapping χ^*. By (3.7) at each point of differentiability of $\chi^* : G \to \mathcal{S}$ we have

$$\begin{aligned} \textstyle\sum_{i=1}^m |\nabla x_i(u)|^2 &= \textstyle\sum_{k=1}^2 \left| \frac{\partial \chi^*}{\partial u_k} \right|^2 = \\ &= 2 \left| \frac{\partial \chi^*}{\partial u_1} \right|^2 = 2 \sqrt{\left| \frac{\partial \chi^*}{\partial u_1} \right|^2 \, \left| \frac{\partial \chi^*}{\partial u_2} \right|^2} \, . \end{aligned}$$

Thus, we obtain

$$\sum_{i=1}^m |\nabla x_i(u)|^2 = 2 \, I(\chi^*, u) \tag{3.15}$$

almost everywhere on G, where

$$I(\chi^*, u) = \sqrt{\left| \frac{\partial \chi^*}{\partial u_1} \right|^2 \left| \frac{\partial \chi^*}{\partial u_2} \right|^2 - \left\langle \frac{\partial \chi^*}{\partial u_1}, \frac{\partial \chi^*}{\partial u_2} \right\rangle^2} \, .$$

Since the area of $\mathcal{S}$ is bounded and the vector function $\chi^* : G \to \mathbf{R}^m$ belongs to ACL^2_{loc} on G, (3.15) yields

$$\int\limits_G \sum_{i=1}^m |\nabla x_i|^2 \, du_1 \, du_2 = 2 \, \text{area} \, (\mathcal{S}) < \infty.$$

Thus on the basis of the inequality (3.14), we arrive at the estimate

$$(3.16)\ \inf_{r\le\tau\le R} l^2(\chi^*(C_\tau)) \le 2\pi \log^{-1} \frac{\sqrt{\rho(p,q;O'',G)}}{\rho(p,q;O'',G)+\epsilon} \int\limits_G \sum_{i=1}^m |\nabla x_i(u)|^2\, du_1\, du_2\,.$$

From (3.16) and letting $\epsilon \to 0$, we deduce

$$\inf_{r\le\tau\le R} l(\chi^*(C_\tau)) \le \sqrt{4\pi\,\mathrm{area}\,(\mathcal{S})}\ \log^{-1/2} \frac{1}{\rho(p,q;O'',G)}.$$

Because each of the arcs C_τ separates point p, q from O'' on G, then each of the arcs $\chi^*(C_\tau)$ separates $\chi^*(p)$, $\chi^*(q)$ from O' on $\mathcal{S}$. Thus we obtain

$$\rho_2(\chi^*(p),\ \chi^*(q); O', \mathcal{S}) \le K\ \log^{-1/2} \frac{1}{\rho(p,q;O'',G)}.$$

This inequality implies (3.10).

The *second case.* We assume that

$$\mathrm{dist}\,(a, \partial G) > \sqrt{\mathrm{diam}\,(\gamma)}\,.$$

Then we have

$$\mathrm{dist}\,(a, \partial G) > \sqrt{\rho(p,\ q; O'', G)} = R\,,$$

and hence, each of the circles C_τ of the family (3.11) does not intersect the boundary ∂G.

As above, we conclude the estimate (3.16).

The mapping $\chi^* : G \to \mathcal{S}$ is homeomorphic and thus each of the curves $\chi^*(C_\tau)$, $r \le \tau \le R$, separates the points $\chi^*(p)$, $\chi^*(q)$ from the boundary $\partial\mathcal{S}$ on $\mathcal{S}$. The vector function χ^* is monotone, so that for each $\tau \in [r, R]$,

$$\rho_1(\chi^*(p),\ \chi^*(q); O', \mathcal{S}) \le \mathrm{diam}\ (\chi^*(C_\tau)) \le l\,(\chi^*(C_\tau))\ .$$

From (3.16) we obtain

$$\rho_1(\chi^*(p),\ \chi^*(q); O', \mathcal{S}) \le K\ \log^{-1/2} \frac{1}{\rho(p,\ q; O'', G)}\,,$$

that implies (3.10). The theorem follows. □

The estimate (3.10) implies that each Cauchy sequence of points $\{a_n\} \subset G$ with respect to $\rho(p,q;O'',G)$ transforms into a Cauchy sequence $\{\chi^*(a_n)\} \subset \mathcal{S}$ with respect to the metric $\rho(\chi^*(p), \chi^*(q); O', \mathcal{S})$. Thus, we obtain

3.17. Corollary. *Under the conditions in Theorem 3.8 the conformal mapping (3.6) is extended up to the continuous about relative distance mapping from $\partial_\rho G$ onto the relative boundary $\partial_\rho\mathcal{S}$.*

Let $\xi(u) = (\xi_1(u), \xi_2(u)) : G \to D$ be the mapping realizing by components $\xi_1(u)$, $\xi_2(u)$ of the vector function (3.6). The following remark is a consequence of Theorem 3.8 and Lemma 2.4.

3.18. Corollary. *If a vector function $\chi^* : G \to \mathbf{R}^m$ is monotone and* area$(\mathcal{S}) < \infty$*, then for an arbitrary pair of points $p, q \in G$ satisfying (3.9),*

$$\rho(\xi(p),\, \xi(q); O, D) \le K \log^{-1/2} \frac{1}{\rho(p,\, q; O'', G)},$$

where K is the constant of Theorem 3.8.

4 Proof of Main Theorem

At first we observe that the statement is trivial for a constant function f. Thus without loss of generality we may assume that $f \not\equiv \mathrm{const}$.

Let $\mathcal{S}$ be a nonparametric surface in $\mathbf{R}^m$ given by a locally Lipschitz mapping (1.2). As above in Section 3, we use isothermal coordinates (u_1, u_2) on $\mathcal{S}$ which are defined by the auxiliary quasiconformal mapping $u = u(\xi) : D \to \mathbf{R}^2$.

We denote by $\xi = \xi(u) = (\xi_1(u), \xi_2(u))$ the inverse mapping of $u = u(\xi)$. Let $\xi_i(u) = g_i(\xi(u))$, $i = 1, 2, \ldots, m-2$. The vector-valued function $\chi^*(u) = (\xi_1(u), \xi_2(u), \xi_3(u), \ldots, \xi_{m-2}(u))$ realizes the one-to-one conformal mapping from a domain $G = u(D)$ onto the graph $\mathcal{S}$.

We prove that the image of D in the plane of isothermal coordinates (u_1, u_2) is $G \ne \mathbf{R}^2$, that is, the following statement is valid.

4.1. Theorem. *Let $D \subset \mathbf{R}^2$ be a simply connected domain and let $\chi : D \to \mathbf{R}^m$ be a nonparametric surface given by (1.2). If the vector function χ is monotone and* area $\mathcal{S} < \infty$*, then $\mathcal{S}$ is of the hyperbolic conformal type.*

Proof. Since (1.2) is monotone on D and the mapping $\xi : G \to D$ is homeomorphic, the vector function $\chi^* = \chi \circ \xi$ is also monotone on G. The mapping ξ is locally quasiconformal and χ is locally Lipschitz and, hence, $\chi^* \in \mathrm{ACL}^2_{\mathrm{loc}}(G)$.

Assume that $G = \mathbf{R}^2$. We fix arbitrarily $R > 0$ and $t > 1$. Let $B(\tau)$ and $S(\tau)$ be a disk and a circle, respectively, with center at origin $u = 0$ and radius τ.

From the Length and Area Principle (3.14), we find

$$\inf_{R \le \tau \le tR} \operatorname{osc}(\chi^*(u),\, S(\tau)) \le \left(\frac{2\pi\, I(R)}{\log t} \right)^{1/2}, \tag{4.2}$$

where

$$I(R) = \int\limits_{|u| < tR} \left(|\chi^*_{u_1}|^2 + |\chi^*_{u_2}|^2 \right) du_1\, du_2 \,.$$

The function $\chi^*(u)$ is monotone and, consequently, for every $t>1$,

$$\begin{aligned} \operatorname{osc}(B(R),\, \chi^*(u)) \ &\le \operatorname{osc}(\chi^*(u),\, S(R)) \le \\ &\le \inf_{R\le\tau\le tR} \operatorname{osc}(\chi^*(u),\, S(\tau))\,. \end{aligned}$$

Thus,

$$\operatorname{osc}(\chi^*(u),\, B(R)) \le \left(\frac{2\pi\, I(\infty)}{\log t}\right)^{1/2}.$$

From (3.7) we find

$$\begin{aligned} I(\infty) \ &= \int_{\mathbf{R}^2} \left(|\chi^*_{u_1}|^2 + |\chi^*_{u_2}|^2\right) du_1\, du_2 = \\ &= 2\int_{\mathbf{R}^2} \sqrt{|\chi^*_{u_1}|^2|\chi^*_{u_2}|^2 - \langle \chi^*_{u_1}, \chi^*_{u_2}\rangle}\, dx_1\, dx_2 \le \\ &\le 2\text{area}\,(\mathcal{S}), \end{aligned}$$

and hence,

$$\operatorname{osc}(\chi^*(u),\, B(R)) \le \left(\frac{4\pi\, \text{area}\,(\mathcal{S})}{\log t}\right)^{1/2}.$$

Letting $t\to\infty$ we conclude $\chi^*(u)\equiv$ const on $\mathbf{R}^2$. It follows that $\chi\equiv$ const on D.

However, if $\chi\equiv$ const in D and area $\mathcal{S}<\infty$, then area $D<\infty$ as well. Thus, D has the hyperbolic conformal type. The mapping $u=u(\xi):D\to\mathbf{R}^2$ is then conformal in the Euclidean metric and consequently, the image of the domain D has to be of hyperbolic type. That is, the domain G can not be the entire plane $\mathbf{R}^2$. This contradiction proves the theorem. □

Since the simply connected domain $G\neq\mathbf{R}^2$ and the mapping $u:D\to G$ is defined up to conformal mappings on the $u-$plane, we may assume that G is the unit disk with center at O''. Moreover,

$$\int_G \left(|\chi^*_{u_1}|^2 + |\chi^*_{u_2}|^2\right) du_1\, du_2 = 2\text{area}\,(\mathcal{S}) < \infty\,.$$

Using Theorem 3.8 and Lemma 2.4, we deduce that *the mapping*

$$x = j\circ\chi^* : G\to D$$

can be extended to a continuous mapping on $\overline{G}$.

Besides, by virtue of Corollary 3.18 for points $p,\, q\in\overline{G}$ with

$$\rho(p,\, q; O'', G) < 1/16$$

the following estimate of the relative distance holds

$$\rho(\tilde{x}(p),\, \tilde{x}(q);\, O, D) \le K \log^{-1/2} \frac{1}{\rho(p,\, q;\, O'', G)}\,,$$

where $K = 2\sqrt{\pi\, \mathrm{area}\,(\mathcal{S})}$.

We denote by $\tilde{x} : \overline{G} \to \overline{D}$ the respective continuous extension mapping of $x : G \to D$ to the boundary ∂G. If points p and q are lying on the unit circle ∂G and $|p-q| < 1/16$, then $\rho(p,\, q; O'', G) = |p-q|$. Thus this estimate has the form

$$\rho(\tilde{x}(p),\, \tilde{x}(q); O, D) \le K\, \log^{-1/2} \frac{1}{|p-q|} \qquad \text{for} \quad |p-q| < \frac{1}{16}. \tag{4.3}$$

Because the mapping $\tilde{x} : \overline{G} \to \overline{D}$ is continuous, then the inverse image $\tilde{x}^{-1}(a)$ of every point $a \in \partial D$ is a closed set on ∂G. Moreover, for an arbitrary pair of points $a \ne b$ on ∂D,

$$\tilde{x}^{-1}(a) \cap \tilde{x}^{-1}(b) = \emptyset.$$

The set $\tilde{x}^{-1}(a)$ is connected. In fact, since the boundary ∂D is a simple Jordan curve, for sufficiently small $\epsilon > 0$ the set $U_\epsilon = \{x \in D : |x - a| < \epsilon\}$ is connected. The mapping $x(u) : G \to D$ is a homeomorphism, and hence, the set $x^{-1}(U_\epsilon)$ are connected for all $\epsilon \in (0, \epsilon_0)$ where $\epsilon_0 > 0$ is sufficiently small. Thus, the set

$$\tilde{x}^{-1}(a) = \cap_{\epsilon > 0} \overline{U}_\epsilon$$

is also connected. This means that for an arbitrary point $a \in \partial D$ the set $\tilde{x}^{-1}(a)$ is either an isolate point or a closed connected arc.

Because the number of mutually disjoint arcs on the circle can be only countable, the inverse mapping $\tilde{x}^{-1}$ is defined in a single value manner on ∂D except at most on some countable set $E \subset \partial G$.

Now we will prove that *every point $a \in \partial D \setminus E$ is a point of the ∂'-continuity* of the vector function χ.

Let $b = \tilde{x}^{-1}(a)$ be a generic point $a \in \partial D \setminus E$. By the Length and Area Principle there exists a sequence of subdomains $\{G_n\}$, $\partial' G_n = S(b, \tau_n)$, with properties: each of arcs $\partial' G_n$ separates the point b from the origin O; $\quad \mathrm{diam}\,(\partial' G_n) \to 0$ and $\mathrm{osc}\,(\partial' G_n, \chi^*) \to 0$ as $n \to \infty$ and $\cap_{n=1}^{\infty} [G_n] = \emptyset$.

We set $D_n = x(G_n)$, $n = 1, 2, \ldots$. Because the mapping $x : G \to D$ is homeomorphic, then each of the arcs $\partial D_n = x(S(b, \tau_n))$ separates the point $a \in \partial D$ from O.

The mapping $\tilde{x}(u) : \overline{G} \to \overline{D}$ is continuous and therefore $\mathrm{diam}\, \partial' D_n \to 0$ and $\cap_{n=1}^{\infty} [D_n] = \emptyset$.

Finally, $\mathrm{osc}\,(\partial' D_n, \chi) = \mathrm{osc}(S(b, \tau_n), \chi^*)$, and hence, $\mathrm{osc}\,(\partial' D_n, \chi) \to 0$ as $n \to \infty$.

We have shown that the point $a \in \partial D \setminus E$ satisfies the properties (α), (β) and (γ) of the ∂'-continuity of χ.

We choose an arbitrary boundary jump point $a \in E$. Its inverse image $\tilde{x}^{-1}(a)$ is some subarc β of the circle $|u| = 1$. We estimate its length $l(\beta)$.

Let $\xi \in \beta$ be the midpoint of this arc. The distance from ξ to the endpoints of the arc is

$$r = 2\sin\frac{l(\beta)}{4}.$$

We assume that $r < 1$, that is, $l(\beta) < \frac{2}{3}\pi$. Consider the family of circles $\{S(\xi,\tau)\}$ with centers at ξ and radii τ, $r < \tau < 1$. Write $C_\tau = G \cap S(\xi,\tau)$.

The standard argument shows,

$$\int_r^1 \frac{l^2(\chi^*(C_\tau))}{\tau}\, d\tau \le 2\pi \int_{\Delta_{r,1}} \sum_{i=1}^3 |\nabla x_i(u)|^2\, du_1\, du_2.$$

Now we observe, that for every $\tau \in [r,1]$,

$$l(\chi^*(C_\tau)) \ge \max\{\mathrm{osc}\,(x(C_\tau),\chi), \mathrm{length}\,(x(C_\tau))\} \ge \delta(a,\chi,O)\,,$$

and therefore,

$$\delta^2(a,\,\chi,\,O) \le 4\pi\,\mathrm{area}\,(\mathcal{S})\,\log^{-1}\frac{1}{r}\,.$$

Thus we arrive at the inequality

$$\delta^2(a,\,\chi,\,O) \le \frac{K^2}{\log 1/r}\,.$$

This implies

$$r = 2\sin\frac{l(\beta)}{4} \ge \exp\{-\frac{K^2}{\delta^2(a,\,\chi,\,O)}\}\,,$$

and we obtain the following estimate of $l(\beta)$ for "small" $\delta(a,\,\chi,\,O)$

$$l(\beta) \ge 4\arcsin\left(\frac{1}{2}\exp\left\{-\frac{K^2}{\delta^2(a,\,\chi,\,O)}\right\}\right) \quad \text{for} \quad l(\beta) < \frac{2}{3}\pi\,. \tag{4.4}$$

Next we estimate $l(\beta)$ for large $\delta(a,\,\chi,\,O)$.

Let $a \in \partial D \cap E$ be a ∂'-jump point, where $\frac{2}{3}\pi \le l(\beta) \le 2\pi$. Without loss of generality, we can assume that the arc $\beta = \tilde{x}^{-1}(a)$ is described by inequalities

$$\beta = \left\{u = (u_1,u_2) : (u_1)^2 + (u_2)^2 = 1,\ -\frac{l(\beta)}{2} \le \mathrm{arctg}\,\frac{u_2}{u_1} \le \frac{l(\beta)}{2}\right\}\,.$$

Fix the segment

$$\gamma = \{u = (u_1,u_2) : 0 \le u_1 \le 1,\ u_2 = 0\}$$

and denote as p_1 and p_2 boundary points of the unit disk G with the cut γ lying on the intersection of $|u| = 1$ and upper and lower edges of the cut γ, respectively.

Let $T : G \to V$ be the conformal mapping from the unit disk $\{|u| < 1\}$ with the cut γ onto the half-disk $V = \{v = (v_1,v_2) : (v_1)^2 + (v_2)^2 < 1,\ v_2 > 0\}$ such that $T(p_1) = (1,0)$, $T(p_2) = (-1,0)$ and $T(0,0) = (0,0)$.

Under the mapping T, the arc $\beta \subset \partial G$ corresponds to the arc

$$\eta = \left\{ v = (v_1, v_2) : \frac{l(\beta)}{4} \leq \operatorname{arctg} \frac{v_2}{v_1} \leq \pi - \frac{l(\beta)}{4} \right\} .$$

For every $0 < k < \cos \frac{l(\beta)}{4}$, the rectilinear segment

$$\zeta(k) = \{v = (v_1, v_2) : -\sqrt{1-k^2} < v_1 < \sqrt{1-k^2},\ v_2 = k\}$$

separates on V the arc η from the point $(0,0)$. Its image $\zeta^*(k) = \tilde{x} \circ T^{-1}(\zeta(k))$ is an arc separating on D points a and O. The mapping $\chi^{**} = \chi^* \circ T^{-1}$ satisfies (3.7). Therefore for every $k \in (0, \cos \frac{l(\beta)}{4})$ we have

$$l^2(\zeta^*(k)) \leq \left(\int\limits_{\zeta(k)} \left| \frac{\partial \chi^{**}}{\partial v_1} \right| dv_1 \right)^2 \leq 2\sqrt{1-k^2} \int\limits_{\zeta(k)} \left| \frac{\partial \chi^{**}}{\partial v_1} \right|^2 dv_1 \,.$$

Next,

$$\int\limits_0^{\cos l(\beta)/4} \frac{l^2(\zeta^*(k))}{\sqrt{1-k^2}}\, dk \leq 2 \int\limits_V \left| \frac{\partial \chi^{**}}{\partial v_1} \right|^2 dv_1\, dv_2 \leq 2 \operatorname{area}(\mathcal{S})$$

and taking into account that

$$l(\zeta^*(k)) \geq \delta(a,\, \chi,\, O) \quad \text{for all} \quad k \in (0, \cos \frac{l(\beta)}{4}) \,,$$

we obtain

$$\delta^2(a, \chi, O) \int\limits_0^{\cos l(\beta)/4} \frac{1}{\sqrt{1-k^2}}\, dk \leq 2 \operatorname{area}(\mathcal{S}) \,,$$

or

$$\delta^2(a,\, \chi,\, O) \left(\frac{\pi}{2} - \frac{l(\beta)}{4} \right) \leq \frac{K^2}{2\pi} \,.$$

Thus we find the estimate

$$l(\beta) \geq \max \left\{ 2\pi - \frac{2K^2}{\pi\, \delta^2(a,\, \chi,\, O)},\, 0 \right\} . \tag{4.5}$$

Let a_k, $k = 1, 2, \ldots$, be ∂'-jump points of the vector function $\chi(\xi)$ and let β_k, $k = 1, 2, \ldots$, be the corresponding close subarcs of the circle $\{|u| = 1\}$. By relabelling if necessary we may assume that

$$l(\beta_1) \geq l(\beta_2) \geq \ldots \geq l(\beta_k) \geq \ldots .$$

Because $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$, then

$$\sum_{k=1}^{\infty} l(\beta_k) \leq 2\pi \,.$$

Starting from $k = 4$, we have $l(\beta_k) < \frac{2}{3}\pi$ and we may use the estimate (4.4). This implies

$$4\sum_{k=4}^{\infty} \arcsin\left(\frac{1}{2}\exp\{-\frac{K^2}{\delta^2(a,\,\chi,\,O)}\}\right) \leq \sum_{k=4}^{\infty} l(\beta_k)\,. \tag{4.6}$$

For each of the points a_1, a_2, a_3 we use (4.5). This gives

$$\sum_{k=1}^{3} \max\{2\pi - \frac{2K^2}{\pi\delta^2(a,\,\chi,\,O)},\, 0\} \leq \sum_{k=1}^{3} l(\beta_k). \tag{4.7}$$

Combining (4.6) and (4.7), we concluse that

$$\begin{aligned}\textstyle\sum_{k=1}^{3} \max\{2\pi - \frac{2K^2}{\pi\,\delta^2(a,\chi,O)},\, 0\} \quad & \textstyle+4\sum_{k=4}^{\infty} \arcsin\left(\frac{1}{2}\exp\{-\frac{K^2}{\delta^2(a,\chi,O)}\}\right) \leq \\ & \textstyle\leq \sum_{k=1}^{3} l(\beta_k) + \sum_{k=4}^{\infty} l(\beta_k) \leq 2\pi\end{aligned}$$

and this completes the proof. $\square$

References

[1] P.P. Belinskii, General properties of quasiconformal mappings, in Russian, Nauka, Sibirskoe otdelenie, Novosibirsk, 1974.

[2] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin, 1969.

[3] M.A. Lavrent'ev, On continuity of schlicht functions on closed domains, in Russian, Dokl. AN SSSR, v. 4, 1936, p. 207-210.

[4] J. Lelong–Ferrand, Représentation conforme et transformations a intégrale de Dirichlet bornée, Gauthier–Villars, Paris, 1955.

[5] V.M. Miklyukov, Boundary properties of minimal surfaces type equations, in Russian, Math. Sb., v. 192, n. 10, 2001, p. 1491-1513.

[6] G.D. Suvorov, Families of planar topological mappings, in Russian, Sibirsk. Otdel. AN SSSR, Novosibirsk, 1965.

Martio and Vuorinen:
Department of Mathematics
University of Helsinki
Yliopistonkatu 5, 00014 Helsinki
FINLAND
Email: `martio@cc.helsinki.fi, vuorinen@csc.fi`
Fax: 358-0-19122879

Miklyukov:
Tulaka 2-120
400119 Volgograd
RUSSIA
E-mail:`miklyuk@hotmail.com`
Fax: 8442-471608