Generalized Wiman and Arima theorems for n-subharmonic functions on cones

Olli Martio, Vladimir Miklyukov, and Matti Vuorinen

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Abstract

Generalized Wiman and Arima theorems about entire holomorphic functions of complex variable are proved for $n-$subharmonic functions on spherical n-dimensional cones.

1 Main Results

The following theorems are well-known:

Theorem A (Wiman, 1905, [20]). *If an entire holomorphic function f has order $\rho < \frac{1}{2}$ then*

$$\limsup_{r\to\infty} \min_{|z|=r} |f(z)| = \infty.$$

Theorem B (Arima, 1952, [1]). *Let $f(z)$ be an entire holomorphic function of order $\rho > 0$ and let l be a number satisfying the following conditions*

$$0 < l \leq 2\pi, \quad l < \frac{\pi}{\rho}.$$

Then there exists a sequence of arcs

$$\{|z| = r_k,\ \theta_k \leq \arg z \leq \theta_k + l\}, \quad r_k \to \infty, \quad 0 \leq \theta_k < 2\pi,$$

along which $|f(z)|$ tends to ∞ uniformly with respect to $\arg z$.

We will prove extensions of these results for $n-$subharmonic functions on n-dimensional spherical cones.

We consider a class of manifolds called spherical cones. Let (r,θ), where $r \geq 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let

$$D = \{(r,\theta) \in \mathbf{R}^n : 0 < r < \infty,\ \theta \in S^{n-1}(1)\}$$

be a domain in $\mathbf{R}^n$ with a metric

$$ds^2_{\mathcal{M}} = \alpha^2\, dr^2 + r^2\, dl^2_\theta, \tag{1.1}$$

where $\alpha > 0$ is a constant and dl_θ is the length element on the unit sphere $S^{n-1}(1)$.

If $\alpha > 1$, then $\mathcal{M}$ is isometric to a spherical cone in $\mathbf{R}^{n+1}$.

If $\alpha = 1$, then $\mathcal{M} \equiv \mathbf{R}^n$. For $0 < \alpha < 1$, we obtain an abstract conic surface which does not have any realization in the form of any revolution surface of $\mathbf{R}^{n+1}$.

We study subsolutions of the following $n-$Laplace-Beltrami equation on $\mathcal{M}$:

$$\Delta_{n,\mathcal{M}}[f] \equiv \operatorname{div}_{\mathcal{M}}(|\nabla_{\mathcal{M}} f|^{n-2}\nabla_{\mathcal{M}} f) = 0\,, \tag{1.2}$$

where $\operatorname{div}_{\mathcal{M}}$ and $\nabla_{\mathcal{M}}[f]$ denote the divergence and the gradient, respectively, in the metric (1.1). The equation (1.2) describes the *$p-$harmonic functions* on $\mathcal{M}$. This reduces to the Laplace-Beltrami equation $\Delta_{\mathcal{M}}[f] = 0$ for $n = 2$ and to the Laplace equation $\Delta[f] = 0$ for $\mathcal{M} = \mathbf{R}^n$, $n = 2$.

The solutions of $\Delta_{n,\mathcal{M}}[f] \geq 0$ are called *n-subharmonic* functions, and the solutions of $\Delta_{n,\mathcal{M}}[f] \leq 0$ are called *$n-$superharmonic* functions.

We define the order ρ of f on a cone $\mathcal{M}$ as

$$\rho = \sup\left\{s : \liminf_{r\to\infty} \frac{M_f(r,D)}{r^s} > 0\right\}\,,$$

where

$$M_f(r,U) = \sup_{\substack{|x|=r\\ x\in U}} \max\{f(x),0\}\,, \quad U \subset D\,.$$

Two following theorems are, respectively, generalized Wiman and Arima theorems.

1.3. Theorem. *Let $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ be an $n-$dimensional spherical cone over the domain $D = \{x \in \mathbf{R}^n : 0 < |x| < \infty\}$ defined by (1.1). Let f be a $n-$subharmonic function on $\mathcal{M}$. Suppose that the order ρ of f satisfies*

$$\rho < \frac{\alpha}{n-1}\lambda_n\left(\dot{S}^{n-1}(1)\right)\,,$$

where $\lambda_n\left(\dot{S}^{n-1}(1)\right)$ is a fundamental frequency of the punctened sphere $\dot{S}^{n-1}(1)$.

If

$$\limsup_{x\to 0} f(x) < \limsup_{x\to\infty} f(x), \tag{1.4}$$

then

$$\limsup_{r\to\infty} \min_{x\in S^{n-1}(r)} f(x) = \infty. \tag{1.5}$$

For metrics (1.1) with $\alpha \neq 1$ in the manifolds $\mathcal{M}$ certainly have a singular point $x = 0$ and hence the condition (1.4) is necessary. If $\mathcal{M} = \mathbf{R}^n$ then (1.4) is superfluous. In this case for $n = 2$ the statement coincides with Theorem A (Wiman).

Let $\Delta^* \subset S^{n-1}(1)$ be the image of $\Delta \subset S^{n-1}(r)$ under the mapping

$$x \in S^{n-1}(r) \to \frac{x}{r} \in S^{n-1}(1)\,.$$

1.6. Theorem. *Let $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ be an $n-$dimensional spherical cone. Let f be a $n-$subharmonic function on $\mathcal{M}$ satisfying (1.4).*

If f has the order ρ, then for an arbitrary $s > \rho$ there is a sequence $\{\Delta_k\}$ of subdomains $\Delta_k \subset S^{n-1}(r_k)$, $r_k \to \infty$, such that

$$\lim_{k\to\infty} \min_{x\in\Delta_k} f(x) = \infty \tag{1.7}$$

and for any $k = 1, 2, \ldots,$

$$\lambda_n(\Delta_k^*) \leq \frac{(n-1)s}{\alpha}\,. \tag{1.8}$$

In the case $\mathcal{M} = \mathbf{R}^n$ and $n = 2$ this statement coincides with Theorem B (Arima).

We employ locally Lipschitz sub- and supersolutions of (1.2). This approch is different from [12] and $(\mathcal{H}^{n-1}, n-1)$-rectifiable sets [9] can be used in an effective way. A brief outline of the essential facts about such sets had been adduced in Appendix.

2 Solutions of Differential Inequalities

Let A be a subset of $\mathbf{R}^n$. Denote by Lip$\,(A)$ the family of the functions $f : A \to \mathbf{R}$ satisfying the Lipschitz condition

$$|f(x'') - f(x')| \leq C\,|x'' - x'| \quad \text{for all} \quad x',\, x'' \in A\,,$$

where $C = C(f, A)$ is a constant.

Let D be as in Section 1 and let $U \subset D$ be a domain. The symbol $\mathrm{Lip}_{\mathrm{loc}}\,(U)$ denotes the set of the functions $f : U \to \mathbf{R}$ such that $f \in \mathrm{Lip}\,(A)$ for each compact set $A \subset U$.

By the Rademacher–Stepanoff theorem [9, Theorem **3.1.9**], every locally Lipschitz function $f : D \to \mathbf{R}$ is differentiable almost everywhere. We set

$$D(f) = \{x \in D : \text{the function } f \text{ is not differentiable at } x\}\,.$$

The volume element in the metric $ds_{\mathcal{M}}$ is given by the expression

$$d\sigma_{\mathcal{M}} = \alpha\, r^{n-1}\, dr\, dS^{n-1}(1)\,. \tag{2.1}$$

If $\phi(r,\theta) \in \mathrm{Lip}_{\mathrm{loc}}\,(D)$ and if ϕ is differentiable at $(r,\theta) \in D$, then we set

$$\nabla_{\mathcal{M}}\,\phi(r,\theta) = (\frac{1}{\alpha}\phi'_r,\ \frac{1}{r}\nabla_\theta\phi)\,. \tag{2.2}$$

Here $\nabla_\theta\phi$ is the gradient in $S^{n-1}(1)$.

The n-Dirichlét integral has the following form

$$\int\limits_{\mathcal{M}} |\nabla_{\mathcal{M}}\phi|^n\, d\sigma_{\mathcal{M}} = \int\limits_{D} \left(\frac{1}{\alpha^2}(\phi'_r)^2 + \frac{1}{r^2}|\nabla_\theta\phi|^2\right)^{n/2} d\sigma_{\mathcal{M}}\,,$$

and extremals of this integral satisfy the Euler-Lagrange equation (1.2).

2.3. Definition. A function $f \in \mathrm{Lip}_{\mathrm{loc}}\,(U)$ is said to be a $n-$*subharmonic* function on a domain U if for each subdomain $V \subset\subset D$ with the countably $(\mathcal{H}^{n-1}, n-1)-$rectifiable boundary ∂V, $\mathcal{H}^{n-1}\,(\partial V \cap D(f)) = 0$, and for each nonnegative Lipschitz function $\phi : \overline{V} \to \mathbf{R}$ the following relation holds:

$$\int\limits_{V} |\nabla_{\mathcal{M}} f|^{n-2}\langle\nabla_{\mathcal{M}} f, \nabla_{\mathcal{M}}\phi\rangle\, d\sigma_{\mathcal{M}} \leq \int\limits_{\partial^* V} \phi\, |\nabla_{\mathcal{M}} f|^{n-2}\langle\nabla_{\mathcal{M}} f, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}. \tag{2.4}$$

Here $n_{\mathcal{M}}$ is an inner approximate normal to ∂V vector of the unit length in the metric (1.1), and $d\mathcal{H}^{n-1}_{\mathcal{M}}$ is the $(n-1)-$area element of $\mathcal{M}$ on the reduced boundary $\partial^* V$ (see Section 6). By Theorem 6.5, this vector exists $\mathcal{H}^{n-1}$ – almost everywhere on $\partial^* V$.

A function $f \in \mathrm{Lip}_{\mathrm{loc}}\,(U)$ is said to be a $n-$*superharmonic* function on U if $-f$ is $n-$subharmonic.

Moreover, it is technically more convenient to consider the inequality

$$f\Delta_{n,\mathcal{M}}[f] \geq 0\,. \tag{2.5}$$

This means that in the set where $f \geq 0$ the function f is $n-$subharmonic and in the set $f \leq 0$ $n-$superharmonic.

2.6. Definition. A function $f \in \mathrm{Lip}_{\mathrm{loc}}(U)$ is said to be a *generalized solution* of (2.5) in a domain $U \subset D$ if for each subdomain $V \subset\subset U$ with the countably $(\mathcal{H}^{n-1}, n-1)-$rectifiable boundary ∂V, $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$, and for each nonnegative Lipschitz function $\phi : \overline{V} \to \mathbf{R}$, the following relation holds:

$$\begin{aligned} &\int\limits_V \phi\,|\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} + \int\limits_V f\,|\nabla_{\mathcal{M}} f|^{n-2}\langle \nabla_{\mathcal{M}} f, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}} \\ &\leq \int\limits_{\partial^* V} \phi\, f\,|\nabla_{\mathcal{M}} f|^{n-2}\langle \nabla_{\mathcal{M}} f, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}\,. \end{aligned} \tag{2.7}$$

If $f \in C^2(U)$ is a solution of (2.5), then (2.7) implies the inequality $f\Delta_{n,\mathcal{M}}[f] \geq 0$ in the standard sense. In fact, let $f \in C^2(U)$ be a generalized solution of (2.5) on U. For an arbitrary subdomain $V \subset\subset U$ with the countably $(\mathcal{H}^{n-1}, n-1)-$rectifiable boundary and for each nonnegative Lipschitz function $\phi : \overline{V} \to \mathbf{R}$ by Theorem 6.10, we may write

$$\begin{aligned} &\int\limits_V \phi\, f\,\Delta_{n,\mathcal{M}}(f)\, d\sigma_{\mathcal{M}} + \int\limits_V \phi\,|\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} + \int\limits_V f\,|\nabla_{\mathcal{M}} f|^{n-2}\langle \nabla_{\mathcal{M}} f, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}} \\ &= \int\limits_{\partial^* V} \phi\, f\,|\nabla_{\mathcal{M}} f|^{n-2}\langle \nabla_{\mathcal{M}} f, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}\,. \end{aligned}$$

The condition (2.7) implies

$$\int\limits_V \phi f\Delta_{n,\mathcal{M}}[f]\, d\sigma_{\mathcal{M}} \geq 0\,.$$

This gives $f\Delta_{p,\mathcal{M}}[f] \geq 0$ everywhere on V. Here we also need to observe : if $f \in C^2(U)$ is a solution of (2.5), then f satisfies (2.7).

The following statement is important for applications.

2.8. Lemma. *If f is a generalized solution of (2.5) on a domain $U \subset D$, then so are $u = \max\{f, 0\}$ and $v = \min\{f, 0\}$.*

Proof. Let E_t be a level set $\{x \in U : u(x) = t\}$. Consider two sets

$$A = \{x \in U : f(x) > 0\} \quad \text{and} \quad B = \{x \in U : f(x) < 0\}$$

At first, we suppose that the level set E_0 is $(\mathcal{H}^{n-1}, n-1)$-rectifiable. For each subdomain $V \subset\subset U$ with the countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary ∂V with $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$, let

$$V_1 = V \cap A, \quad V_2 = V \cap B.$$

Functions $f^* = f|_{\overline{V}_1}$ and $g \equiv 0$ are generalized solutions of (2.5) on V_1 and V_2, respectively. Thus, from (2.7) for each nonnegative function ϕ,

$$\int_{V_1} \phi\, |\nabla_{\mathcal{M}} f^*|^n d\sigma_{\mathcal{M}} + \int_{V_1} f^*\, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}}$$

$$\leq \int_{\partial^* V_1} \phi\, f^*\, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}$$

and because $f^* = 0$ on $\partial V_1 \setminus (\partial V_1 \cap A)$,

$$\int_{V_1} \phi\, |\nabla_{\mathcal{M}} f^*|^n d\sigma_{\mathcal{M}} + \int_{V_1} f^*\, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}}$$

$$\leq \int_{\partial^* V_1 \cap A} \phi\, f^*\, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}\,.$$

Trivially,

$$\int_{V_2} \phi\, |\nabla_{\mathcal{M}} g|^n d\sigma_{\mathcal{M}} + \int_{V_2} g\, |\nabla_{\mathcal{M}} g|^{n-2} \langle \nabla_{\mathcal{M}} g, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}}$$

$$\leq \int_{\partial^* V_2 \cap B} \phi\, g\, |\nabla_{\mathcal{M}} g|^{n-2} \langle \nabla_{\mathcal{M}} g, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}\,.$$

Observe that

$$V \subset (V_1 \cup V_2 \cup E_0)\,,$$

$$\mathcal{H}^n(\partial A) = \mathcal{H}^n(\partial B) = \mathcal{H}^n(E_0) = 0\,,$$

and

$$f^* = g = 0 \quad \text{for all } x \in \partial V \setminus [(\partial V_1 \cap A) \cup (\partial V_2 \cap B)].$$

Summing both inequalities, we see that

$$\int_{V} \phi\, |\nabla_{\mathcal{M}} u|^n d\sigma_{\mathcal{M}} + \int_{V} u\, |\nabla_{\mathcal{M}} u|^{n-2} \langle \nabla_{\mathcal{M}} u, \nabla_{\mathcal{M}} \phi\rangle\, d\sigma_{\mathcal{M}}$$

$$\leq \int_{\partial^* V} \phi\, u\, |\nabla_{\mathcal{M}} u|^{n-2} \langle \nabla_{\mathcal{M}} u, n_{\mathcal{M}}\rangle\, d\mathcal{H}^{n-1}_{\mathcal{M}}\,. \tag{2.9}$$

The inequality (2.9) implies that u satisfies (2.5) in the generalized sense.

Now we suppose, that the boundary set E_0 is not $(\mathcal{H}^{n-1}, n-1)$-rectifiable. Using Theorem 6.3, we conclude that for almost all t the boundaries of the sets E_t are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable.

On the other hand, let $V \subset\subset D$ be a subdomain. By 6.2,

$$\int\limits_{\mathbf{R}} dt \int\limits_{E_t \cap V} d\mathcal{H}^{n-1}(x) = \int\limits_{V} |\nabla f(x)| \, dx < \infty \,,$$

and for almost all $t \in \mathbf{R}$, sets $E_t \cap V$ have a finite $(n-1)$-dimensional Hausdorff measure.

Moreover, (6.2) gives

$$\int\limits_{\mathbf{R}} dt \int\limits_{E_t \cap D(f)} d\mathcal{H}^{n-1}(x) = 0 \,, \tag{2.10}$$

and hence $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$ for almost all $t \in \mathbf{R}$.

Therefore, for almost all t, sets $E_t \cap V$ are $(\mathcal{H}^{n-1}, n-1)$-rectifiable and such that $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$.

For each t, let

$$A_t = \{x \in U : f(x) > t\} \quad \text{and} \quad B_t = \{x \in U : f(x) < t\}.$$

Fix an arbitrary subdomain $V \subset\subset U$ with $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary, $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$. For each $\epsilon > 0$ such that the set $E_\epsilon \cap V$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, we let

$$V_{1\,\epsilon} = V \cap A_\epsilon, \quad V_{2\,\epsilon} = V \cap B_\epsilon.$$

By (2.7) for an arbitrary nonnegative $\phi \in \operatorname{Lip} V$, we can write

$$\begin{aligned}
&\int\limits_{V_{1\epsilon}} \phi \, |\nabla_{\mathcal{M}} f^*|^n d\sigma_{\mathcal{M}} + \int\limits_{V_{1\epsilon}} f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, \nabla_{\mathcal{M}} \phi \rangle \, d\sigma_{\mathcal{M}} \\
&\le \int\limits_{\partial^* V_{1\epsilon}} \phi \, f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&= \int\limits_{\partial^* V_{1\epsilon} \cap A_\epsilon} \phi \, f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad + \epsilon \left| \int\limits_{\gamma^*_\epsilon} \phi \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}^{n-1}_{\mathcal{M}} \right|
\end{aligned} \tag{2.11}$$

where $\gamma_\epsilon = \partial V_{1\epsilon} \setminus (\partial V_{1\epsilon} \cap A_\epsilon)$ and γ^*_ϵ is the subset of γ_ϵ where γ_ϵ has an approximate normal.

We estimate the last integral. We have

$$|I_\epsilon| \equiv \left| \int\limits_{\gamma^*_\epsilon} \phi \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}^{n-1}_{\mathcal{M}} \right| \le \max_{\overline{V}_1} |\phi| \int\limits_{\gamma^*_\epsilon} |\nabla f|^{n-1} \, d\mathcal{H}^{n-1}_{\mathcal{M}} \,.$$

Let

$$\Delta_\epsilon = \{x \in V : 0 < f(x) < \epsilon\} \quad \text{and} \quad J(\epsilon) = \int\limits_{\Delta_\epsilon} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \,.$$

Using Theorem 6.1, we have

$$J(\epsilon) = \int\limits_0^\epsilon dt \int\limits_{\gamma_t^*} |\nabla_{\mathcal{M}} f|^{n-1} d\mathcal{H}_{\mathcal{M}}^{n-1} \,.$$

From here, there exists a sequence $\{\gamma_{\epsilon_k}\}$, $\epsilon_k \to 0$, along which:

$$\int\limits_{\gamma_{\epsilon_k}^*} |\nabla_{\mathcal{M}} f|^{n-1} d\mathcal{H}_{\mathcal{M}}^{n-1} \le \frac{2}{\epsilon_k} J(\epsilon_k) \,.$$

Thus, we find

$$|I_{\epsilon_k}| \le \frac{2}{\epsilon_k} \max_{\overline{V}_1} |\phi| \, J(\epsilon_k) \,. \tag{2.12}$$

Now because

$$V_{1\epsilon'} \subset V_{1\epsilon''} \quad \text{for} \quad \epsilon' > \epsilon'' \quad \text{and} \quad \cup_{\epsilon > 0} V_{1\epsilon} = V_1 \,,$$

we observe that

$$\int\limits_{V_{1\epsilon}} \phi |\nabla_{\mathcal{M}} f^*|^n d\sigma_{\mathcal{M}} \to \int\limits_{V_1} \phi |\nabla_{\mathcal{M}} f^*|^n d\sigma_{\mathcal{M}} \,. \tag{2.13}$$

Also clearly,

$$\left| \int\limits_{V_1 \setminus V_{1\epsilon}} f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, \nabla \phi_{\mathcal{M}} \rangle \, d\sigma_{\mathcal{M}} \right|$$

$$\le \max_{V_1 \setminus V_{1\epsilon}} |f| \int\limits_{V_1 \setminus V_{1\epsilon}} |\nabla_{\mathcal{M}} f|^{n-1} |\nabla_{\mathcal{M}} \phi| \, d\sigma_{\mathcal{M}}$$

$$\le \epsilon \int\limits_{V_1 \setminus V_{1\epsilon}} |\nabla_{\mathcal{M}} f|^{n-1} |\nabla_{\mathcal{M}} \phi| \, d\sigma_{\mathcal{M}} \le \text{const}\, \epsilon. \tag{2.14}$$

Finally, for the first integral at the right side of (2.11) we can write

$$\int\limits_{\partial^* V_{1\epsilon} \cap A_\epsilon} \phi \, f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}_{\mathcal{M}}^{n-1}$$

$$= \int\limits_{\partial^* V_1 \cap A_\epsilon} \phi \, f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}_{\mathcal{M}}^{n-1}$$

$$+ \int\limits_{\partial^* V_1 \cap \{0 < f(x) < \epsilon\}} \phi \, f^* \, |\nabla_{\mathcal{M}} f^*|^{n-2} \langle \nabla_{\mathcal{M}} f^*, n_{\mathcal{M}} \rangle \, d\mathcal{H}_{\mathcal{M}}^{n-1} \,. \tag{2.15}$$

Moreover,

$$
\begin{aligned}
&\left| \int\limits_{\partial^* V_1 \cap \{0<f(x)<\epsilon\}} \phi\, f^* \, |\nabla_{\mathcal M} f^*|^{n-2} \langle \nabla_{\mathcal M} f^*, n_{\mathcal M}\rangle \, d\mathcal{H}^{n-1}_{\mathcal M} \right| \\
&\le \epsilon \int\limits_{\partial^* V_1 \cap A_\epsilon} \phi \, |\nabla_{\mathcal M} f^*|^{n-1} d\mathcal{H}^{n-1}_{\mathcal M} \le \mathrm{const}\, \epsilon \, .
\end{aligned} \tag{2.16}
$$

Combining (2.11) – (2.16), we arrive at the following estimate

$$
\begin{aligned}
&\int\limits_{V_1} \phi \, |\nabla_{\mathcal M} f^*|^n d\sigma_{\mathcal M} + \int\limits_{V_1} f^* \, |\nabla_{\mathcal M} f^*|^{n-2} \langle \nabla_{\mathcal M} f^*, \nabla_{\mathcal M}\phi\rangle \, d\sigma_{\mathcal M} \\
&\le \int\limits_{\partial^* V_1} \phi\, f^* \, |\nabla_{\mathcal M} f^*|^{n-2} \langle \nabla_{\mathcal M} f^*, n_{\mathcal M}\rangle \, d\mathcal{H}^{n-1}_{\mathcal M} + \mathrm{const}\, \epsilon_k + 2\, \max\nolimits_{\overline{V}} |\phi| \, J(\epsilon_k) \, .
\end{aligned}
$$

Since $J(\epsilon_k) \to 0$ as $k \to 0$, we obtain

$$
\begin{aligned}
&\int\limits_{V_1} \phi \, |\nabla_{\mathcal M} f^*|^n d\sigma_{\mathcal M} + \int\limits_{V_1} f^* \, |\nabla_{\mathcal M} f^*|^{n-2} \langle \nabla_{\mathcal M} f^*, \nabla_{\mathcal M}\phi\rangle \, d\sigma_{\mathcal M} \\
&\le \int\limits_{\partial^* V_1} \phi\, f^* \, |\nabla_{\mathcal M} f^*|^{n-2} \langle \nabla_{\mathcal M} f^*, n_{\mathcal M}\rangle \, d\mathcal{H}^{n-1}_{\mathcal M} \, .
\end{aligned} \tag{2.17}
$$

Observe that

$$V = V_1 \cup V_2 \cup (E_0 \cap V),$$

and moreover, $f|_{E_0} \equiv 0$.

At each point $x \in V_2$, we have $|\nabla g| = 0$.

For $\mathcal{H}^{n-1}$-almost all points $x \in E_0$ for a contingent of E_0,

$$\mathrm{cont}_x \, E_0 = \mathbf{R}^n \, .$$

At each point $x \in E_0$ with this property, the partial derivative $\frac{\partial f}{\partial \mathbf{u}}$ with respect to an directional $\mathbf{u}$ equals 0 for almost all directions at this point.

Consequently, $\mathcal{H}^{n-1}$-almost everywhere on E_0,

$$\nabla f(x) = 0 \, .$$

Thus, from (2.17), we obtain (2.9). Hence, u is a generalized solution of (2.5).

To prove the corresponding statement for $v = \min\{f(x), 0\}$, it is sufficient to observe that

$$v = \max\{-f(x), 0\}$$

and to reason as above. □

3 Maximum Principle

The point $\{r=0\}$ is a singular point of the cone $\mathcal{M}=(D, ds^2_{\mathcal{M}})$. In the following maximum principle we assume a special behavior of solutions close to this point.

3.1. Lemma. *Let $U \subset D$ be a bounded domain with a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary, $\mathcal{H}^n(\partial U \cap D(f)) = 0$. Let f be a generalized solution of (2.5) in U such that $f|_{\partial U} = 0$ and, if the point $\{r=0\}$ lies on ∂U, let, in addition,*

$$\int\limits_0 dr \left(\int\limits_{U \cap S^{n-1}(r)} |f|^n \, dS^{n-1}(r) \right)^{1/(1-n)} = \infty \,. \tag{3.2}$$

Then $f \equiv 0$ on U.

Proof. Fix $r_0 > d = \min_{x \in \partial U} |x|$ such that $U \cap \{|x| > r_0\} \neq \emptyset$. Let $\phi(r) : [0,\infty) \to [0,\infty)$ be a Lipschitz function,

$$\phi(r) = \begin{cases} 1 & \text{for} \quad r \geq r_0 \\ \phi^*(r) & \text{for} \quad d < r < r_0 \\ 0 & \text{for} \quad 0 \leq r \leq d \end{cases}$$

with a Lipschitz function $\phi^* : [d, r_0] \to \mathbf{R}$ for which

$$\operatorname*{ess\,inf}_{r \in E} |(\phi^*)'(r)| > 0 \quad \text{for any compact set} \quad E \subset (d, r_0). \tag{3.3}$$

Since $f|_{\partial U} = 0$, using (2.7), we find

$$\int\limits_U \phi^n |\nabla_{\mathcal{M}} f|^n \, d\sigma_{\mathcal{M}} \leq -n \int\limits_U \phi^{n-1} \, f \, |\nabla_{\mathcal{M}} f|^{n-2} \langle \nabla_{\mathcal{M}} f, \nabla_{\mathcal{M}} \phi \rangle \, d\sigma_{\mathcal{M}}.$$

From here, by the Hölder inequality, we have

$$\int\limits_U \phi^n |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \leq n \int\limits_U |f| \, \phi^{n-1} |\nabla_{\mathcal{M}} f|^{n-1} \, |\nabla_{\mathcal{M}} \phi| \, d\sigma_{\mathcal{M}}$$

$$\leq n \left(\int\limits_U \phi^n |\nabla_{\mathcal{M}} f|^n \, d\sigma_{\mathcal{M}} \right)^{(n-1)/n} \left(\int\limits_U |f|^n \, |\nabla_{\mathcal{M}} \phi|^n \, d\sigma_{\mathcal{M}} \right)^{1/n} .$$

The function $\phi \equiv 1$ on $U \cap \{r_0 < |x|\}$. Thus, we obtain

$$\int\limits_{U \cap \{r_0 < |x|\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \leq n^n \int\limits_{U \cap \{d < |x| < r_0\}} |f|^n \, |\nabla_{\mathcal{M}} \phi|^n \, d\sigma_{\mathcal{M}} \,. \tag{3.4}$$

Find the minimum of the right integral. Granting (3.3) and using Theorem 6.1, we may write

$$\begin{aligned} I(\phi) &= \int\limits_{U\cap\{d<|x|<r_0\}} |f|^n \, |\nabla_{\mathcal{M}}\phi|^n \, d\sigma_{\mathcal{M}} \\ &= \alpha^{-n} \int\limits_{U\cap\{d<|x|<r_0\}} |\phi'|^n \, |f|^n \, d\sigma_{\mathcal{M}} \\ &= \alpha^{1-n} \int\limits_d^{r_0} |\phi'|^n \, dr \int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \, . \end{aligned}$$

Here we had used also that by (2.2), the relation $|\nabla_{\mathcal{M}}\phi(r)| = \frac{1}{\alpha}|\phi'(r)|$ holds.

Since $\phi(r_0) - \phi(d) = 1$, by the Hölder inequality, we have

$$\begin{aligned} 1 \leq \left(\int\limits_d^{r_0} |\phi'(r)| \, dr \right)^n & \leq \left(\int\limits_d^{r_0} |\phi'|^n \, dr \int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right) \\ & \times \left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{n-1} . \end{aligned}$$

Hence for each admissible ϕ,

$$\left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{1-n} \leq \alpha^{n-1} I(\phi) \, .$$

Choosing $\phi^*(r) = \beta(r)/\beta(r_0)$, where

$$\beta(r) = \int\limits_d^{r} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \qquad r \in (d, r_0) \, ,$$

we see that

$$\min_{\phi} I(\phi) = \alpha^{1-n} \left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{1-n} . \tag{3.5}$$

Thus, from (3.4), it follows

$$\int\limits_{U\cap\{r_0<|x|\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \leq n^n \, \alpha^{1-n} \left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{1-n} . \tag{3.6}$$

The inequality (3.6) shows that (3.2) implies

$$\int\limits_{U\cap\{r_0<|x|\}} |\nabla_{\mathcal{M}}|^n \, d\sigma_{\mathcal{M}} = 0.$$

This means that $f \equiv \text{const}$ on $U \cap \{r_0 < |x|\}$. Since $r_0 > d$ is arbitrary and $f|_{\partial U} = 0$, we conclude that $f \equiv 0$ everywhere on U. $\Box$

Below we will need an analog of the Liouville Theorem for solutions of the differential inequality (2.5).

3.1. Theorem. *Let f be a generalized solution of (2.5) on D. Suppose that f satisfies (3.2) in D and*

$$\int^{\infty} dr \left(\int\limits_{S^{n-1}(r)} |f|^n \, dS^{n-1}(r) \right)^{\frac{1}{1-n}} = \infty \,. \tag{3.2}$$

Then $f \equiv \mathrm{const}$ in D.

3.3. Corollary. *Each generalized bounded solution of (2.5) in D is the identical constant.*

Let $U \subset D$ be a domain, and let f be a locally Lipschitz function on U. We set

$$E(r, U) = \int\limits_{U \cap \{0<|x|<r\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \,.$$

Before the proof we establish an useful estimate.

3.4. Lemma. *Let $U \subset D$ be a domain having a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary, for which $\mathcal{H}^{n-1}(\partial V \cap D(f)) = 0$ and*

$$U \cap S^{n-1}(t) \neq \emptyset \quad \text{for all sufficiently big } t > 0 \,. \tag{3.5}$$

Let f be a generalized solution of (2.5) in U satisfying $f|_{\partial U} = 0$. Suppose also that f satisfies (3.2) on U if $\{r = 0\} \in \partial U$.

Then for arbitrary r_1, r_2 such that $d = \min_{x \in U} |x| < r_1 < r_2 < \infty$, the following estimate holds

$$E(r_1, U) \le \alpha^{1-n} n^n \left(\int_{r_1}^{r_2} dr \left(\int\limits_{S^{n-1}(r)} |f|^n \, dS^{n-1}(r) \right)^{\frac{1}{1-n}} \right)^{1-n} \,. \tag{3.6}$$

Proof. Fix arbitrarily r_0, r_1, r_2 such that $d < r_0 < r_1 < r_2 < \infty$. Consider two Lipschitz functions:

$$\psi(r) = \begin{cases} 0 & \text{for} \quad 0 \le r \le d \\ \psi^*(r) & \text{for} \quad d < r < r_0 \\ 1 & \text{for} \quad r_0 \le r \end{cases} \tag{3.7}$$

and

$$\eta(r) = \begin{cases} 1 & \text{for} \quad r \le r_1 \\ \eta^*(r) & \text{for} \quad r_1 < r < r_2 \\ 0 & \text{for} \quad r_2 \le r < \infty \end{cases} \,, \tag{3.8}$$

where ψ^*, η^* are satisfy (3.3) and $0 \le \psi^*, \eta^* \le 1$.

By (2.7) with $\phi = \psi^n \eta^n$, we have

$$\begin{aligned}\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n d\sigma_{\mathcal M} &\le -n \int_U \psi^n \eta^{n-1} \eta' f |\nabla_{\mathcal M} f|^{n-2} \langle \nabla_{\mathcal M} f, \nabla_{\mathcal M} r\rangle \, d\sigma_{\mathcal M} \\ &\quad -n \int_U \psi^{n-1} \psi' \eta^n f |\nabla_{\mathcal M} f|^{n-2} \langle \nabla_{\mathcal M} f, \nabla_{\mathcal M} r\rangle \, d\sigma_{\mathcal M} .\end{aligned}$$

From here,

$$\begin{aligned}\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n d\sigma_{\mathcal M} &\le n \int_U \psi^n \eta^{n-1} |\eta'| \, |f| \, |\nabla_{\mathcal M} f|^{n-1} |\nabla_{\mathcal M} r| \, d\sigma_{\mathcal M} \\ &\quad + n \int_U \psi^{n-1} |\psi'| \, \eta^n |f| \, |\nabla_{\mathcal M} f|^{n-1} |\nabla_{\mathcal M} r| \, d\sigma_{\mathcal M} .\end{aligned}$$

Using (2.2), we see that $|\nabla_{\mathcal M} r| \equiv 1/\alpha$ and, consequently,

$$\begin{aligned}\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n d\sigma_{\mathcal M} &\le \tfrac{n}{\alpha} \int_U \psi^n \eta^{n-1} |\eta'| \, |f| \, |\nabla_{\mathcal M} f|^{n-1} \, d\sigma_{\mathcal M} \\ &\quad + \tfrac{n}{\alpha} \int_U \psi^{n-1} |\psi'| \, \eta^n |f| \, |\nabla_{\mathcal M} f|^{n-1} \, d\sigma_{\mathcal M}\end{aligned}$$

$$\le \tfrac{n}{\alpha} \left(\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n \, d\sigma_{\mathcal M} \right)^{(n-1)/n} \left(\int_U \psi^n |\eta'|^n \, |f|^n \, d\sigma_{\mathcal M} \right)^{1/n}$$

$$+ \tfrac{n}{\alpha} \left(\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n \, d\sigma_{\mathcal M} \right)^{(n-1)/n} \left(\int_U \eta^n |\psi'|^n \, |f|^n \, d\sigma_{\mathcal M} \right)^{1/n} .$$

Now, since $|\psi| \, , |\eta| \le 1$, we obtain

$$\left(\int_U \psi^n \eta^n |\nabla_{\mathcal M} f|^n d\sigma_{\mathcal M} \right)^{1/n} \le \tfrac{n}{\alpha} \left(\int_U |\eta'|^n \, |f|^n \, d\sigma_{\mathcal M} \right)^{1/n}$$

$$+ \tfrac{n}{\alpha} \left(\int_U |\psi'|^n \, |f|^n \, d\sigma_{\mathcal M} \right)^{1/n} \equiv \tfrac{n}{\alpha} \, I_1^{1/n}(\eta) + \tfrac{n}{\alpha} \, I_2^{1/n}(\psi) \, ,$$

or

$$\left(\int_{\substack{\{r_0 < |x| < r_1\} \\ x \in U}} |\nabla_{\mathcal M} f|^n d\sigma_{\mathcal M} \right)^{1/n} \le \frac{n}{\alpha} \, I_1^{1/n}(\eta) + \frac{n}{\alpha} \, I_2^{1/n}(\psi) \, . \tag{3.9}$$

As above at (3.5), we find

$$\min_\eta I_1(\eta) = \left(\int_{r_1}^{r_2} dr \left(\int_{U \cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{1-n}$$

and

$$\min_{\psi} I_2(\psi) = \left(\int\limits_{d}^{r_0} dr \left(\int\limits_{U \cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{1/(1-n)} \right)^{1-n} .$$

Thus, the inequality (3.9) implies

$$\left(\int\limits_{\substack{r_0<|x|<r_1 \\ x\in U}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \right)^{\frac{1}{n}} \leq \tfrac{p}{\alpha} \left(\int\limits_{r_1}^{r_2} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{\frac{1}{1-n}} \right)^{\frac{1-n}{n}}$$

$$+ \tfrac{n}{\alpha} \left(\int\limits_{d}^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{\frac{1}{1-n}} \right)^{\frac{1-n}{n}} .$$

Granting (3.2), we arrive at

$$\left(\int\limits_{\substack{r_0<|x|<r_1 \\ x\in U}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \right)^{1/n} \leq \frac{n}{\alpha} \left(\int\limits_{r_1}^{r_2} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{\frac{1}{1-n}} \right)^{\frac{1-n}{n}} ,$$

and, next, because r_0 is arbitrary we find

$$\int\limits_{\substack{0<|x|<r_1 \\ x\in U}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \leq \frac{n^n}{\alpha} \left(\int\limits_{r_1}^{r_2} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r) \right)^{\frac{1}{1-n}} \right)^{1-n} .$$

The necessary estimate is proved. □

Proof of Theorem 3.1. Suppose that $f \not\equiv \text{const}$. Let m and M be infimum and, respectively, supremum of f on D. By Theorem 6.1 and the relation (2.10) for almost all τ such that

$$\frac{m+M}{2} < \tau < M,$$

level sets $E_\tau = \{x \in D : f(x) = \tau\}$ are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable with

$$\mathcal{H}^{n-1}\left(E_\tau \cap D(f)\right) = 0 .$$

The inequality (3.6) implies $E(\tau; D) = 0$ for almost all τ. This means that $\nabla f \equiv 0$. Thus, $f \equiv \text{const}$ on D. □

For the **proof of Corollary 3.3** is it sufficient to observe that boundness of f implies (3.2) and (3.2). □

4 Saint-Venant Principle

Let $U \subset S^{n-1}(r)$ be an open set with $\partial U \neq \emptyset$. We need further the following quantity

$$\lambda_n(U) = \inf \frac{\left(\int\limits_U |\nabla_S \phi|^n \, dS^{n-1}(r)\right)^{1/n}}{\left(\int\limits_U \phi^n dS^{n-1}(r)\right)^{1/n}}, \tag{4.1}$$

where the infimum is taken over all functions $\phi \in \mathrm{Lip}\,(U)$, $\phi|_{\partial U} = 0$. Here $\nabla_S \phi$ is gradient of ϕ in the metric of $S = S^{n-1}(r)$.

This quantity is well-known and can be interpreted, in particular, as the best constant in the Poincaré inequality

$$\lambda_n(U) \left(\int\limits_U \phi^n \, dS^{n-1}(r)\right)^{1/n} \leq \left(\int\limits_U |\nabla_S \phi|^n \, dS^{n-1}(r)\right)^{1/n}.$$

The quantity $\lambda_n(U)$ is said to be the *fundamental frequency* of U [2, Chapter III].

4.2. Example. In the case $n = 2$ for any connected arc $U \subset S^1(r)$, we have

$$\lambda_2(U) = \frac{\pi}{\mathcal{H}^1(U)}. \tag{4.3}$$

(for example, see [4, Chapter 5, §10]). □

4.4. Example. Let $n \geq 3$. Let $U \subset S^{n-1}(r)$ be an open set with $\partial U \neq \emptyset$. Denote by

$$y = \frac{x}{r} : \mathbf{R}^n \to \mathbf{R}^n$$

a homothety of $\mathbf{R}^n$ transforming the sphere $S^{n-1}(r)$ onto the sphere $S^{n-1}(1)$. Let U^* be image of U. Using concepts of homothety in (4.1), we have

$$\lambda_n(U) = \frac{1}{r}\lambda_n(U^*).$$

Since $\lambda_n(U^*) \geq \lambda_n(\dot{S})$, where $\dot{S} = \dot{S}^{n-1}(1)$ is an unit punctured sphere, we find

$$\lambda_n(U) \geq \frac{1}{r}\lambda_n\left(\dot{S}\right).$$

If $\phi \in \mathrm{Lip}\,(\dot{S})$ and $\phi = 0$ at the punctured point, the following Sobolev $C-$embedding inequality holds:

$$\max_{\dot{S}} |\phi| \leq c(\dot{S}) \left(\int\limits_{\dot{S}} |\nabla_{S^{n-1}(1)} \phi|^n \, dS^{n-1}(1)\right)^{1/n},$$

where $c(\dot{S})$ is a constant.

From here,

$$\lambda_n(U^*) \geq \inf_{\phi} \frac{\left(\int\limits_{\dot{S}} |\nabla_{S^{n-1}(1)}\phi|^n dS^{n-1}(1)\right)^{1/n}}{\left(\int\limits_{\dot{S}} \phi^n dS^{n-1}(1)\right)^{1/n}} \geq c^{-1}(\dot{S})\, \omega_{n-1}^{-1/n}\,.$$

Thus, we obtain

$$\lambda_n(U) \geq \frac{1}{r} c^{-1}(\dot{S})\, \omega_{n-1}^{-1/n} \quad \text{for all} \quad U \subset S^{n-1}(r)\,. \tag{4.5}$$

For the constant $\dot{S}$, see [18, Lemma 3.3] and [19, Lemma 6.26]. □

Observe a usefull property of the fundamental frequency.

4.6. Lemma. *Let $U \subset S^{n-1}(r)$ be an open set and let U_k be connected components of U, $k = 1, 2, \ldots$. Then*

$$\lambda_p(U) = \inf_k \lambda_p(U_k).$$

See proof of this statement in [16, Lemma 2.1]. □

Let $U \subset D$ be a domain satisfying (3.5) with an countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary,

$$\mathcal{H}^{n-1}\left(\partial U \cap D(f)\right) = 0 \quad and \quad f|_{\partial U \cap D} = 0\,.$$

For arbitrary r_0, r such that $d < r_0 < r < \infty$, we set

$$\Lambda_n(r_0, r; U) = \exp\left\{-\frac{\alpha n}{n-1} \int\limits_{r_0}^{r} \lambda_n(U \cap S^{n-1}(\tau))\, d\tau\right\}\,.$$

In the elasticity theory, the following type inequalities are known as the Saint-Venant principle (for example, see [17], [16], [14]).

4.7. Lemma. *Let f be a generalized solution of (2.5) on U and f satisfies (3.2) if the point $\{r = 0\}$ lies on ∂U.*

Then

$$\int\limits_{U\cap\{|x|<r_0\}} |\nabla_{\mathcal{M}} f|^p d\sigma_{\mathcal{M}} \leq \Lambda_n(r_0, r; U) \int\limits_{U\cap\{|x|<r\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}}\,. \tag{4.8}$$

Proof. The idea of the proof goes back to [3]. Without loss of generality we assume that $\{r = 0\}$ be lying on ∂U. Let r_0, r, $d < r_0 < r < \infty$. Choose arbitrarily a Lipschitz function ψ by (3.7). With $\phi = \psi^p$ and $V = U \cap \{|x| < r\}$ in (2.7), we have

$$\int\limits_{U\cap\{|x|<r\}} \psi^n \, |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \le -n \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^{n-1}\psi' \, f \, |\nabla_{\mathcal{M}} f|^{n-2} \langle \nabla_{\mathcal{M}} f, \nabla_{\mathcal{M}} r\rangle \, d\sigma_{\mathcal{M}}$$

$$+ \int\limits_{\partial^*(U\cap\{d<|x|<r\})} \psi^n f \, |\nabla_{\mathcal{M}} f|^{n-2} \langle \nabla_{\mathcal{M}} f, n_{\mathcal{M}}\rangle \, d\mathcal{H}^{n-1}_{\mathcal{M}} \, .$$

Since $f|_{\partial U} = 0$ granting (2.2), we find

$$\int\limits_{U\cap\{|x|<r\}} \psi^n \, |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \le \tfrac{n}{\alpha} \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^{n-1} |\psi'| \, |f| \, |\nabla_{\mathcal{M}} f|^{n-1} d\sigma_{\mathcal{M}}$$

$$(4.9) \qquad + \tfrac{1}{\alpha} \int\limits_{U\cap S^{n-1}(r)} |f| \, |\nabla_{\mathcal{M}} f|^{n-2} |f'_r| \, dS^{n-1}(r) \, .$$

We use Cauchy inequality 'with $\epsilon > 0$' for arbitrary $a \ge 0, \; b \ge 0$,

$$ab \le \frac{\epsilon^n}{n} \, a^n + \frac{n-1}{n} \left(\frac{a}{\epsilon}\right)^{n/(n-1)} \, .$$

We obtain

$$\tfrac{n}{\alpha} \int\limits_{U\cap\{d<|x|<r_0\}} \psi^{n-1} |\psi'| \, |f| \, |\nabla_{\mathcal{M}} f|^{n-1} \, d\sigma_{\mathcal{M}} \le \left(\tfrac{\epsilon}{\alpha}\right)^n \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} |\psi'|^n \, |f|^n \, d\sigma_{\mathcal{M}}$$

$$+ (n-1) \, \epsilon^{-n/(n-1)} \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^n \, |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \, .$$

Further, for

$$\epsilon = (n-1)^{(n-1)/n} \, ,$$

we have

$$\tfrac{n}{\alpha} \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^{n-1} |\psi'| \, |f| \, |\nabla_{\mathcal{M}} f|^{n-1} \, d\sigma_{\mathcal{M}} \le \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^n \, |\nabla_{\mathcal{M}} f|^n \, d\sigma_{\mathcal{M}}$$

$$+ \left(\tfrac{\epsilon}{\alpha}\right)^n \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} |\psi'|^n \, |f|^n \, d\sigma_{\mathcal{M}} \, .$$

Substituting this estimate in (4.9), we arrive at the inequality

$$\int\limits_{U\cap\{|x|<r\}} \psi^n \, |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \le \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} \psi^n \, |\nabla_{\mathcal{M}} f|^n \, \sigma_{\mathcal{M}}$$

$$+ \left(\tfrac{\epsilon}{\alpha}\right)^n \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in D}} |\psi'|^n \, |f|^n \, d\sigma_{\mathcal{M}}$$

$$+ \tfrac{1}{\alpha} \int\limits_{U\cap S^{n-1}(r)} |f| \, |\nabla_{\mathcal{M}} f|^{n-2} |f'_r| \, dS^{n-1}(r) \, ,$$

or,

$$\int\limits_{U\cap\{r_0<|x|<r\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} \le \left(\tfrac{\epsilon}{\alpha}\right)^n \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} |\psi'|^n\,|f|^n\,d\sigma_{\mathcal{M}}$$

$$+\tfrac{1}{\alpha}\int\limits_{U\cap S^{n-1}(r)} |f|\,|\nabla_{\mathcal{M}} f|^{n-2}|f_r'|\,dS^{n-1}(r)\,.$$

As at (3.5), we have

$$\min_{\psi} \int\limits_{\substack{\{d<|x|<r_0\}\\ x\in U}} |\psi'|^n\,|f|^n\,d\sigma_{\mathcal{M}} = \left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r)\right)^{1/(1-n)}\right)^{1-n},$$

and

$$\int\limits_{\substack{\{r_0<|x|<r\}\\ x\in U}} |\nabla_{\mathcal{M}} f|^n\,d\sigma_{\mathcal{M}} \le \left(\tfrac{\epsilon}{\alpha}\right)^n \left(\int\limits_d^{r_0} dr \left(\int\limits_{U\cap S^{n-1}(r)} |f|^n dS^{n-1}(r)\right)^{1/(1-n)}\right)^{1-n}$$

$$+\tfrac{1}{\alpha}\int\limits_{U\cap S^{n-1}(r)} |f|\,|\nabla_{\mathcal{M}} f|^{n-2}\,|f_r'|\,dS^{n-1}(r)\,.$$

The condition (3.2) and arbitrariness in choose of $r_0 > d$ imply

$$\text{(4.10)}\qquad \int\limits_{\substack{\{d<|x|<r\}\\ x\in U}} |\nabla_{\mathcal{M}} f|^n\,d\sigma_{\mathcal{M}} \le \frac{1}{\alpha}\int\limits_{U\cap S^{n-1}(r)} |f|\,|\nabla_{\mathcal{M}} f|^{n-2}\,|f_r'|\,dS^{n-1}(r)\,.$$

(In the case when $\{r=0\}$ does not lie on ∂U, the inequality (4.10) follows from (2.7) immediately.)

Observing that for an arbitrary $\epsilon > 0$:

$$\alpha^{-1}\,|f|\,|\nabla_{\mathcal{M}} f|^{n-2}|f_r'| \le \frac{\epsilon^n}{n}|f|^n + \frac{n-1}{n}\,(\epsilon\alpha)^{-\frac{n}{n-1}}\,|\nabla_{\mathcal{M}} f|^{\frac{n(n-2)}{n-1}}\,|f_r'|^{\frac{n}{n-1}}\,,$$

we find

$$\tfrac{1}{\alpha}\int\limits_{U\cap S^{n-1}(r)} |f|\,|\nabla_{\mathcal{M}} f|^{n-2}\,|f_r'|\,dS^{n-1}(r) \le \tfrac{\epsilon^n}{n}\int\limits_{U\cap S^{n-1}(r)} |f|^n\,dS^{n-1}(r)$$

$$+\tfrac{n-1}{n}(\epsilon\alpha)^{-\frac{n}{n-1}}\int\limits_{U\cap S^{n-1}(r)} |\nabla_{\mathcal{M}} f|^{\frac{n(n-2)}{n-1}}\,|f_r'|^{\frac{n}{n-1}}\,dS^{n-1}(r) \equiv \tfrac{\epsilon^n}{n}\,I_1 + \tfrac{n-1}{n}(\epsilon\alpha)^{-\frac{n}{n-1}}\,I_2\,.$$

From (4.1) and $f|_{\partial U} = 0$, we may write

$$I_1 \le \frac{1}{\lambda_n^n(U\cap S^{n-1}(r))}\int\limits_{U\cap S^{n-1}(r)} |\nabla_S f|^n\,dS^{n-1}(r)\,,$$

and (4.10) implies

$$\text{(4.11)}\qquad \int\limits_{\substack{\{d<|x|<r\}\\ x\in U}} |\nabla_{\mathcal{M}} f|^n\,d\sigma_{\mathcal{M}} \le \frac{n-1}{n}(\epsilon\alpha)^{-\frac{n}{n-1}}\,I_2 + \frac{\epsilon^n}{n\lambda_n^n(r)}\int\limits_{U\cap S^{n-1}(r)} |\nabla_S f|^n\,dS^{n-1}(r)\,,$$

where $\lambda_n(r)=\lambda_n(U\cap S^{n-1}(r))$.

To estimate I_2 we use the inequality

$$ab\le\frac{1}{n-1}\,a^{n-1}+\frac{n-2}{n-1}\,b^{\frac{n-1}{n-2}},\quad a,\,b\ge 0\,.$$

We observe that

$$\alpha^{-\frac{n}{n-1}}\,|f_r'|^{\frac{n}{n-1}}\,|\nabla_{\mathcal{M}}f|^{\frac{n(n-2)}{n-1}}\le\frac{1}{n-1}\,|f_r'|^n\,\alpha^{-n}+\frac{n-2}{n-1}\,|\nabla_{\mathcal{M}}f|^n\,.$$

Thus we arrive at the estimate

$$\alpha^{-\frac{n}{n-1}}\,I_2\le\frac{1}{n-1}\int\limits_{U\cap S^{n-1}(r)}\alpha^{-n}\,|f_r'|^n\,dS^{n-1}(r)+\frac{n-2}{n-1}\int\limits_{U\cap S^{n-1}(r)}|\nabla_{\mathcal{M}}f|^n\,dS^{n-1}(r)\,.$$

(4.12)

Combining inequalities (4.11) and (4.12), we get

$$\begin{aligned}\int\limits_{\substack{\{d<|x|<r\}\\ x\in U}}|\nabla_{\mathcal{M}}f|^n\,d\sigma_{\mathcal{M}}\ &\le\tfrac{1}{n}(\epsilon)^{-\frac{n}{n-1}}\int\limits_{U\cap S^{n-1}(r)}\alpha^{-n}|f_r'|^n\,dS^{n-1}(r)\\ &+\tfrac{n-2}{n}\epsilon^{-\frac{n}{n-1}}\int\limits_{U\cap S^{n-1}(r)}|\nabla_{\mathcal{M}}f|^n\,dS^{n-1}(r)\\ &+\tfrac{\epsilon^n}{n\lambda_n^n(r)}\int\limits_{U\cap S^{n-1}(r)}|\nabla_S f|^n\,dS^{n-1}(r)\,.\end{aligned}$$

Choosing now $\epsilon^{\frac{n}{n-1}}=\lambda_n(r)$, we have

$$\begin{aligned}\int\limits_{\substack{\{d<|x|<r\}\\ x\in U}}|\nabla_{\mathcal{M}}f|^n\,d\sigma_{\mathcal{M}}\ &\le\tfrac{1}{n}\lambda_n^{-1}(r)\int\limits_{U\cap S^{n-1}(r)}\left(\tfrac{1}{\alpha^n}|f_r'|^n+|\nabla_S f|^n\right)\,dS^{n-1}(r)\\ &+\tfrac{n-2}{n}\,\lambda_n^{-1}(r)\int\limits_{U\cap S^{n-1}(r)}|\nabla_{\mathcal{M}}f|^n\,dS^{n-1}(r)\,.\end{aligned}$$

Since for $n\ge 2$

$$\frac{1}{\alpha^n}|f_r'|^n+|\nabla_S f|^n\le|\nabla_{\mathcal{M}}f|^n\,,$$

we find

$$\int\limits_{\substack{\{d<|x|<r\}\\ x\in U}}|\nabla_{\mathcal{M}}f|^n\,d\sigma_{\mathcal{M}}\le\frac{n-1}{n}\lambda_n^{-1}(r)\int\limits_{U\cap S^{n-1}(r)}|\nabla_{\mathcal{M}}f|^n\,dS^{n-1}(r)\,.\tag{4.13}$$

Finally, from (4.13), we find

$$\int\limits_{U\cap\{d<|x|<r\}}|\nabla_{\mathcal{M}}f|^n d\sigma_{\mathcal{M}}\le\frac{n-1}{n}\,\lambda_n^{-1}(r)\int\limits_{U\cap S^{n-1}(r)}|\nabla_{\mathcal{M}}f|^n\,dS^{n-1}(r)\,.$$

Let

$$J(r)=\int\limits_{U\cap\{d<|x|<r\}}|\nabla_{\mathcal{M}}f|^n\,d\sigma_{\mathcal{M}}\,.$$

Using by (2.1) for almost all $r > d$, we can write

$$J'(r) = \alpha \int\limits_{U \cap S^{n-1}(r)} |\nabla_{\mathcal{M}} f|^n \, dS^{n-1}(r)$$

and, next, for almost all $r > d$

$$J(r) \leq \frac{n-1}{\alpha n} \lambda_p^{-1}(r) \, J'(r) \, .$$

Integrating this differential inequality, we find

$$\log \frac{J(r)}{J(r_0)} \geq \frac{\alpha n}{n-1} \int\limits_{r_0}^{r} \lambda_p(r) \, dr.$$

From this relation we arrive at (4.8). $\Box$

5 Generalized Arima's Inequality

Here we provide an inequality that generalizes the Arima's inequality for entire holomorphic functions of complex variable [1, Theorem 2].

5.1. Theorem. *Let $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ be an $n-$dimensional cone over the domain $D = \{x \in \mathbf{R}^n : 0 < |x| < \infty\}$ defined by (1.1). Let $U \subset D$ be a domain satisfying (3.5) and having the countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary with $\mathcal{H}^{n-1}(\partial U \cap D(f)) = 0$. Let f, $f|_{\partial U \cap D} = 0$, be a nonconstant nonnegative n-subharmonic function on U.*

Then, for an arbitrary $0 < \gamma < 1$, we have

$$(5.2)\quad \log M_f(r,U) \geq \frac{\alpha}{n-1} \int\limits_{r_0}^{\gamma r} \lambda_n(U \cap S^{n-1}(\tau))d\tau + \frac{n-1}{n} \log\log\frac{1}{\gamma} + c(r_0,n)\,,$$

where $d < r_0 < \gamma r$, $d = \min_{x \in U}|x|$, and

$$c(r_0,n) = \frac{1}{n}\log\frac{1}{n^n\,\omega_{n-1}} \int\limits_{U \cap \{|x|<r_0\}} |\nabla_{\mathcal{M}} f|^n d\sigma_{\mathcal{M}} + \frac{n-1}{n}\log\alpha.$$

In particular, for $n = 2$,

$$(5.3)\qquad \log M_f(r,U) \geq \alpha\,\pi \int\limits_{r_0}^{\gamma r} \frac{d\tau}{\mathcal{H}^1(U \cap S^1(\tau))} + \frac{1}{2}\log\log\frac{1}{\gamma} + c(r_0,2)\,.$$

Proof. Combining relations (3.6) and (4.8) we have for arbitrary r_0, r_1, r, such that $0 < r_0 < r_1 < r < \infty$, the following estimate holds

$$(5.4)\qquad E(r_0;U) \leq n^n\,\alpha^{1-n}\omega_{n-1}\,M_f^n(r,U)\left(\log\frac{r}{r_1}\right)^{1-n} \Lambda_p(r_0,r_1;U)\,.$$

Our theorem follows immediately from 5.4 and (4.3). $\square$

Proof of Theorem 1.3. Fix a number C such that

$$\limsup_{x\to 0} f(x) < C < \limsup_{x\to\infty} f(x).$$

The condition (1.4) guarantees it is possible.

By Theorem 6.3 for almost all C, boundaries of sets $D_C = \{x \in D : f(x) > C\}$ are countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable. Fix arbitrarily a connected component U_C of D_C with such property. Since $\mathcal{H}^n(D(f)) = 0$ without loss of generality, we may assume that U_C satisfies the property $\mathcal{H}^{n-1}(\partial U \cap D(f)) = 0$. Consider the function

$$f^*(x) = \begin{cases} f(x) - C & \text{for} \quad x \in U_C \\ \quad 0 & \text{for} \quad x \in D \setminus U_C\,. \end{cases}$$

By Lemma 2.8 the function f^* satisfies (2.5) on D. Because $f^* \equiv 0$ close to the point $x = 0$, this function has the property (3.2). By Theorem 3.1, we have

$$\limsup_{x \to \infty} f^*(x) = \infty\,,$$

and, hence, f is not bounded in D close to the point ∞.

Suppose that (1.5) is not true, that is,

$$\limsup_{r \to \infty} \min_{x \in S^{n-1}(r)} |f(x)| = M < \infty\,.$$

Let C_1 be an arbitrary number such that $\max\{C, M\} < C_1 < \infty$. The set U_{C_1} has the property (3.5).

We will use Theorem 5.1. For any r, $\gamma r > r_0 > d$, we have $S^{n-1}(r) \cap \partial U_{C_1} \neq \emptyset$. From this relation, we find

$$\lambda_n(U_{C_1} \cap S^{n-1}(r)) \geq \lambda_n(\dot{S}^{n-1}(r)) = \frac{1}{r}\lambda_n(\dot{S}^{n-1}(1))\,,$$

and the inequality (5.2) implies

$$\log\left(M_f(r, U_{C_1}) - C_1\right) \geq \frac{\alpha}{n-1}\lambda_n(\dot{S}^{n-1}(1)) \log \frac{\gamma r}{r_0} + \frac{n-1}{n} \log\log \frac{1}{\gamma} + c(r_0, n)\,.$$

Therefore,

$$M_f(r, U_{C_1}) - C_1 \geq e^{c(r_0,n)} \left(\frac{\gamma r}{r_0}\right)^{\frac{\alpha}{n-1}\lambda_n(\dot{S}^{n-1}(1))} \left(\log \frac{1}{\gamma}\right)^{\frac{n-1}{n}}\,,$$

and

$$\rho \geq \frac{\alpha}{n-1}\lambda_n(\dot{S}^{n-1}(1))\,.$$

We arrive at a contradiction with assumptions of the theorem. Hence, (1.5) is proved. □

Proof of Theorem 1.6. At first, as above we prove that

$$\limsup_{x \to \infty} f(x) = \infty.$$

Next, choose an arbitrary sequence of numbers $\{C_k\}_{k=1}^{\infty}$, $C_k \to \infty$. Let U_k be a connected component of the set $\{x \in D : f(x) > C_k\}$ for $k = 1, 2, \ldots$. By Theorem 5.1 for each r_{0k},

$$d_{0k} = \min_{x \in U_k} |x| < r_{0k} < \gamma r_k, \quad 0 < \gamma < 1,$$

we may write

$$(M_f(r, U_k) - C_k) \exp\left\{-\frac{\alpha}{n-1} \int\limits_{r_{0k}}^{\gamma r_k} \lambda_n(U_k \cap S^{n-1}(\tau))\, d\tau\right\} \geq \epsilon_k \quad k = 1, 2, \ldots,$$

with some constants $\epsilon_k > 0$.

We observe that

$$\lambda_n(U_k \cap S^{n-1}(\tau)) = \frac{1}{\tau}\lambda_n\left[(U_k \cap S^{n-1}(\tau))^*\right],$$

where the set $(U_k \cap S^{n-1}(\tau))^* \subset S^{n-1}(1)$ is defined in Section 1.

Let $s > s' > \rho$ be arbitrary numbers. For any fixed $k = 1, 2, \ldots$ there exists a sequence $\{r_{km}\}_{m=1}^{\infty}$, $r_{km} \to \infty$, such that

$$M_f(r_{km}, U_k) - C_k \le L_k\, r_{km}^{s'}.$$

Now,

$$r_{km}^{s'} \exp\left\{-\frac{\alpha}{n-1}\int\limits_{r_{0k}}^{\gamma r_{km}} \lambda_n\left[(U_k \cap S^{n-1}(\tau))^*\right]\frac{d\tau}{\tau}\right\} \ge \epsilon_k \Big/ L_k\,,$$

or

$$\exp\left\{\frac{\alpha}{n-1}\int\limits_{r_{0k}}^{\gamma r_{km}}\left(s'\frac{n-1}{\alpha} - \lambda_n\left[(U_k \cap S^{n-1}(\tau))^*\right]\right)\frac{d\tau}{\tau}\right\} \ge \epsilon'_{km}.$$

Thus, we find

$$\int\limits_{r_{0k}}^{\infty}\left(s\frac{n-1}{\alpha} - \lambda_n\left[(U_k \cap S^{n-1}(\tau))^*\right]\right)\frac{d\tau}{\tau} > -\infty, \tag{5.5}$$

and there exists a sequence $\{\tau_k\} \to \infty$ for which

$$s\frac{n-1}{\alpha} > \lambda_n\left[(U_k \cap S^{n-1}(\tau_k))^*\right]. \tag{5.6}$$

In fact, suppose opposition, that is,

$$s\frac{n-1}{\alpha} \le \lambda_n\left[(U_k \cap S^{n-1}(\tau_k))^*\right].$$

Now, we have

$$s'\frac{n-1}{\alpha} - \lambda_n\left[(U_k \cap S^{n-1}(\tau_k))^*\right] \le -(s-s')\frac{n-1}{\alpha}\,,$$

and from (5.5) we obtain

$$-(s-s')\frac{n-1}{\alpha}\int\limits_{r_{0k}}^{\infty}\frac{d\tau}{\tau} > -\infty.$$

We obtained a contradiction.

By Lemma 4.6, each of the sets U_k contains a connected component Δ_k such that

$$\lambda_n(U_k \cap S^{n-1}(\tau)) = \lambda_n(\Delta_k) \quad (k = 1, 2, \ldots).$$

Along $\{\Delta_k\}$ the function f tends to ∞. The inequality (5.6) implies (1.8). $\square$

6 Appendix

We need the co-area formula [9, Section **3.2**].

6.1. Theorem. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a Lipschitz function. Let $E \subset \mathbf{R}^n$ be a measurable set and $g : E \to \mathbf{R}$ be a nonnegative measurable function. Then the following co-area formula holds*

$$\int_E g(x)\,|\nabla f(x)|\,dx = \int_{\mathbf{R}} \left(\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^{n-1}(x) \right) dy, \tag{6.2}$$

where ∇f is the gradient of f and $dx = dx_1 \cdots dx_n$ is the volume element on $\mathbf{R}^n$.

Let $E \subset \mathbf{R}^n$. We will use the following terminology and results (Federer [9, Sections **3.2.14**, **3.2.15**]):

(i) E is called *$(n-1)$-rectifiable*, if there is a Lipschitz function mapping a bounded subset of $\mathbf{R}^m$ onto E;

(ii) E is called *countably $(n-1)$-rectifiable*, if E is union of a countable family, all elements of which are $(n-1)$-rectifiable;

(iii) E is called *countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable*, if there is a countably $(n-1)$-rectifiable set containing $\mathcal{H}^{n-1}$-almost all points of E;

(iv) E is called *$(\mathcal{H}^{n-1}, n-1)$-rectifiable*, if E is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and $\mathcal{H}^{n-1}(E) < \infty$.

6.3. Theorem. *If a map $f : \mathbf{R}^n \to \mathbf{R}$ is Lipschitz, then any set $f^{-1}(y)$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable for $\mathcal{H}^1$-almost all y.*

We next explain a geometric sense of $(\mathcal{H}^{n-1}, n-1)$-rectifiable sets. Let $\mu \geq 1$ and $k \geq 1$ be some integers. At first, we recall that a *μ-dimensional submanifold of class k* of $\mathbf{R}^n$ is a subset $B \subset \mathbf{R}^n$ satisfying the following condition:
For each point $b \in B$ there are a neighborhood T of b in $\mathbf{R}^n$, a C^k-diffeomorphism

$$\sigma : T \to \mathbf{R}^n$$

and a μ-dimensional vector subspace Z of $\mathbf{R}^n$ such that

$$\sigma(B \cap T) = Z \cap \sigma(T).$$

6.4. Theorem. *A set $W \subset \mathbf{R}^n$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable if and only if $\mathcal{H}^{n-1}$-almost all points of W are contained in union of a countable family of $(n-1)$-dimensional submanifolds of class 1.*

See proof in [9, Section **3.2.29**].

We say that $\mathcal{H}^n$-measurable sets $A_j \subset \mathbf{R}^n$ *converge* to A, if for any compact set $Q \subset \mathbf{R}^n$,

$$\mathcal{H}^n\left[((A \setminus A_j) \cup (A_j \setminus A)) \cup Q\right] \to 0 \quad \text{as } j \to \infty.$$

The quantity

$$P(A) = \inf\ \liminf_{j\to\infty} \mathcal{H}^{n-1}(\partial D_j)$$

is called the *perimeter* of A. Here the infimum is taken over all sets $D_j \subset \mathbf{R}^n$ such that the D_j converges to A and each ∂D_j is either smooth or polyhedral.

With a view to explain a geometric meaning of the perimeter $P(A)$, we introduce the following definition. Let $B(x,\varepsilon) \subset \mathbf{R}^n$ be a ball with center at $x \in A$ and radius $\varepsilon > 0$. For an arbitrary unit vector ν we set

$$\begin{aligned} B_\nu^-(x,\varepsilon) &= B(x,\varepsilon) \cap \{y \in \mathbf{R}^n : \langle \nu, y - x\rangle < 0\}, \\ B_\nu^+(x,\varepsilon) &= B(x,\varepsilon) \setminus B_\nu^-(x,\varepsilon). \end{aligned}$$

An unit vector $\nu(x)$ is said to be *approximate normal* to $A \subset \mathbf{R}^n$ at a point $x \in A$ if

$$\lim_{\varepsilon\to 0} \varepsilon^{-n}\mathcal{H}^n\left(A \cap B_\nu^-(x,\varepsilon)\right) = \frac{1}{2}\, v_n, \quad \lim_{\varepsilon\to 0} \varepsilon^{-n}\mathcal{H}^n\left(A \cap B_\nu^+(x,\varepsilon)\right) = 0,$$

where $v_n = \mathcal{H}^n\left(B(0,1)\right)$. If this vector does not exist, then we set $\nu(x) = 0$.

The set $\partial^* A = \{x \in \mathbf{R}^n : |\nu(x)| = 1\}$ is called the *reduced boundary* of A. Clearly, $\partial^* A \subset \partial A$.

6.5. Theorem. *Assume A has finite perimeter in $\mathbf{R}^n$. Then*

$$\partial^* A = \cup_{k=1}^{\infty} K_k \,\cup\, N,$$

where

$$\mathcal{H}^{n-1}(N) = 0$$

K_k is a compact subset of a C^1-hypersurface S_k and the approximate normal $\nu(x)|_{S_k}$ is normal to S_k $(k = 1, 2 \ldots)$.

See proof in [6, Section **5.7.3**]

The following useful statement follows from the results of De Giorgi [5] and Federer [8].

6.6. Theorem. *For an arbitrary set $A \subset \mathbf{R}^n$,*

$$P(A) = \mathcal{H}^{n-1}(\partial^* A) \le \mathcal{H}^{n-1}(\partial A).$$

Our nearest aim is the Gauss-Green theorem for Lipschitz functions. We are going to obtain this theorem as a corollary of the fundamental Federer result [9, Theorem **4.5.6**].

We need an *integral-geometric measure* $\mathcal{J}_1^{n-1}$ (called sometimes Favard's measure). In this descripcion we will follow Hajlasz [11].

Let $E \subset \mathbf{R}^n$ be a Borel set. We set

$$(6.7)\quad \mathcal{J}_1^{n-1}(E) = \frac{1}{\beta(n,n-1)} \int\limits_{p\in O^*(n,n-1)} \int\limits_{y\in \operatorname{Im} p} N_p(y,E)\, d\mathcal{H}^{n-1}(y)\, d\theta^*_{n,n-1}(p),$$

where $O^*(n,n-1)$ denotes the space of orthogonal projections p from $\mathbf{R}^n$ onto $(n-1)$-dimensional linear subspaces, $\operatorname{Im} p$ is the image of the projection p and $\theta^*_{n,n-1}$ is the Haar measure on $O^*(n,n-1)$ invariant under the action of $O(n)$, normalized to have total mass 1. By $N_p(y,E)$ we denote the *Banach indicatrix*[1].

Roughly speaking $\mathcal{J}^{n-1}$ is defined as follows. Fix an $(n-1)$-dimensional linear subspace of $\mathbf{R}^n$ and denote by p the orthogonal projection from $\mathbf{R}^n$ onto that subspace. We compute the measure of the image of the projection of the set E taking into account the multiplicity function N_p, and then we average the resulting integral over all $(n-1)$-dimensional linear subspaces of $\mathbf{R}^n$.

We still need to define the coefficient $\beta(n,n-1)$. If $E = Q^{n-1}$ is an $(n-1)$-dimensional cube in $\mathbf{R}^n$, then the double integral, which stands on the right hand side of (6.7) is finite and positive. The coefficient $\beta(n,n-1)$ is defined in such a way that $\mathcal{J}_1^{n-1}(Q^{n-1}) = \mathcal{H}^{n-1}(Q^{n-1})$.

We extend the definition of $\mathcal{J}^{n-1}$ to all sets $A \subset \mathbf{R}^n$ by the formula

$$\mathcal{J}_1^{n-1}(A) = \inf\left\{\mathcal{J}_1^{n-1}(E) : A \subset E,\ E\, a\, \text{Borel set}\right\}.$$

It follows easily from the definition that $\mathcal{J}_1^m$ coincides with $\mathcal{H}^m$ on polyhedral sets. The following theorem says that both measures coincide on all regular sets. This is Crofton's theorem (Federer [7, Theorem **5.14**], Ivanov [13, Theorem 3, Chapter III] or Hajlasz [11, Theorem 7]).

6.8. Theorem. *If $A \subset \mathbf{R}^n$ is countably $(n-1)$-rectifiable, then*

$$(6.9)\qquad \mathcal{J}_1^{n-1}(A) = \mathcal{H}^{n-1}(A).$$

Now we can formulate the Gauss–Green formula used in this paper.

6.10. Theorem. *Let $D \subset \mathbf{R}^n$ be a bounded domain with the $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary ∂D. Then for an arbitrary Lipschitz map $\xi : \overline{D} \to \mathbf{R}^n$,*

$$(6.11)\qquad \int\limits_D \operatorname{div}\xi\, dx^1 \cdots dx^n = \int\limits_{\partial^* D} \langle \xi, \mathbf{n}\rangle\, d\mathcal{H}^{n-1}.$$

[1]The Banach indicatrix defined as a number of points in the set $p^{-1}(y) \cap E$

Proof. In fact, this statement is a special case of [9, Theorem **4.5.6**]. To see this note that the condition on the rectifiability of ∂D implies that ∂D is countably $\mathcal{H}^{n-1}$-rectifiable.

Next by Theorem 6.8 we see that the integral-geometric measure $\mathcal{J}_1^{n-1}(\partial D)$ equals the Hausdorff measure $\mathcal{H}^{n-1}(\partial D)$ and $(\mathcal{H}^{n-1}, n-1)$-rectifiability of ∂D implies $\mathcal{J}_1^{n-1}(\partial D) < \infty$.

The proof then follows from the first remark of Section **4.5.12** and the fundamental Theorem **4.5.6** of [9]. $\square$

O. Martio and M. Vuorinen:
Department of Mathematics
P.O.Box 4 (Yliopistonkatu 5)
FIN-00014 University of Helsinki, FINLAND
Fax: 3589191 22879
Email: `martio@cc.helsinki.fi` and `vuorinen@csc.fi`

V.M. Miklyukov
Tulaka 2-120, 400119
Volgograd, RUSSIA
Fax: (7-8442)471608
Email: `miklyuk@vlink.ru`

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