LARGE DEVIATIONS IN TESTING OF MODELS WITH FRACTIONAL BROWNIAN MOTION

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Abstract. We consider a problem of distinguishing of two models of processes from financial mathematics containing a fractional Brownian motion. Large deviation theorems for logarithm of likelihood ratio in these models are obtained. By these limit theorems the rate of decrease of error probabilities is obtained for Neyman-Pearson test, Bayes test and minimax test.

1. Introduction

A *fractional Brownian motion* (fBm) $Z = (Z_t)_{t\geq 0}$ is a continuous Gaussian process with stationary increments on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the properties

1. $Z_0 = 0$,
2. $\mathbf{E}Z_t = 0 \; \forall \, t > 0$,
3. $\mathbf{E}Z_s Z_t = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}) \; \forall \, s,t \geq 0$.

Here $H \in (0,1)$ is a *Hurst index* (more details about fBm see, for example, Norros et al. (1999)).

In financial mathematics the following models of observations are considered

$$\xi_s = \sigma Z_s + \mu s, \; 0 \leq s \leq t, \tag{A}$$

$$\xi_s = \sigma Z_s + \mu s - \frac{\sigma^2}{2} s^{2H}, \; 0 \leq s \leq t, \tag{B}$$

where $\sigma > 0$ and $\mu \in (-\infty, \infty)$ are constants. Bellow we assume that $H \in (1/2, 1)$. About models (A) and (B) see Kukush et al. (2001), where some statistical problems are considered for these models.

By $\mathbf{P}^t_{\mu,\sigma}$ we denote a distribution of observation $\xi^t = (\xi_s)_{0\leq s\leq t}$ of process of the form (A), and by $\mathbf{P}^t_{\mu,\sigma,\sigma}$ we denote a distribution of observation ξ^t of process of the form (B). The distributions $\mathbf{P}^t_{\mu,\sigma}$ and $\mathbf{P}^t_{\mu,\sigma,\sigma}$ are the measures in the space $C^t = C[0,t]$ of continuous functions. We introduce the simple hypotheses $H_{\mu,\sigma}$ and $H_{\mu,\sigma,\sigma}$ under which a distribution of observation ξ^t is $\mathbf{P}^t_{\mu,\sigma}$ and $\mathbf{P}^t_{\mu,\sigma,\sigma}$ respectively.

We consider the problems of testing hypothesis $H_{\mu,\sigma}$ under observations ξ^t when an alternative is either the simple hypothesis $H_{\widetilde{\mu},\widetilde{\sigma}}$ or the simple hypothesis $H_{\widetilde{\mu},\widetilde{\sigma},\widetilde{\sigma}}$. Also, we consider the problems of testing hypothesis $H_{\mu,\sigma,\sigma}$ under observations ξ^t when an alternative is either the simple hypothesis $H_{\widetilde{\mu},\widetilde{\sigma}}$ or the simple hypothesis $H_{\widetilde{\mu},\widetilde{\sigma},\widetilde{\sigma}}$. Here, in general, $\widetilde{\mu} \neq \mu$ and $\widetilde{\sigma} \neq \sigma$.

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We consider the cases $\widetilde{\sigma} = \sigma$ and $\widetilde{\sigma} \neq \sigma$ separately. If $\widetilde{\sigma} = \sigma$ then we use the observations ξ^t and assume that $t \to \infty$. In this case we prove large deviation theorems for logarithm of corresponding likelihood ratio and apply these theorems to investigation of error probabilities of Neyman-Pearson tests, Bayes tests and minimax tests.

If $\widetilde{\sigma} \neq \sigma$ then we use the observation $\xi^n = (\xi_{t_1}, \xi_{t_2}, \dots, \xi_{t_n})$ where $0 = t_0 < t_1 < t_2 < \dots < t_n = t$ and assume that $n \to \infty$ and $\max_i (t_{i+1} - t_i) \to 0$ as $n \to \infty$. In this case we prove also large deviation theorems for logarithm of corresponding likelihood ratio and apply these theorems to investigation of error probabilities of Neyman-Pearson tests, Bayes tests and minimax tests, too.

The paper is organized as follows. In Sections 2 and 3 we give well-known results about large deviation theorems in hypotheses testing problem for general binary statistical experiments. Section 2 is devoted to large deviation theorems for the logarithm of likelihood ratio in the scheme of general binary statistical experiments. In Section 3 limit theorems from Section 2 are applied to investigation of rate of decrease of error probabilities for Neyman-Pearson tests, Bayes tests and minimax tests in the scheme of general binary statistical experiments. Sections 4 and 5 contain main results of this paper. In Section 4 and 5 we apply limit theorems from Sections 2 and 3 to models (A) and (B) in the case $\widetilde{\sigma} = \sigma$ (Section 4) and in the case $\widetilde{\sigma} \neq \sigma$ (Section 5).

2. General large deviation theorems

Let $(X^t, \mathcal{B}^t, \mathbf{P}^t, \widetilde{\mathbf{P}}^t)$, $t \in R_+$, be a family of general binary statistical experiments generated by some general observation ξ^t. By H^t and $\widetilde{H}^t$ we denote two statistical hypotheses according to which the distribution of the observation ξ^t is defined by the measures $\mathbf{P}^t$ and $\widetilde{\mathbf{P}}^t$, respectively.

Let $\mathbf{Q}^t$ is a measure on $(X^t, \mathcal{B}^t)$ such that $\mathbf{P}^t \ll \mathbf{Q}^t$ and $\widetilde{\mathbf{P}}^t \ll \mathbf{Q}^t$ and let $\mathfrak{z}_t = d\mathbf{P}^t / d\mathbf{Q}^t$ and $\tilde{\mathfrak{z}}_t = d\widetilde{\mathbf{P}}^t / d\mathbf{Q}^t$ be Radon-Nikodym derivatives of the measures $\mathbf{P}^t$ and $\widetilde{\mathbf{P}}^t$ with respect to $\mathbf{Q}^t$.

We introduce the *Hellinger integral* of order $\varepsilon \in (-\infty, \infty)$ by the formula

$$H_t(\varepsilon) = H(\varepsilon; \widetilde{\mathbf{P}}^t, \mathbf{P}^t) = \mathbf{E}^t_Q \tilde{\mathfrak{z}}_t^{\varepsilon} \mathfrak{z}_t^{1-\varepsilon}$$

where $\mathbf{E}^t_Q$ is an expectation with respect to $\mathbf{Q}^t$ (Jacod and Shiryaev (1987), Lin'kov (1993a)).

We introduce the *likelihood ratio* $z_t = \tilde{\mathfrak{z}}_t / \mathfrak{z}_t$ setting for definiteness $0/0 = 0$. Let $\Lambda_t = \ln z_t$ with $\ln 0 = -\infty$.

Obviously, Λ_t is an extended random variable and

$$\mathbf{P}^t(\Lambda_t = -\infty) = \mathbf{P}^t(\tilde{\mathfrak{z}}_t = 0) \geq 0, \quad \mathbf{P}^t(\Lambda_t = +\infty) = 0,$$
$$\widetilde{\mathbf{P}}^t(\Lambda_t = +\infty) = \widetilde{\mathbf{P}}^t(\mathfrak{z}_t = 0) \geq 0, \quad \widetilde{\mathbf{P}}^t(\Lambda_t = -\infty) = 0.$$

It is easy to show that

$$H_t(\varepsilon) = \mathbf{E}^t \exp(\varepsilon \Lambda_t)$$

where $\mathbf{E}^t$ is an expectation with respect to the measure $\mathbf{P}^t$ with appropriate definition this expectation in the case $\varepsilon = 0$, i. e. the Hellinger integral $H_t(\varepsilon)$ is a moment generating function of the extended random variable Λ_t (see Lin'kov (1999a,b)).

We introduce the following condition for Hellinger integral

($\mathbf{H}$) *for all* $\varepsilon \in (-\infty, \infty)$ *there exists the limit*

$$\lim_{t\to\infty} \psi_t^{-1} \ln H_t(\varepsilon) = \varkappa(\varepsilon), \tag{2.1}$$

where $\psi_t \to \infty$ *as* $t \to \infty$, *the function* $\varkappa(\varepsilon)$ *is proper convex and differentiable on* $(\varepsilon_-, \varepsilon_+)$ *with*

$$\gamma_- = \lim_{\varepsilon\downarrow\varepsilon_-} \varkappa'(\varepsilon) < \gamma_+ = \lim_{\varepsilon\uparrow\varepsilon_+} \varkappa'(\varepsilon),$$

where

$$\varepsilon_- = \inf\{\varepsilon : \varkappa(\varepsilon) < \infty\}, \qquad \varepsilon_+ = \sup\{\varepsilon : \varkappa(\varepsilon) < \infty\}.$$

Obviously, $\varepsilon_- \le 0$ and $\varepsilon_+ \ge 1$. If $\varepsilon_- < 0$ then the derivative $\varkappa'(0) = \gamma_0$ is well defined. If $\varepsilon_+ > 1$ then the derivative $\varkappa'(1) = \gamma_1$ is well defined, too.

We denote by $I(\gamma)$ a *Legendre-Fenchel transform* of the function $\varkappa(\varepsilon)$, i. e.

$$I(\gamma) = \sup_{\varepsilon}(\varepsilon\gamma - \varkappa(\varepsilon))$$

(about the properties of Legendre-Fenchel transform see Rockafellar (1970)).

In addition, we introduce the quantities

$$\Gamma_0 = \gamma_0 I(\varepsilon_- < 0) + \gamma_- I(\varepsilon_- = 0),$$
$$\Gamma_1 = \gamma_1 I(\varepsilon_+ > 1) + \gamma_+ I(\varepsilon_+ = 1).$$

The following theorem is a large deviation theorem of Chernoff type for Λ_t with respect to $\mathbf{P}^t$.

Theorem 2.1. *Let condition* ($\mathbf{H}$) *be satisfied. Then the following conclusions hold:*

(1) *if* $\Gamma_0 < \gamma_+$ *then for all* $\gamma \in (\Gamma_0, \gamma_+)$

$$\begin{aligned}\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t(\psi_t^{-1}\Lambda_t > \gamma) &= \\ = \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t(\psi_t^{-1}\Lambda_t \ge \gamma) &= -I(\gamma);\end{aligned} \tag{2.2}$$

(2) *if* $\varepsilon_- < 0$ *and* $\gamma_- < \gamma_0$ *then for all* $\gamma \in (\gamma_-, \gamma_0)$

$$\begin{aligned}\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t(\psi_t^{-1}\Lambda_t < \gamma) &= \\ = \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t(\psi_t^{-1}\Lambda_t \le \gamma) &= -I(\gamma).\end{aligned} \tag{2.3}$$

Using the relation $H(\varepsilon; \mathbf{P}^t, \widetilde{\mathbf{P}}^t) = H(1-\varepsilon; \widetilde{\mathbf{P}}^t, \mathbf{P}^t)$ by analogy with Theorem 2.1 we get the following large deviation theorem of Chernoff type for Λ_t with respect to $\widetilde{\mathbf{P}}^t$.

Theorem 2.2. *Let condition* (**H**) *be satisfied. Then the following statements hold*:

(1) *if* $\gamma_- < \Gamma_1$ *then for all* $\gamma \in (\gamma_-, \Gamma_1)$

$$\begin{aligned} &\lim_{t\to\infty} \psi_t^{-1} \ln \widetilde{\mathbf{P}}^t(\psi_t^{-1}\Lambda_t < \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \widetilde{\mathbf{P}}^t(\psi_t^{-1}\Lambda_t \le \gamma) = \gamma - I(\gamma); \end{aligned} \tag{2.4}$$

(2) *if* $\varepsilon_+ > 1$ *and* $\gamma_+ > \gamma_1$ *then for all* $\gamma \in (\gamma_1, \gamma_+)$

$$\begin{aligned} &\lim_{t\to\infty} \psi_t^{-1} \ln \widetilde{\mathbf{P}}^t(\psi_t^{-1}\Lambda_t > \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \widetilde{\mathbf{P}}^t(\psi_t^{-1}\Lambda_t \ge \gamma) = \gamma - I(\gamma). \end{aligned} \tag{2.5}$$

Theorems 2.1 and 2.2 are proved using large deviation theorems for extended random variables which are proved by Lin'kov (1999a,b).

Remark 2.1. In Lin'kov (1993b, 1994, 1995a,b), Linkov and Medvedeva (1995a,b), the statements of Theorems 2.1 and 2.2 have been proved without using large deviation theorems for extended random variables, but under more restrictive conditions than condition (**H**).

3. The rate of decrease of error probabilities for some tests

Theorems 2.1 and 2.2 permit to investigate the rate of decrease of error probabilities for some statistical tests. In this section we give theorems about behavior of error probabilities for Neyman-Pearson tests, Bayes tests and minimax tests. The proof of these theorems can be found in Lin'kov (1997a,b; 1999a,b). These theorems will be applied to models (A) and (B) in Sections 4 and 5.

Neyman-Pearson tests. Let δ_t be a *Neyman-Pearson test* of level $\alpha_t \in (0,1)$ for testing hypotheses H^t and $\widetilde{H}^t$ under the observation ξ^t. We denote by β_t a probability of 2nd type error for the test δ_t. The following theorem describes the rate of decrease of the error probabilities α_t and β_t for the test δ_t under condition (**H**).

Theorem 3.1. *Let condition* (**H**) *be satisfied with* $\Gamma_0 < \Gamma_1$. *Then the following statements hold*:

(1) *for all* $a \in (I(\Gamma_0), I(\Gamma_1))$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = -a \Leftrightarrow \lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = -b(a), \tag{3.1}$$

where

$$b(a) = a - \gamma(a) \in (I(\Gamma_1) - \Gamma_1, I(\Gamma_0) - \Gamma_0)$$

and $\gamma(a)$ *is a unique solution of the equation* $I(\gamma) = a$ *with respect to* $\gamma \in (\Gamma_0, \Gamma_1)$;

(2) *for all* $a \in [0, I(\Gamma_0)]$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = -a \Rightarrow \limsup_{t\to\infty} \psi_t^{-1} \ln \beta_t \le \Gamma_0 - I(\Gamma_0),$$

and for all $a \in [I(\Gamma_1), \infty]$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = -a \Rightarrow \liminf_{t\to\infty} \psi_t^{-1} \ln \beta_t \geq \Gamma_1 - I(\Gamma_1);$$

(3) *for all* $b \in [0, I(\Gamma_1) - \Gamma_1]$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = -b \Rightarrow \limsup_{t\to\infty} \psi_t^{-1} \ln \alpha_t \leq -I(\Gamma_1),$$

and for all $b \in [I(\Gamma_0) - \Gamma_0, \infty]$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = -b \Rightarrow \liminf_{t\to\infty} \psi_t^{-1} \ln \alpha_t \geq -I(\Gamma_0).$$

Remark 3.1. Theorem 3.1 under more restrictive conditions have been proved in Lin'kov (1993b, 1995b), Lin'kov and Medvedeva (1995a,b) for general statistical experiments. First, the implication $\Rightarrow$ in (3.1) have been proved in Birgé (1981) for scheme observations of i. i. d. random variables.

Bayes tests. Let δ_t^π be a *Bayes test* for testing hypotheses H^t and $\widetilde{H}^t$, where π and $\widetilde{\pi}$ are the priori probabilities of the hypotheses H^t and $\widetilde{H}^t$ and $\pi + \widetilde{\pi} = 1$. Denote by $\alpha(\delta_t^\pi)$, $\beta(\delta_t^\pi)$ and $e(\delta_t^\pi)$ the error probabilities of 1st and 2nd type and the risk of the test δ_t^π.

Theorem 3.2. *Let condition* (**H**) *be satisfied with* $\Gamma_0 < 0 < \Gamma_1$ *and let* π *be independent on* t. *Then the following relations hold*:

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \ln \beta(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \ln e(\delta_t^\pi) = -I(0). \tag{3.2}$$

Remark 3.2. Under other conditions, the last equality in (3.2) have been proved in Vajda (1990). For i. i. d. random variables, Theorem 3.2 was proved by Chernoff (1952).

Remark 3.3. The statements of Theorem 3.2 remain true also in the case where a priori distribution $(\pi_t, \widetilde{\pi}_t)$ depends on t so that

$$\psi_t^{-1} \ln(\pi_t/\widetilde{\pi}_t) \to 0 \text{ as } t \to \infty.$$

It is easy to deduce the corresponding result when

$$\psi_t^{-1} \ln(\pi_t/\widetilde{\pi}_t) \to c \in (\Gamma_0, \Gamma_1) \text{ as } t \to \infty.$$

Minimax test. Let δ_t^* be a *minimax test* for testing hypotheses H^t and $\widetilde{H}^t$ under the observation ξ^t. We denote by $\alpha(\delta_t^*)$, $\beta(\delta_t^*)$ and e_t^* the error probabilities of 1st and 2nd type and the minimax risk for the test δ_t^*.

Theorem 3.3. *If condition* (**H**) *is satisfied with* $\Gamma_0 < 0 < \Gamma_1$, *then*

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \ln \beta(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \ln e_t^* = -I(0).$$

4. The models (A) and (B), the case $\sigma = \widetilde{\sigma}$

In this section we consider the statistical experiments $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, generated by the observations ξ^t of process ξ from models (A) and (B). To apply the results of Sections 2 and 3 we need Girsanov theorem for fBm.

Let $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion with respect to $(\mathbf{F}, \mathbf{P})$.

Introduce the following constants

$$\begin{gathered} B_1 = B\left(3/2 - H,\ 3/2 - H\right), \quad B_2 = B\left(H + 1/2,\ 3/2 - H\right), \\ B_3 = B\left(H - 1/2, 3/2 - H\right), \quad B(\mu,\nu) = \Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu), \\ C_0 = 2^{-1}\left(\left(H - 1/2\right) H(1-H) B_1 B_3\right)^{-1/2}. \end{gathered}$$

Here $\Gamma(\mu)$ and $B(\mu,\nu)$ are a gamma-function and beta-function respectively. Consider the kernel $z(t,s) = C_0 s^{1/2-H}(t-s)^{1/2-H}$ for $s \in (0,t)$ and $z(t,s) = 0$ otherwise. Let φ be a $\mathbf{F}$-adapted process and

$$\int_0^t z(t,s)|\varphi_s|\, ds < \infty \quad (\mathbf{P}\text{-a. s.})\ \forall t > 0 \tag{4.1}$$

We assume that

$$\int_0^t z(t,s)\varphi_s\, ds = \int_0^t \delta_s\, ds, \quad \forall t > 0 \tag{4.2}$$

where δ is a $\mathbf{F}$-adapted process satisfying

$$\int_0^t |\delta_s|\, ds < \infty \quad (\mathbf{P}\text{-a. s.})\ \forall t > 0. \tag{4.3}$$

¿From Norros et al. (1999) we have such result

$$\int_0^t z(t,s)\, dZ_s = \int_0^t s^{1/2-H} dW_s, \tag{4.4}$$

where the left integral exists as a pathwise integral w.r.t. fBm Z.

Moreover, we assume that

$$\mathbf{E}\int_0^t s^{2H-1}\delta_s^2 < \infty\ \forall t > 0. \tag{4.5}$$

Then the following process

$$L_t = \int_0^t s^{H-1/2}\delta_s\, dW_s,\ t > 0, \tag{4.6}$$

is a square integrable martingale w.r.t. $(\mathbf{F}, \mathbf{P})$.

The following lemma is a Girsanov theorem for fBm (for the proof of this theorem see Kukush et al. (2001)).

Lemma 4.1. *Let the conditions* (4.1)-(4.3) *and* (4.5) *be satisfied and we assume that*

$$\mathbf{E}\exp(L_t - 2^{-1}\langle L\rangle_t) = 1 \ \forall t > 0. \tag{4.7}$$

Then the process $Z_t - \int_0^t \varphi_s\, ds$ *is a fBm w.r.t. the measure* $\mathbf{Q}$ *which is defined by*

$$\frac{d\mathbf{Q}^t}{d\mathbf{P}^t} = \exp(L_t - 2^{-1}\langle L\rangle_t), \tag{4.8}$$

where $\mathbf{Q}^t$ *and* $\mathbf{P}^t$ *are restrictions of* $\mathbf{Q}$ *and* $\mathbf{P}$ *to* σ*-field* $\mathcal{F}_t$*.*

¿From Lemma 4.1 it follows that $\mathbf{P}^t_{\mu,\sigma} \ll \mathbf{P}^t_{0,\sigma}$ and $\mathbf{P}^t_{\mu,\sigma,\sigma} \ll \mathbf{P}^t_{0,\sigma}$. And we denote

$$\mathfrak{z}_t = \frac{d\mathbf{P}^t_{\mu,\sigma}}{d\mathbf{P}^t_{0,\sigma}}, \quad \tilde{\mathfrak{z}}_t = \frac{d\mathbf{P}^t_{\mu,\sigma,\sigma}}{d\mathbf{P}^t_{0,\sigma}}. \tag{4.9}$$

The following lemma gives us the representations for densities $\mathfrak{z}_t$ and $\tilde{\mathfrak{z}}_t$ which will be used as follows.

Lemma 4.2. *For the densities* $\mathfrak{z}_t$ *and* $\tilde{\mathfrak{z}}_t$ *defined by (4.9) the following representations hold*

$$\mathfrak{z}_t = \exp(v_t\eta_t - 2^{-1}v_t^2), \tag{4.10}$$

$$\tilde{\mathfrak{z}}_t = \exp(\widetilde{v}_t\widetilde{\eta}_t - 2^{-1}\widetilde{v}_t^2), \tag{4.11}$$

where

$$\mathcal{L}(\eta_t|\mathbf{P}^t_{0,\sigma}) = \mathcal{L}(\widetilde{\eta}_t|\mathbf{P}^t_{0,\sigma}) = \mathcal{N}(0,1), \tag{4.12}$$

$$v_t^2 = 2(1-H)C_0^2\, B_1^2\, \frac{\mu^2}{\sigma^2}\, t^{2(1-H)}, \tag{4.13}$$

(4.14)

$$\widetilde{v}_t^2 = \frac{1}{2}\,\sigma^2\, H\, C_0^2\, B_2^2\, t^{2H} - 4\mu H(1-H)C_0^2B_1B_2t + 2(1-H)C_0^2B_1^2\left(\frac{\mu}{\sigma}\right)^2 t^{2(1-H)}.$$

Here $\mathcal{L}(\eta_t|\mathbf{P}^t_{0,\sigma})$ is a distribution law of η_t w.r.t. $\mathbf{P}^t_{0,\sigma}$ and $\mathcal{N}(0,1)$ is a standard normal law.

Proof. We will prove (4.10), (4.12) for $\mathcal{L}(\eta_t|\mathbf{P}^t_{0,\sigma})$ and (4.13). For this we apply Lemma 4.1 to the process $\sigma^{-1}X$ setting $\mathbf{P}^t = \mathbf{P}^t_{0,\sigma}$ and $\mathbf{Q}^t = \mathbf{P}^t_{\mu,\sigma,\sigma}$. Then $\int_0^t \varphi_s ds = \mu\sigma^{-1}t$. Obviously, the conditions (4.1)-(4.3) are satisfied with $\delta_s = \mu\sigma^{-1}(2-2H)C_0B_1s^{1-2H}$. From Lemma 4.1 we obtain the equality (4.10) where $\eta_t = L_t/v_t$ and

$$v_t^2 = \langle L\rangle_t = 2(1-H)C_0^2B_1^2\mu^2\sigma^{-2}t^{2(1-H)}.$$

Note that $\mathcal{L}(L_t|\mathbf{P}^t_{0,\sigma}) = \mathcal{N}(0,v_t^2)$. Thus, the relations (4.10), (4.12) for $\mathcal{L}(\eta_t|\mathbf{P}^t_{0,\sigma})$ and (4.13) are proved. The relations (4.11), (4.12) for $\mathcal{L}(\widetilde{\eta}_t|\mathbf{P}^t_{0,\sigma})$ and (4.14) can be proved by analogy with above using the proof of Theorem 3.1 from Kukush et al. (2001). $\square$

Now from Lemma 4.2 we deduce the following lemma which establishes a validity of condition **(H)** for statistical experiments $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$.

Lemma 4.3. *For statistical experiments* $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, *the condition* **(H)** *is satisfied with*

$$\psi_t = \frac{1}{2}\sigma^2 H C_0^2 B_2^2 t^{2H}, \qquad \varkappa(\varepsilon) = -\frac{\varepsilon(1-\varepsilon)}{2}, \tag{4.15}$$

$$\varepsilon_- = -\infty,\ \varepsilon_+ = \infty, \quad \gamma_- = -\infty,\ \gamma_+ = \infty. \tag{4.16}$$

Proof. ¿From definition of Hellinger integral $H_t(\varepsilon)$, using (4.10)-(4.12), we have for all $\varepsilon \in (-\infty, \infty)$

$$H_t(\varepsilon) = H(\varepsilon; \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma}) = \exp\left\{-\frac{\varepsilon(1-\varepsilon)}{2}(\widetilde{v}_t - v_t)^2\right\}. \tag{4.17}$$

But from (4.13) and (4.14) we get

$$(\widetilde{v}_t - v_t)^2 = 2^{-1}\sigma^2 H C_0^2 B_2^2 t^{2H}(1 + o(1)),\ t \to \infty. \tag{4.18}$$

The equalities (4.17) and (4.18) give us that the condition **(H)** is satisfied and the inequalities (4.15) and (4.16) are valid. □

¿From Lemma 4.3 we have

$$\Gamma_0 = \gamma_0 = -1/2, \qquad \Gamma_1 = \gamma_1 = 1/2, \qquad I(\gamma) = 2^{-1}(\gamma + 2^{-1})^2.$$

¿From here using Lemma 4.3 and Theorems 2.1 and 2.2 we obtain the two following large deviation theorems for $\Lambda_t = \ln \tilde{\mathfrak{z}}_t / \mathfrak{z}_t$ as $t \to \infty$.

Theorem 4.1. *For statistical experiments* $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, *the following statements hold with* $\psi_t = 2^{-1}\sigma^2 H C_0^2 B_2^2 t^{2H}$:

(1) *for all* $\gamma \in (-2^{-1}, \infty)$

$$\begin{aligned}&\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma}(\psi_t^{-1}\Lambda_t > \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma}(\psi_t^{-1}\Lambda_t \geq \gamma) = -2^{-1}(\gamma + 2^{-1})^2;\end{aligned}$$

(2) *for all* $\gamma \in (-\infty, -2^{-1})$

$$\begin{aligned}&\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma}(\psi_t^{-1}\Lambda_t < \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma}(\psi_t^{-1}\Lambda_t \leq \gamma) = -2^{-1}(\gamma + 2^{-1})^2.\end{aligned}$$

Theorem 4.2. *For statistical experiments* $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, *the following statements hold with* $\psi_t = 2^{-1}\sigma^2 H C_0^2 B_2^2 t^{2H}$:

(1) *for all* $\gamma \in (-\infty, 2^{-1})$

$$\begin{aligned}&\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma,\sigma}(\psi_t^{-1}\Lambda_t < \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma,\sigma}(\psi_t^{-1}\Lambda_t \leq \gamma) = -2^{-1}(\gamma - 2^{-1})^2;\end{aligned}$$

(2) *for all* $\gamma \in (2^{-1}, \infty)$

$$\begin{aligned}&\lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma,\sigma}(\psi_t^{-1}\Lambda_t > \gamma) = \\ &= \lim_{t\to\infty} \psi_t^{-1} \ln \mathbf{P}^t_{\mu,\sigma,\sigma}(\psi_t^{-1}\Lambda_t \geq \gamma) = -2^{-1}(\gamma - 2^{-1})^2.\end{aligned}$$

Theorems 4.1 and 4.2 permit us to formulate following result about behavior of error probabilities for tests δ_t, δ_t^{π} and δ_t^* as $t \to \infty$.

Theorem 4.3. *For statistical experiments* $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, *the following statements hold with* $\psi_t = 2^{-1}\sigma^2 H C_0^2 B_2^2 t^{2H}$:

(1) *for all* $a \in (0, 1/2)$

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = -a \Longleftrightarrow \lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = -\frac{(1-\sqrt{2a})^2}{2}; \tag{4.19}$$

(2) *if* π *is independent on* t, *then*

$$\lim_{t\to\infty} \frac{\ln \alpha(\delta_t^\pi)}{\psi_t} = \lim_{t\to\infty} \frac{\ln \beta(\delta_t^\pi)}{\psi_t} = \lim_{t\to\infty} \frac{\ln e(\delta_t^\pi)}{\psi_t} = -\frac{1}{8}; \tag{4.20}$$

(3) *we have*

$$\lim_{t\to\infty} \frac{\ln \alpha(\delta_t^*)}{\psi_t} = \lim_{t\to\infty} \frac{\ln \beta(\delta_t^*)}{\psi_t} = \lim_{t\to\infty} \frac{\ln e_t^*}{\psi_t} = -\frac{1}{8}. \tag{4.21}$$

Proof. The relations (4.20) and (4.21) follow from Theorems 3.2 and 3.3 since $\Gamma_0 = -2^{-1} < 0 < 2^{-1} = \Gamma_1$ and $I(0) = 1/8$. To prove the statement (1) we apply the statement (1) from Theorem 3.1. Note that $I(\Gamma_0) = 0$, $I(\Gamma_1) = 2^{-1}$. Moreover, a solution of the equation $I(\gamma) = a$ with respect to $\gamma \in (-2^{-1}, 2^{-1})$ for $a \in (0, 2^{-1})$ has the form $\gamma(a) = \sqrt{2a} - 2^{-1}$. From here using (3.1) we obtain the relation (4.19). $\square$

Remark 4.1. In statement (2) of Theorem 3.1 applied to the statistical experiments $(C^t, \mathcal{B}^t, \mathbf{P}^t_{\mu,\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$, $t \in R_+$, we have $I(\Gamma_0) = 0$ and $\Gamma_0 = -2^{-1}$. Then we get

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = 0 \implies \limsup_{t\to\infty} \psi_t^{-1} \ln \beta_t \le -2^{-1}.$$

But, if we use Theorem 2.3.3 from Lin'kov (1993a) we obtain more exact result

$$\lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = \lim_{t\to\infty} \psi_t^{-1} \ln(1-\alpha_t) = 0 \implies \lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = -2^{-1}.$$

By analogy, in statement (3) of Theorem 3.1 we get more exact result

$$\lim_{t\to\infty} \psi_t^{-1} \ln \beta_t = \lim_{t\to\infty} \psi_t^{-1} \ln(1-\beta_t) = 0 \implies \lim_{t\to\infty} \psi_t^{-1} \ln \alpha_t = -2^{-1}.$$

Using Lemma 4.2 we can obtain more strong results than statements (2) and (3) in Theorem 3.1.

Remark 4.2. Lemma 4.1 permits us to consider the following statistical experiments

$$(C^t, \mathcal{B}^t, \mathbf{P}^t_{\widetilde{\mu},\sigma}, \mathbf{P}^t_{\mu,\sigma}), \quad (C^t, \mathcal{B}^t, \mathbf{P}^t_{\widetilde{\mu},\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma,\sigma}), \quad (C^t, \mathcal{B}^t, \mathbf{P}^t_{\widetilde{\mu},\sigma,\sigma}, \mathbf{P}^t_{\mu,\sigma})$$

where $\widetilde{\mu} \ne \mu$. From Lemma 4.1, it is easy to get that for these statistical experiments the condition **(H)** is satisfied with the same function $\varkappa(\varepsilon) = -2^{-1}\varepsilon(1-\varepsilon)$ and the normalizing functions

$$\psi_t = 2(1-H)C_0^2 B_1^2 (\widetilde{\mu}-\mu)^2 \sigma^{-2} t^{2(1-H)},$$
$$\psi_t = 8H(1-H)^2 C_0^2 B_1^2 (\widetilde{\mu}-\mu)^2 \sigma^{-2} t^{2(1-H)}, \quad \psi_t = 2^{-1}\sigma^2 C_0^2 B_2^2 t^{2H}$$

respectively. Thus, we can formulate the corresponding statements of Theorems 4.1-4.3 for these statistical experiments, too.

5. The models (A) and (B), the case $\sigma \neq \widetilde{\sigma}$

In this section we consider the models (A) and (B) for $s \in [0,1]$. The corresponding distributions of observation $(\xi_s)_{0\le s\le 1}$ will be denoted by $\mathbf{P}_{\mu,\sigma}$ for the model (A) and by $\mathbf{P}_{\mu,\sigma,\sigma}$ for the model (B). Thus, the distributions $\mathbf{P}_{\mu,\sigma}$ and $\mathbf{P}_{\mu,\sigma,\sigma}$ are the measures in the space $C = C[0,1]$.

At first we consider the hypotheses $H_{\mu,\sigma}$ and $H_{\widetilde{\mu},\widetilde{\sigma}}$, where $\widetilde{\sigma} \neq \sigma$ and, in general, $\widetilde{\mu} \neq \mu$. Let $\Delta_n = \{t_{n0}, t_{n1}, \dots, t_{nn}\}$ be a partition of the interval $[0,1]$ such that $0 = t_{n0} < t_{n1} < \dots < t_{nn} = 1$, $\Delta_n \subset \Delta_{n+1}$ and $\max\limits_{0\le i\le n-1}(t_{n,i+1} - t_{ni}) \to 0$ as $n \to \infty$. We consider the problems of testing hypotheses $H_{\mu,\sigma}$ and $H_{\mu,\widetilde{\sigma}}$ under observations $\xi^n = (\xi_{t_1}, \xi_{t_2}, \dots, \xi_{t_n})$ as $n \to \infty$. By $\mathbf{P}^n_{\mu,\sigma}$ we denote a distribution of ξ^n under hypothesis $H_{\mu,\sigma}$. Thus, we have the family of binary statistical experiments $(R^n, \mathcal{B}^n, \mathbf{P}^n_{\mu,\sigma}, \mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma}})$, $n = 1, 2, \dots$. Obviously, the measure $\mathbf{P}^n_{\mu,\sigma}$ is absolutely continuous with respect to a Lebesgue measure with density

$$p_n(x) = (2\pi)^{-n/2}(\det A)^{-1/2}\exp\{-\frac{1}{2}(x-\mu\bar{t})'A^{-1}(x-\mu\bar{t})\},\ x \in R^n, \tag{5.1}$$

and the measure $\mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma}}$ is absolutely continuous with respect to a Lebesgue measure with density

$$\widetilde{p}_n(x) = (2\pi)^{-n/2}(\det \widetilde{A})^{-1/2}\exp\{-\frac{1}{2}(x-\widetilde{\mu}\bar{t})'\widetilde{A}^{-1}(x-\widetilde{\mu}\bar{t})\}.\ x \in R^n, \tag{5.2}$$

Here $x = (x_1, \dots, x_n)'$, $\bar{t} = (t_{n1}, \dots, t_{nn})'$, $A = \sigma^2 C_n$, $\widetilde{A} = \widetilde{\sigma}^2 C_n$ where $C_n = \|\mathbf{E} Z_{t_{ni}} Z_{t_{nj}}\|_{1\le i,j\le n}$.

Lemma 5.1. *For statistical experiments $(R^n, \mathcal{B}^n, \mathbf{P}^n_{\mu,\widetilde{\sigma}}, \mathbf{P}^n_{\mu,\sigma})$, $n \ge 1$ the condition* **(H)** *is satisfied with*

$$\psi_n = n, \quad \varkappa(\varepsilon) = -\frac{1}{2}\left\{\ln\left(\left(\frac{\sigma^2}{\widetilde{\sigma}^2} - 1\right)\varepsilon + 1\right) - \varepsilon \ln\frac{\sigma^2}{\widetilde{\sigma}^2}\right\}, \tag{5.3}$$

$$\varepsilon_- = \begin{cases} -\infty,\ \sigma < \widetilde{\sigma}, \\ -\left(\dfrac{\sigma^2}{\widetilde{\sigma}^2} - 1\right)^{-1},\ \sigma > \widetilde{\sigma}, \end{cases} \qquad \varepsilon_+ = \begin{cases} \left(1 - \dfrac{\sigma^2}{\widetilde{\sigma}^2}\right)^{-1},\ \sigma < \widetilde{\sigma}, \\ \infty,\ \sigma > \widetilde{\sigma}, \end{cases} \tag{5.4}$$

$$\gamma_- = \begin{cases} \ln\dfrac{\sigma}{\widetilde{\sigma}},\ \sigma < \widetilde{\sigma}, \\ -\infty,\ \sigma > \widetilde{\sigma}, \end{cases} \qquad \gamma_+ = \begin{cases} \infty,\ \sigma < \widetilde{\sigma}, \\ \ln\dfrac{\sigma}{\widetilde{\sigma}},\ \sigma > \widetilde{\sigma}. \end{cases} \tag{5.5}$$

Proof. Using (5.1) and (5.2) we obtain that

$$\begin{aligned} H(\varepsilon; \mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma}}, \mathbf{P}^n_{\mu,\sigma}) &= \left(\frac{\sigma}{\widetilde{\sigma}}\right)^{n\varepsilon}\left[\left(\frac{\sigma^2}{\widetilde{\sigma}^2} - 1\right)\varepsilon + 1\right]^{-n/2} \times \\ &\times \exp\left\{-\frac{\varepsilon(1-\varepsilon)}{2}\frac{(\widetilde{\mu}-\mu)^2}{\varepsilon\sigma^2 + (1-\varepsilon)\widetilde{\sigma}^2}\bar{t}' C_n^{-1}\bar{t}\right\} \end{aligned} \tag{5.6}$$

for all $\varepsilon \in (\varepsilon_-, \varepsilon_+)$ and $H_n(\varepsilon) = \infty$ for all $\varepsilon \notin (\varepsilon_-, \varepsilon_+)$, where ε_- and ε_+ is defined by (5.4).

Consider the statistical experiment $(C, \mathcal{B}, \mathbf{P}_{\widetilde{\mu},\sigma}, \mathbf{P}_{\mu,\sigma})$. By Lemma 4.2 we get that $0 < H(\varepsilon) = H(\varepsilon; \mathbf{P}_{\widetilde{\mu},\sigma}, \mathbf{P}_{\mu,\sigma}) < \infty$ for all $\varepsilon \in (-\infty, \infty)$. Now using (5.1) and (5.2), we obtain for all $\varepsilon \in (-\infty, \infty)$

$$H_n(\varepsilon) = H(\varepsilon;\, \mathbf{P}^n_{\widetilde{\mu},\sigma}, \mathbf{P}^n_{\mu,\sigma}) = \exp\left\{-\frac{\varepsilon(1-\varepsilon)}{2}\left(\frac{(\widetilde{\mu}-\mu)}{\sigma}\right)^2 \bar{t}' C_n^{-1}\bar{t}\right\}. \tag{5.7}$$

But

$$H(\varepsilon;\, \mathbf{P}^n_{\widetilde{\mu},\sigma}, \mathbf{P}^n_{\mu,\sigma}) = \mathbf{E}_Q \tilde{\mathfrak{z}}_n^\varepsilon \mathfrak{z}_n^{1-\varepsilon}, \tag{5.8}$$

where $\mathbf{Q} = \mathbf{P}_{0,\sigma}$ and

$$\tilde{\mathfrak{z}}_n = \frac{d\mathbf{P}^n_{\widetilde{\mu},\sigma}}{d\mathbf{P}^n_{0,\sigma}}(\xi^n), \quad \mathfrak{z}_n = \frac{d\mathbf{P}^n_{\mu,\sigma}}{d\mathbf{P}^n_{0,\sigma}}(\xi^n).$$

Obviously, $\tilde{\mathfrak{z}}_n = \mathbf{E}_Q(\tilde{\mathfrak{z}}/\mathcal{F}_n)$ and $\mathfrak{z}_n = \mathbf{E}_Q(\mathfrak{z}/\mathcal{F}_n)$ where $\mathcal{F}_n = \sigma\{\xi_{t_{n1}}, \xi_{t_{n2}}, \dots, \xi_{t_{nn}}\}$ and

$$\tilde{\mathfrak{z}} = \frac{d\mathbf{P}_{\widetilde{\mu},\sigma}}{d\mathbf{P}_{0,\sigma}}(\xi), \quad \mathfrak{z} = \frac{d\mathbf{P}_{\mu,\sigma}}{d\mathbf{P}_{0,\sigma}}(\xi), \quad \xi = (\xi_s)_{0\le s\le 1}.$$

Using the convexity of the function $x^\varepsilon y^{1-\varepsilon}$ on R^2_+ for $\varepsilon \notin [0,1]$, by Jensen's inequality we have

$$\begin{aligned} H(\varepsilon) &= \mathbf{E}_Q \tilde{\mathfrak{z}}^\varepsilon \mathfrak{z}^{1-\varepsilon} = \mathbf{E}_Q \mathbf{E}_Q(\tilde{\mathfrak{z}}^\varepsilon \mathfrak{z}^{1-\varepsilon}/\mathcal{F}_n) \ge \\ &\ge \mathbf{E}_Q(\mathbf{E}_Q(\tilde{\mathfrak{z}}/\mathcal{F}_n))^\varepsilon(\mathbf{E}_Q(\mathfrak{z}/\mathcal{F}_n))^{1-\varepsilon} = \mathbf{E}_Q \tilde{\mathfrak{z}}_n^\varepsilon \mathfrak{z}_n^{1-\varepsilon} = H_n(\varepsilon) \end{aligned} \tag{5.9}$$

i. e. $H_n(\varepsilon) \le H(\varepsilon) < \infty$ for all $n = 1, 2, \dots$, if $\varepsilon \notin [0,1]$. Now, for $\varepsilon \in (0,1)$, by Hölder's inequality we have

$$\begin{aligned} H(\varepsilon) &= \mathbf{E}_Q \tilde{\mathfrak{z}}^\varepsilon \mathfrak{z}^{1-\varepsilon} = \mathbf{E}_Q \mathbf{E}_Q(\tilde{\mathfrak{z}}^\varepsilon \mathfrak{z}^{1-\varepsilon}/\mathcal{F}_n) \le \\ &\le \mathbf{E}_Q(\mathbf{E}_Q(\tilde{\mathfrak{z}}/\mathcal{F}_n))^\varepsilon(\mathbf{E}_Q(\mathfrak{z}/\mathcal{F}_n))^{1-\varepsilon} = \mathbf{E}_Q \tilde{\mathfrak{z}}_n^\varepsilon \mathfrak{z}_n^{1-\varepsilon} = H_n(\varepsilon) \end{aligned} \tag{5.10}$$

i. e. $H_n(\varepsilon) \ge H(\varepsilon) > 0$ for all $n = 1, 2, \dots$, if $\varepsilon \in (0,1)$. From here, using (5.7), we get that $\limsup\limits_{n\to\infty} \bar{t}' C_n^{-1}\bar{t} < \infty$. Thus, the equality (5.6) implies that the condition **(H)** is satisfied with (5.3), (5.4) and (5.5). $\square$

Remark 5.1. By arguments from the proof of Lemma 5.1 we can prove that

$$\lim_{n\to\infty} \bar{t}' C_n^{-1}\bar{t} = 2(1-H)C_0^2 B_1^2. \tag{5.11}$$

In fact, by Lemma 4.2 we get for all $\varepsilon \in (-\infty, \infty)$

$$H(\varepsilon) = H(\varepsilon;\, \mathbf{P}_{\widetilde{\mu},\widetilde{\sigma}}, \mathbf{P}_{\mu,\sigma}) = \exp\left\{-\frac{\varepsilon(1-\varepsilon)}{2}(\widetilde{v} - v)^2\right\}$$

where

$$\widetilde{v}^2 = 2(1-H)C_0^2B_1^2\left(\frac{\widetilde{\mu}}{\sigma}\right)^2, \quad v^2 = 2(1-H)C_0^2B_1^2\left(\frac{\mu}{\sigma}\right)^2,$$

i. e. we have

$$H(\varepsilon) = \exp\left\{-\frac{\varepsilon(1-\varepsilon)}{2}2(1-H)C_0^2B_1^2\left(\frac{\widetilde{\mu}-\mu}{\sigma}\right)^2\right\}. \tag{5.12}$$

Using (5.1) and (5.2) we obtain (5.7) for $\varepsilon(-\infty,\infty)$. Note that $\tilde{\mathfrak{z}} = (\tilde{\mathfrak{z}}_n)$ and$\mathfrak{z} = (\mathfrak{z}_n)$ are the $(\mathbf{P}_{0,\sigma}, \mathbf{F})$-martingales where $\mathbf{F} = (\mathcal{F}_n)$. Then for all $\varepsilon \notin [0,1]$ by Jensen's inequality we have

$$\begin{aligned} H_{n+1}(\varepsilon) &= \mathbf{E}_Q\tilde{\mathfrak{z}}_{n+1}^{\varepsilon}\mathfrak{z}_{n+1}^{1-\varepsilon} = \mathbf{E}_Q\mathbf{E}_Q\{\tilde{\mathfrak{z}}_{n+1}^{\varepsilon}\mathfrak{z}_{n+1}^{1-\varepsilon}/\mathcal{F}_n\} \geq \\ &\geq \mathbf{E}_Q(\mathbf{E}_Q(\tilde{\mathfrak{z}}_{n+1}/\mathcal{F}_n))^{\varepsilon}(\mathbf{E}_Q(\mathfrak{z}_{n+1}/\mathcal{F}_n))^{1-\varepsilon} = \mathbf{E}_Q\tilde{\mathfrak{z}}_n^{\varepsilon}\mathfrak{z}_n^{1-\varepsilon} = H_n(\varepsilon). \end{aligned} \tag{5.13}$$

And by Hölder's inequality we get for all $\varepsilon(0,1)$

$$\begin{aligned} H_{n+1}(\varepsilon) &= \mathbf{E}_Q\tilde{\mathfrak{z}}_{n+1}^{\varepsilon}\mathfrak{z}_{n+1}^{1-\varepsilon} = \mathbf{E}_Q\mathbf{E}_Q\{\tilde{\mathfrak{z}}_{n+1}^{\varepsilon}\mathfrak{z}_{n+1}^{1-\varepsilon}/\mathcal{F}_n\} \leq \\ &\leq \mathbf{E}_Q(\mathbf{E}_Q(\tilde{\mathfrak{z}}_{n+1}/\mathcal{F}_n))^{\varepsilon}(\mathbf{E}_Q(\mathfrak{z}_{n+1}/\mathcal{F}_n))^{1-\varepsilon} = \mathbf{E}_Q\tilde{\mathfrak{z}}_n^{\varepsilon}\mathfrak{z}_n^{1-\varepsilon} = H_n(\varepsilon). \end{aligned} \tag{5.14}$$

But $\tilde{\mathfrak{z}}_n \to \tilde{\mathfrak{z}}$ and $\mathfrak{z}_n \to \mathfrak{z}$ ($\mathbf{P}_{0,\sigma}$-a. s.) as $n \to \infty$. Using (5.9), (5.10), (5.13) and (5.14), from relations (5.7), (5.8) and (5.12) we get (5.11).

¿From Lemma 5.1 we obtain

(5.15)

$$\Gamma_0 = \gamma_0 = -\frac{1}{2}\left(\frac{\sigma^2}{\widetilde{\sigma}^2} - 1 - \ln\frac{\sigma^2}{\widetilde{\sigma}^2}\right), \qquad \Gamma_1 = \gamma_1 = \frac{1}{2}\left(\frac{\widetilde{\sigma}^2}{\sigma^2} - 1 - \ln\frac{\widetilde{\sigma}^2}{\sigma^2}\right),$$

$$I(\gamma) = \frac{1}{2}(z(\gamma) - 1 - \ln z(\gamma)), \qquad z(\gamma) = \left(2\gamma - \ln\frac{\sigma^2}{\widetilde{\sigma}^2}\right)\left(1 - \frac{\sigma^2}{\widetilde{\sigma}^2}\right)^{-1}. \tag{5.16}$$

Using Lemma 5.1 from these relations we obtain that the large deviation theorems 2.1 and 2.2 are valid for $\Lambda_n = \ln(\widetilde{p}_n(\xi^n)/p_n(\xi^n))$ as $n \to \infty$. These theorems permit us to formulate the following theorem about behavior of error probabilities for the tests δ_n, δ_n^{π} and δ_n^* as $n \to \infty$ for binary statistical experiments $(R^n, \mathcal{B}^n, \mathbf{P}_{\mu,\sigma}^n, \mathbf{P}_{\widetilde{\mu},\widetilde{\sigma}}^n)$, $n \geq 1$.

Theorem 5.1. *For statistical experiments $(R^n, \mathcal{B}^n, \mathbf{P}_{\mu,\sigma}^n, \mathbf{P}_{\widetilde{\mu},\widetilde{\sigma}}^n)$, $n \geq 1$ the following statements hold:*

(1) *for all $a \in (0, \gamma_1)$*

$$\lim_{n\to\infty}\frac{1}{n}\ln\alpha_n = -a \Longleftrightarrow \lim_{n\to\infty}\frac{1}{n}\ln\beta_t = -b(a), \tag{5.17}$$

where

$$b(a) = a - \frac{1}{2}z_a\left(1 - \frac{\sigma^2}{\widetilde{\sigma}^2}\right) - \frac{1}{2}\ln\frac{\sigma^2}{\widetilde{\sigma}^2} \tag{5.18}$$

and z_a is a solution of the equation $z - 1 - \ln z = 2a$ for $z > 1$;

(2) *if π is independent on n, then*

$$\lim_{n\to\infty}\frac{\ln\alpha(\delta_n^\pi)}{n}=\lim_{n\to\infty}\frac{\ln\beta(\delta_n^\pi)}{n}=\lim_{n\to\infty}\frac{\ln e(\delta_n^\pi)}{n}=-I(0); \tag{5.19}$$

(3) *we have*

$$\lim_{n\to\infty}\frac{\ln\alpha(\delta_n^*)}{n}=\lim_{n\to\infty}\frac{\ln\beta(\delta_n^*)}{n}=\lim_{n\to\infty}\frac{\ln e_n^*}{n}=-I(0), \tag{5.20}$$

where

$$I(0)=\frac{1}{2}\left[\left(\frac{\sigma^2}{\widetilde{\sigma}^2}-1\right)^{-1}\ln\frac{\sigma^2}{\widetilde{\sigma}^2}-1-\ln\left(\left(\frac{\sigma^2}{\widetilde{\sigma}^2}-1\right)^{-1}\ln\frac{\sigma^2}{\widetilde{\sigma}^2}\right)\right]. \tag{5.21}$$

Proof. The relations (5.19) and (5.20) follow from Theorems 3.2 and 3.3 since $\Gamma_0=\gamma_0<0<\gamma_1=\Gamma_1$ and $I(0)$ has the form (5.21). To prove the statement (1) we apply the statement (1) of Theorem 3.1. Note that $I(\Gamma_0)=0$ and $I(\Gamma_1)=\gamma_1$. Moreover, a solution of equation $I(\gamma)=a$ with respect to $\gamma\in(\gamma_0,\gamma_1)$ for all $a\in(0,\gamma_1)$, by the relations (5.16), has the form

$$\gamma(a)=\frac{1}{2}z_a\left(1-\frac{\sigma^2}{\widetilde{\sigma}^2}\right)+\frac{1}{2}\ln\frac{\sigma^2}{\widetilde{\sigma}^2}$$

where z_a is a solution of the equation $z-1-\ln z=2a$. From here, using the statement (1) of Theorem 3.1, we obtain the relations (5.17) and (5.18). □

Remark 5.2. About statements (2) and (3) from Theorem 3.1 for statistical experiments $(R^n,\mathcal{B}^n,\mathbf{P}^n_{\mu,\sigma},\mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma}})$, $n\geq 1$, we can make the same remarks as in Remark 4.1.

Remark 5.3. Consider the problems of testing hypotheses $H_{\widetilde{\mu},\widetilde{\sigma},\widetilde{\sigma}}$ and $H_{\mu,\sigma,\sigma}$ under the observations $\xi^n=(\xi_{t_{n1}},\dots,\xi_{t_{nn}})$, $n\geq 1$, where $\widetilde{\sigma}\neq\sigma$ and, in general, $\widetilde{\mu}\neq\mu$. By $\mathbf{P}^n_{\mu,\sigma,\sigma}$ we denote a distribution of the vector ξ^n under the hypothesis $H_{\mu,\sigma,\sigma}$. By analogy with Lemma 5.1 we can prove that the condition **(H)** holds for the statistical experiments $(R^n,\mathcal{B}^n,\mathbf{P}^n_{\mu,\sigma,\sigma},\mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma},\widetilde{\sigma}})$, $n\geq 1$ and $\psi_n,\varkappa(\varepsilon),\varepsilon_-,\varepsilon_+,\gamma_-$ and γ_+ are the same as in Lemma 5.1. Thus, we get that the statements of Theorem 5.1 are valid for the statistical experiments $(R^n,\mathcal{B}^n,\mathbf{P}^n_{\mu,\sigma,\sigma},\mathbf{P}^n_{\widetilde{\mu},\widetilde{\sigma},\widetilde{\sigma}})$, $n\geq 1$, too.

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