REGULARITY OF THE FRACTIONAL MAXIMAL FUNCTION

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Abstract. The purpose of this work is to show that the fractional maximal operator has somewhat unexpected regularity properties. Our main result shows that the fractional maximal operator maps L^p-spaces boundedly into certain first oder Sobolev spaces. We also prove that the fractional maximal operator preserves first order Sobolev spaces. This extends known results for the Hardy-Littlewood maximal operator.

1. Introduction

Let $0 \le \alpha \le n$. The fractional maximal function of a locally integrable function $f\colon \mathbf{R}^n \to [-\infty, \infty]$ is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\,dy. \tag{1.1}$$

For $\alpha = 0$ we obtain the Hardy-Littlewood maximal function. The fractional maximal operator has applications in potential theory and partial differential equations. There is a close connection between the fractional maximal function and the Riesz potential

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy.$$

It is easy to see that

$$\mathcal{M}_\alpha f(x) \le \Omega_n^{-1} I_\alpha f(x) \tag{1.2}$$

for every $x \in \mathbf{R}^n$, where Ω_n is the volume of the unit ball in $\mathbf{R}^n$, but there is no pointwise inequality in the reverse direction. However, such an inequality holds in average by a result of Muckenhoupt and Wheeden. Indeed, for every $1 < p < \infty$ there is a constant $c = c(n,p)$ so that

$$c^{-1} \|\mathcal{M}_\alpha f\|_p \le \|I_\alpha f\|_p \le c \|\mathcal{M}_\alpha f\|_p \tag{1.3}$$

for every locally integrable f. The first inequality is an obvious consequence of (1.2), but the second inequality is more involved.

On the other hand, the fractional maximal function of the gradient bounds the oscillation of the function. To be more precise, suppose that $u \in C_0^\infty(\mathbf{R}^n)$ and let $0 \le \alpha < p$. Then one can show that there is a constant $c = c(n)$ such that

$$|u(x) - u(y)| \le c|x-y|^{1-\alpha/p} \big(\mathcal{M}_{\alpha/p}|Du|(x) + \mathcal{M}_{\alpha/p}|Du|(y)\big)$$

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for every $x, y \in \mathbf{R}^n$.

Let us recall the standard boundedness properties of the Riesz potential. If $1 < p < \infty$ and $0 \le \alpha < n/p$, the Sobolev imbedding theorem (see [St2]) gives

$$\|I_\alpha f\|_q \le c(n, p, \alpha)\, \|f\|_p \tag{1.4}$$

with $q = np/(n - \alpha p)$ and if $p = 1$ then we have the weak type estimate

$$|\{x \in \mathbf{R}^n \colon I_\alpha f(x) > \lambda\}| \le \big(c\lambda^{-1}\|f\|_1\big)^{n/(n-\alpha)} \tag{1.5}$$

for every $\lambda > 0$. Using (1.2) we observe that the corresponding results hold for the fractional maximal operator as well. If $\alpha = 0$ then we have the classical maximal function theorem of Hardy, Littlewood and Wiener.

The purpose of this work is to show that the fractional maximal operator has somewhat unexpected regularity properties. Our motivation is the following example: If $f \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$, then the Riesz potential I_1 has the first weak partial derivatives and they are given almost everywhere by

$$D_i I_1 f(x) = -R_i f(x),$$

where

$$R_i f(x) = \lim_{\varepsilon \to 0} c(n) \int_{|y| > \varepsilon} \frac{y_i}{|y|^{n+1}} f(x - y)\, dy,$$

$i = 1, 2, \dots, n$, are the Riesz transforms. Since the Riesz transforms are bounded in L^p when $1 < p < \infty$, we see that I_1 maps $L^p(\mathbf{R}^n)$ into a first order Sobolev space.

Keeping in mind the close connection between the Riesz potential and the fractional maximal operator it is natural to ask whether the fractional maximal operator has similar properties. Our main result shows that the fractional maximal operator is smoothing in the sense that it maps L^p-spaces into certain first oder Sobolev spaces. We also show that the fractional maximal operator is bounded on first order Sobolev spaces. This latter fact is a simple consequence of the Sobolev embedding theorem and extends known results for the Hardy-Littlewood maximal operator in [K]. We close the paper by recording boundedness results for the fractional spherical maximal operator. These results are closely related to the regularity problems studied in this paper.

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2. Fractional maximal operator on Sobolev spaces

Recall that the Sobolev space $W^{1,p}(\mathbf{R}^n)$, $1 \le p \le \infty$, consists of functions in $L^p(\mathbf{R}^n)$ whose first distributional partial derivatives are also in $L^p(\mathbf{R}^n)$, see [GT] or [St2]. The following result shows that the fractional maximal operator preserves the first order Sobolev spaces.

2.1. Theorem. *Suppose that $1 < p < \infty$ and let $0 \le \alpha < n/p$. If $u \in W^{1,p}(\mathbf{R}^n)$, then $\mathcal{M}_\alpha u \in W^{1,q}(\mathbf{R}^n)$ with $q = np/(n - \alpha p)$. Moreover, there is $c = c(n, p, \alpha)$ such that*

$$\|\mathcal{M}_\alpha u\|_{1,q} \le c\, \|u\|_{1,p}.$$

Proof. We use the characterization of Sobolev spaces by integrated difference quotients, see 7.11 of [GT]. Denote $f_h(x) = f(x+h)$ for $h \in \mathbf{R}^n$. We observe that the fractional maximal operator commutes with translations and is sublinear. Thus the Sobolev embedding theorem together with (1.3) gives

$$\begin{aligned}\big\|(\mathcal{M}_\alpha u)_h - \mathcal{M}_\alpha u\big\|_q &= \big\|\mathcal{M}_\alpha(u_h) - \mathcal{M}_\alpha u\big\|_q \le \big\|\mathcal{M}_\alpha(u_h - u)\big\|_q \\ &\le c\,\|u_h - u\|_p \le c\,\|Du\|_p |h|.\end{aligned}$$

This proves the claim. □

2.2. Remark. A slightly more careful analysis would yield the pointwise estimate

$$|D_i \mathcal{M}_\alpha u| \le \mathcal{M}_\alpha D_i u, \qquad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$. The proof is an easy modification of the argument in [K].

Observe that $q > p$ if $\alpha > 0$ in the previous theorem. Hence $\mathcal{M}_\alpha u$ belongs to a higher Sobolev space than u. This shows that the fractional maximal operator has certain smoothing properties. The most basic example of this phenomenon is that $\mathcal{M}_\alpha f$, $\alpha > 0$, is continuous for every $f \in L^\infty(\mathbf{R}^n)$ which has a compact support. This is clearly not true when $\alpha = 0$.

3. Fractional maximal operator on Lebesgue spaces

The following theorem is our main result.

3.1. Theorem. *Suppose that $f \in L^p(\mathbf{R}^n)$ with $1 < p < n$ and let $1 \le \alpha < n/p$.*

(i) *Then the weak partial derivatives $D_i \mathcal{M}_\alpha f$, $i = 1, 2, \dots, n$, exist almost everywhere and there is a constant $c = c(n, \alpha)$ such that*

$$|D_i \mathcal{M}_\alpha f| \le c\,\mathcal{M}_{\alpha-1} f, \qquad i = 1, 2, \dots, n,$$

almost everywhere in $\mathbf{R}^n$.

(ii) *Let $q = np/(n-(\alpha-1)p)$ and $q^* = np/(n-\alpha p)$. Then $\mathcal{M}_\alpha f \in L^{q^*}(\mathbf{R}^n)$ and $D_i \mathcal{M}_\alpha f \in L^q(\mathbf{R}^n)$, $i = 1, 2, \dots, n$. Moreover, there is $c = c(n, p, \alpha)$ such that*

$$\|\mathcal{M}_\alpha f\|_{q^*} \le c\,\|f\|_p \quad \text{and} \quad \|D_i \mathcal{M}_\alpha f\|_q \le c\,\|f\|_p, \quad i = 1, 2, \dots, n.$$

Proof. Suppose first that $f \in C_0^\infty(\mathbf{R}^n)$ and $f \not\equiv 0$. We verify that $\mathcal{M}_\alpha f$ is Lipschitz continuous on any compact subset $K \subset \mathbf{R}^n$. Since f is bounded and not identically zero we easily deduce that there are constants r_0 and R_0, with $0 < r_0 < R_0$, depending only on f such that

$$\mathcal{M}_\alpha f(x) = \sup_{r_0 < r < R_0} r^\alpha \fint_{B(x,r)} |f(y)|\,dy$$

for all $x \in K$. On the other hand, it is obvious that $\chi_{B(0,r)} * f$ is Lipschitz continuous with a constant that stays bounded for $r_0 \le r \le R_0$. The claim follows, since the

Lipschitz constant of a supremum does not exceed the supremum of the individual Lipschitz constants.

Let us fix $x \in \mathbf{R}^n$ and $h \in \mathbf{R}^n$ such that $|h| < 1$. Let $K = B(x,2)$ and let r_0 and R_0 be as above corresponding to this particular set K. We denote

$$D^+\mathcal{M}_\alpha f(x) = \limsup_{h\to 0} \frac{\mathcal{M}_\alpha f(x+h) - \mathcal{M}_\alpha f(x)}{|h|}.$$

Since $\mathcal{M}_\alpha f$ is Lipschitz continuous, we have $D^+\mathcal{M}_\alpha f(x) < \infty$.

We want to deduce an upper bound for $D^+\mathcal{M}_\alpha f(x)$. There is a sequence $h_k \in \mathbf{R}^n$, $k = 1, 2, \dots$, such that $h_k \to 0$ as $k \to \infty$ and

$$\frac{\mathcal{M}_\alpha f(x+h_k) - \mathcal{M}_\alpha f(x)}{|h_k|} \ge D^+\mathcal{M}_\alpha f(x) - \frac{1}{k}.$$

Let us fix h_k. Clearly it is enough to consider the case $\mathcal{M}_\alpha f(x+h_k) \ge \mathcal{M}_\alpha f(x)$. Then

$$\mathcal{M}_\alpha f(x+h_k) \ge |h_k| D^+\mathcal{M}_\alpha f(x) - \frac{|h_k|}{k} + \mathcal{M}_\alpha f(x).$$

Choose $r_j > 0$, $j = 1, 2, \dots$, such that

$$r_j^\alpha \fint_{B(x+h_k, r_j)} |f|\, dy \ge \mathcal{M}_\alpha f(x+h_k) - \frac{1}{j}.$$

We may assume that $r_0 \le r_j \le R_0$. Since $B(x+h_k, r_j) \subset B(x, r_j + |h_k|)$, we have

$$\begin{aligned} \mathcal{M}_\alpha f(x) &\ge (r_j + |h_k|)^\alpha \fint_{B(x, r_j+|h_k|)} |f|\, dy \\ &\ge \Omega_n^{-1}(r_j + |h_k|)^{\alpha-n} \int_{B(x+h_k, r_j)} |f|\, dy. \end{aligned}$$

By combining the last three estimates we obtain

$$|h_k| D^+\mathcal{M}_\alpha f(x) \le \Omega_n^{-1}\big(r_j^{\alpha-n} - (r_j+|h_k|)^{\alpha-n}\big) \int_{B(x+h_k,r_j)} |f|\, dy + \frac{1}{j} + \frac{|h_k|}{k}.$$

By the mean value theorem there is $\zeta_{j,k}$, $r_j < \zeta_{j,k} < r_j + |h_k|$ such that

$$r_j^{\alpha-n} - (r_j+|h_k|)^{\alpha-n} \le (n-\alpha-1)\zeta_{j,k}^{\alpha-n-1}|h_k|.$$

This implies that

$$\begin{aligned} |h_k| D^+\mathcal{M}_\alpha f(x) &\le c\zeta_{j,k}^{\alpha-n+1} |h_k| \int_{B(x+h_k,r_j)} |f|\, dy + \frac{1}{j} + \frac{|h_k|}{k} \\ &\le c r_j^{\alpha-n-1} |h_k| \int_{B(x, r_j+|h_k|)} |f|\, dy + \frac{1}{j} + \frac{|h_k|}{k} \\ &\le c\Big(\frac{r_j}{r_j+|h_k|}\Big)^{\alpha-n-1} |h_k| \mathcal{M}_{\alpha-1} f(x) + \frac{1}{j} + \frac{|h_k|}{k}, \end{aligned}$$

where $c = c(n, \alpha)$. We choose j so large that $1/j \le |h_k|/k$. Then

$$\begin{aligned} D^+\mathcal{M}_\alpha f(x) &\le c\Big(\frac{r_j}{r_j + |h_k|}\Big)^{\alpha-n-1}\mathcal{M}_{\alpha-1}f(x) + \frac{2}{k} \\ &\le c\Big(\frac{r_0}{r_0 + |h_k|}\Big)^{\alpha-n-1}\mathcal{M}_{\alpha-1}f(x) + \frac{2}{k}. \end{aligned}$$

Letting $k \to \infty$ we arrive at

$$D^+M_\alpha f(x) \le c\mathcal{M}_{\alpha-1}f(x), \tag{3.3}$$

where $c = c(n, \alpha)$. Since $x \in \mathbf{R}^n$ was arbitrary, inequality (3.3) holds for every $x \in \mathbf{R}^n$.

Suppose then that $f \in L^p(\mathbf{R}^n)$. Let $f_j \in C_0^\infty(\mathbf{R}^n)$, $j = 1, 2, \dots$, be such that $f_j \to f$ in $L^p(\mathbf{R}^n)$ as $j \to \infty$. Since $\mathcal{M}_\alpha f_j$ is Lipschitz, it is differentiable at almost every $x \in \mathbf{R}^n$. At each point of differentiability, and hence almost everywhere in $\mathbf{R}^n$, (3.3) yields the inequality

$$|D_i\mathcal{M}_\alpha f_j| \le D^+\mathcal{M}_\alpha f_j \le c\mathcal{M}_{\alpha-1}f_j$$

for each partial derivative $D_i\mathcal{M}_\alpha f_j$. By the Sobolev embedding theorem we have

$$\|D_i\mathcal{M}_\alpha f_j\|_q \le c\|\mathcal{M}_{\alpha-1}f_j\|_q \le c\|f_j\|_p,$$

where $c = c(n, p, \alpha)$. Hence $(D_i\mathcal{M}_\alpha f_j)$ is a bounded sequence in $L^q(\mathbf{R}^n)$ for every $i = 1, 2, \dots, n$. On the other hand, the Sobolev embedding theorem implies that

$$\|\mathcal{M}_\alpha f_j - \mathcal{M}_\alpha f\|_{q^*} \le \|\mathcal{M}_\alpha(f_j - f)\|_{q^*} \le c\|f_j - f\|_p,$$

where $c = c(n, p, \alpha)$, and therefore $\mathcal{M}_\alpha f_j \to \mathcal{M}_\alpha f$ in $L^{q^*}(\mathbf{R}^n)$ as $j \to \infty$. From this we conclude that $D_i\mathcal{M}_\alpha f \in L^q(\mathbf{R}^n)$, $i = 1, 2, \dots, n$, and, by extracting a subsequence if needed, we may assume that $D_i\mathcal{M}_\alpha f_j \to D_i\mathcal{M}_\alpha f$, $i = 1, 2, \dots$, weakly in $L^q(\mathbf{R}^n)$ as $j \to \infty$. Hence for any $i = 1, \dots, n$ we have $D_i\mathcal{M}_\alpha f \in L^q(\mathbf{R}^n)$. Moreover, it is easy to see that the almost everywhere pointwise estimate

$$|D_i\mathcal{M}_\alpha f_j| \le c\mathcal{M}_{\alpha-1}f_j$$

is preserved up to the weak limit and we have

$$|D_i\mathcal{M}_\alpha f| \le c\mathcal{M}_{\alpha-1}f$$

for almost everywhere in $\mathbf{R}^n$. This completes the proof. □

3.4. Remark. Observe that in the definition of the factional maximal function balls can be replaced, for example, by cubes and Theorem 3.1 is still true with an analoguous proof. This observation is interesting when it is combined with remark 4.3 below.

4. Fractional spherical maximal operator

Suppose that $f\colon \mathbf{R}^n \to (-\infty,\infty)$ is a continuous function and let $0 \le \alpha \le n-1$. The fractional spherical maximal function of f is defined as

$$\mathcal{S}_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{\partial B(x,r)} |f(y)|\, d\mathcal{H}^{n-1}(y).$$

Here the bar on the integral sign denotes the average with respect to the normalized $(n-1)$-dimensional Hausdorff measure

$$\fint_{\partial B(x,r)} |f(y)|\, d\mathcal{H}^{n-1}(y) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B(x,r)} |f(y)|\, d\mathcal{H}^{n-1}(y),$$

where $\omega_{n-1} = \mathcal{H}^{n-1}(\partial B(x,r))$. If $\alpha = 0$, we have the spherical maximal function studied originally by E.M. Stein, see [St1], [SWein]. In general it is much more difficult to obtain bounds for $\mathcal{S}_\alpha$ than for $\mathcal{M}_\alpha$. This difficulty is visible already when $\alpha = 0$. It is clear that

$$\mathcal{M}_\alpha f(x) \le \frac{n}{n-\alpha} \mathcal{S}_\alpha f(x), \qquad 0 \le \alpha \le n-1, \tag{4.1}$$

for every $x \in \mathbf{R}^n$, but there is no inequality in the reverse direction. However, the the following result is true.

4.2. Theorem. *Let $n \ge 2$, $n/(n-1) < p < n$ and $0 \le \alpha < n/p - 1$. Then there exists $c = c(n,p,\alpha)$ such that*

$$\|\mathcal{S}_\alpha f\|_q \le c \|f\|_p$$

with $q = np/(n-\alpha p)$ for all $f \in L^p(\mathbf{R}^n)$.

For $\alpha = 0$, this was proved by by Stein [St1] (see also [SWein]) when $n \ge 3$ and by Bourgain [B] in the case $n = 2$. For $\alpha > 0$ this result is due to Schlag when $n = 2$, see Theorem 1.3 in [Sc], and Schlag and Sogge when $n \ge 3$, see Theorem 4.1 in [ScSo]. The result actually holds for a slightly larger range of α's, but we do not consider this refinement here. Both references state the analogue of Theorem 4.2 for maximal means over radii $1 < r < 2$ only but the result for the corresponding fractional maximal operator can be obtained from this result by the Littlewood-Paley theory, see pages 71–73 of [B] or pages 72–74 of [So].

For readers' convenience we sketch here a different argument in the easier case $n \ge 3$. This proof interpolates between the Sobolev embedding theorem and an estimate from [SWein]. We stress that the proof of Stein's estimate only uses simple g-function techniques in contrast with more intrigue Theorem 4.1 in [ScSo].

Proof of Theorem 4.2. Let $n \ge 3$ be the dimension of the underlying space $\mathbf{R}^n$, $r > 0$, α, β be complex numbers and f be a smooth function. We define the operator $M^\beta_{\alpha,r}$ as

$$(\widehat{M^\beta_{\alpha,r} f})(\xi) = r^\alpha m_\beta(r\xi) \widehat{f}(\xi),$$

and

$$m_\beta(\xi) = 2^{n/2+\alpha-1}\Gamma(n/2+1)(2\pi\xi)^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi\xi),$$

see page 1270 of [SWein]. Here J denotes the Bessel function. The corresponding maximal function $\mathcal{M}_\alpha^\beta f$ is defined as

$$\mathcal{M}_\alpha^\beta f(x) = \sup_{r>0} M_{\alpha,r}^\beta f(x).$$

Notice that the values $\beta = 0$ and $\beta = 1$ correspond to the fractional spherical and the fractional Hardy-Littlewood maximal functions, respectively. According to Theorem 9 on p. 1270 of [SWein] we have

$$\|\mathcal{M}_0^\beta f\|_2 \le c_\beta \|f\|_2 \tag{4.3}$$

for $\operatorname{Re}\beta > 1 - n/2$ and smooth f . Moreover, the Sobolev embedding theorem can be rewritten as

$$\|\mathcal{M}_\alpha^1 f\|_q \le c\|f\|_p \tag{4.4}$$

for $1 < p \le q$ and $\alpha = n(1/p - 1/q)$.

We next apply Stein's analytic interpolation theorem, see page 205 of [SWeis], in a standard way to estimates (4.3) and (4.4). For this end, let $1 < p_0 \le q_0 < \infty$ and denote $\alpha_0 = n(1/p_0 - 1/q_0)$. Assume that the measurable function r defined on $\mathbf{R}^n$ is strictly positive and takes only finitely many values. For $\varepsilon > 0$ consider the linear operator $G(\beta)$, where

$$(G(\beta)f)(x) = (M_{g(\beta),r(x)}^\beta f)(x),$$

where $g(\beta) = \alpha_0(\beta+n/2-1-\varepsilon)/(n/2-\varepsilon)$. Clearly $G(\beta)$ is analytic with respect to β and obeys bounds similar to (4.3) and (4.4). Since c_β grows moderately enough as $\operatorname{Im}\beta \to \pm\infty$ we may interpolate in the strip $\varepsilon + 1 - n/2 \le \operatorname{Re}\beta \le 1$. Observe that the norm bounds do not depend on the function r. The claim is obtained at $\beta = 0$, however, for a smaller range for the indexes. The stated range is obtained by letting $\varepsilon \to 0$. □

4.3. Remark. We may also use the method of[K]to establish directly that

$$|D_i \mathcal{M}_\alpha f| \le c\,\mathcal{S}_{\alpha-1} f, \qquad i = 1, 2, \ldots, n, \tag{4.5}$$

almost everywhere in $\mathbf{R}^n$. Let us scetch the idea. We compute the derivative of the convolutions corresponding to the integral averages and observe that the derivative of the supremums is bounded by the supremum of the individual derivatives. By (4.1) this is weaker than the estimate in Theorem 3.1, but we feel that this observation is of independent interest. In particular, if $\alpha = 1$, then on the right hand side of (4.5) we have Stein's spherical maximal function, which is known to be bounded in $L^p(\mathbf{R}^n)$ if and only if $n/(n-1) < p < \infty$. Hence by this argument we obtain a smaller range of exponents. In addition, this method fails if ball are replaced by cubes (compare to Remark 3.4), since Theorem 4.2 is obviously not true for the boundaries of cubes.

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