Joining large deviations upper bounds of random vectors

Harri Nyrhinen
Department of Mathematics
P.O. Box 4
FIN-00014 University of Helsinki
FINLAND

November 2001

Abstract

Let $Y_{1i}, Y_{2i}, Y_{3i}, \ldots$ be a sequence of random vectors in $\mathbf{R}^{d_i}$ for $i = 1, \ldots, m$ and

$$Y_n = (Y_{n1}, \ldots, Y_{nm})$$

for $n \in \mathbf{N}$. Write $d = d_1 + \cdots + d_m$. Then Y_n may be regarded as a random vector in $\mathbf{R}^d$. Suppose that the components $Y_{n1}, \ldots, Y_{nm}$ are independent for every n and that for each i, the large deviations upper bounds are satisfied for the sequence $\{Y_{ni}/n\}$ with a given rate function. We derive large deviations upper bounds for the joined sequence $\{Y_n/n\}$ in this environment. Extensions of Cramér's theorem and the Gärtner-Ellis theorem can be deduced from the results. This viewpoint is in particular considered in the case where the components $Y_{n1}, \ldots, Y_{nm}$ are one-dimensional. Then under appropriate large deviations principles for the component sequences, necessary and sufficient conditions for the large deviations principle for the joined sequence can be given. Two classes of processes are studied in detail. Firstly, we prove that the large deviations principle is always satisfied when $\{Y_n\}$ is a random walk with independent one-dimensional components. Secondly, simple necessary and sufficient conditions are derived for the large deviations principle in the case where the components are appropriate increments of a one-dimensional random walk.

Key words: Large deviations theory, Cramér's theorem, Gärtner-Ellis theorem

AMS 1991 subject classification: primary 60F10

1. Introduction

Let m and $d_1, \ldots, d_m$ be positive integers and $d = d_1 + \cdots + d_m$. Let $Y_{1i}, Y_{2i}, Y_{3i}, \ldots$ be a sequence of random vectors in $\mathbf{R}^{d_i}$ for $i = 1, \ldots, m$. Suppose that for each $n \in \mathbf{N}$, the vectors $Y_{n1}, \ldots, Y_{nm}$ are defined on the same probability space $(\Omega_n, S_n, \mathbf{P}_n)$. Write

$$Y_n = (Y_{n1}, \ldots, Y_{nm}) \tag{1.1}$$

for $n \in \mathbf{N}$. Then Y_n may be regarded in the natural way as a random vector in $\mathbf{R}^d$. We assume throughout that the vectors $Y_{n1}, \ldots, Y_{nm}$ are independent for every $n \in \mathbf{N}$.

General principles of large deviations theory can be used to obtain limit results for the joined sequence $\{Y_n \mid n = 1, 2, \ldots\}$ from the corresponding results for the component sequences. To describe this, recall some basic concepts of the theory. A function $I : \mathbf{R}^d \to [0, \infty]$ is a *rate function* if it is lower semicontinuous. The sequence $\{Y_n/n\}$ satisfies the *large deviations principle* with the rate function I if

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F) \leq -\inf\{I(x) \mid x \in F\} \tag{1.2}$$

for every closed set $F \subseteq \mathbf{R}^d$ and

$$\liminf_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in G) \geq -\inf\{I(x) \mid x \in G\} \tag{1.3}$$

for every open set $G \subseteq \mathbf{R}^d$.

Suppose that $\{Y_{ni}/n \mid n = 1, 2, \ldots\}$ satisfies the large deviations principle in $\mathbf{R}^{d_i}$ with the rate function I_i for $i = 1, \ldots, m$. Every vector $x \in \mathbf{R}^d$ can be understood as a composition $x = (x_1, \ldots, x_m)$ as in (1.1) where $x_i \in \mathbf{R}^{d_i}$ for each i. Corresponding to this, define the rate function I on $\mathbf{R}^d$ by

$$I(x) = I_1(x_1) + \cdots + I_m(x_m) \tag{1.4}$$

for $x \in \mathbf{R}^d$. Then upper bound (1.2) holds for every compact set $F \subseteq \mathbf{R}^d$ and lower bound (1.3) for every open set $G \subseteq \mathbf{R}^d$. The sequence $\{Y_n/n\}$ satisfies the large deviations principle with the rate function I if the sequence of the distributions of $\{Y_{ni}/n\}$ is *exponentially tight* i.e. if

$$\lim_{M\to\infty} \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(| Y_{ni}/n |> M) = -\infty \tag{1.5}$$

for $i = 1, \ldots, m$. We refer to Dembo and Zeitouni (1998), Lemma 1.2.18, Theorems 4.1.11 and 4.1.18 and Exercise 4.2.7 for the background. Conditions (1.5) are essential for upper bounds (1.2) for the closed sets. An interesting counterexample is given in Dinwoodie (1991).

Our interest is in refinements of upper bounds (1.2) for the closed sets in the above environment. We take as hypotheses strengthened versions of the large deviations upper bounds for the component sequences. These strengthenings replace conditions (1.5) and lead to extensions of (1.2).

We specify the results in the framework of the Gärtner-Ellis theorem in the case where the component sequences are one-dimensional. Given the large deviations principles for the components, we derive necessary and sufficient conditions for the large deviations principle for the joined sequence. A detailed analysis is carried out in two classes of processes. Firstly, we show that the large deviations principle is always satisfied when $\{Y_n\}$ is a random walk with independent one-dimensional components. Secondly, we take the components of Y_n to be consecutive increments of a single one-dimensional random walk. Large deviations for this process will be completely described by means of simple conditions.

The paper is organized as follows. General sufficient conditions for (1.2) and a converse result are stated in Section 2. In Section 3, we consider extensions and special cases in the framework of the Gärtner-Ellis theorem. Section 4 consists of the proofs.

2. General upper bounds

We derive in this section general sufficient conditions for upper bounds (1.2) for the joined sequence with the rate function of type (1.4). We also state a converse result.

We begin by describing in detail the structure of the process of our interest. Let $m \geq 2$ and $d_1, \ldots, d_m$ be positive integers. Write $d = d_1 + \cdots + d_m$. Let Y_{ni} be a random vector on the probability space $(\Omega_n, S_n, \mathbf{P}_n)$ taking values in $\mathbf{R}^{d_i}$ for $n \in \mathbf{N}$ and $i = 1, \ldots, m$. We assume throughout that the vectors $Y_{n1}, \ldots, Y_{nm}$ are independent for each n.

Let

$$x_i = (a_{i1}, \ldots, a_{id_i}) \in \mathbf{R}^{d_i} \tag{2.1}$$

be an arbitrary vector for $i = 1, \ldots, m$. Write

$$x = (x_1, \ldots, x_m) \tag{2.2}$$

and identify x with the vector

$$(a_{11}, \ldots, a_{1d_1}, \ldots, a_{m1}, \ldots, a_{md_m}) \in \mathbf{R}^d. \tag{2.3}$$

Each vector Y_{ni} can be written componentwise as in (2.1). Corresponding to (2.2), we write

$$Y_n = (Y_{n1}, \ldots, Y_{nm}) \tag{2.4}$$

for $n \in \mathbf{N}$ and consider Y_n as a d-dimensional vector as in (2.3).

To describe the results, we need some notations and definitions. Let k be a positive integer and A a subset of $\mathbf{R}^k$. We denote by $\operatorname{int} A$, $\operatorname{cl} A$ and A^c the interior, the closure and the complement of A in $\mathbf{R}^k$ respectively. For any function $f : \mathbf{R}^k \to \mathbf{R} \cup \{\pm\infty\}$, write

$$\mathcal{D}(f) = \{y \in \mathbf{R}^k \mid f(y) < \infty\}. \tag{2.5}$$

For $\alpha \in \mathbf{R}$, the *level set* $\Psi_f(\alpha)$ is defined by

$$\Psi_f(\alpha) = \{y \in \mathbf{R}^k \mid f(y) \le \alpha\}. \tag{2.6}$$

Theorem 2.1. Let the processes $\{Y_{ni} \mid n = 1, 2, \ldots\}$ for $i = 1, \ldots, m$ and $\{Y_n \mid n = 1, 2, \ldots\}$ be as described above. Let $I_1, \ldots, I_m$ be rate functions on $\mathbf{R}^{d_1}, \ldots, \mathbf{R}^{d_m}$ respectively and

$$I(x) = I_1(x_1) + \cdots + I_m(x_m) \tag{2.7}$$

for $x = (x_1, \ldots, x_m) \in \mathbf{R}^d$. Assume that for every $\alpha > 0$ and $i = 1, \ldots, m$,

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in \Psi_{I_i}(\alpha)^c) \le -\alpha. \tag{2.8}$$

Then upper bound (1.2) holds for every closed set $F \subseteq \mathbf{R}^d$.

Condition (2.8) describes a strengthened version of the large deviations upper bounds for the sequence $\{Y_{ni}/n\}$. Namely, for given $i \in \{1, \ldots, m\}$, we have

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in F_i) \le -\inf\{I_i(x_i) \mid x_i \in F_i\} \tag{2.9}$$

for every closed set $F_i \subseteq \mathbf{R}^{d_i}$ if and only if for every $\alpha > 0$,

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in F_i) \le -\alpha \tag{2.10}$$

for every closed set $F_i \subseteq \Psi_{I_i}(\alpha)^c$. We refer to Dembo and Zeitouni (1998), Section 1.2, for the background. Requirement (2.8) is stronger than (2.10). Thus the assumptions of Theorem 2.1 imply upper bound (2.9) for every closed set $F_i \subseteq \mathbf{R}^{d_i}$ for $i = 1, \ldots, m$.

The pair of conditions (1.5) and (2.9) was assumed in Section 1 for upper bounds (1.2) for the closed sets. It is clarifying to compare requirement (2.8) with this pair. If $\{Y_{ni}\}$ is a one-dimensional random walk then (2.8) is always satisfied with the rate function of Cramér's theorem. This is a consequence of Theorem 2.2.3 of Dembo and Zeitouni (1998) and Theorems 2.2 and 3.4 below. It is seen that there are interesting processes such that (2.8) holds but (1.5) fails. On the other hand, let $Y_{ni} \equiv 1$ for every $n \in \mathbf{N}$. Take $I_i(0) = 0$ and $I_i(x_i) = +\infty$ for $x_i \neq 0$. Then (1.5) and (2.9) are satisfied but (2.8) fails for every $\alpha > 0$. In this sense, requirement (2.8) is independent of the above pair of conditions.

Upper bounds (2.9) are much more studied in the literature than (2.8). To get some examples, see Ellis (1984), Theorem II.2, Slaby (1988), Corollary 3.2, and Nyrhinen (2001), Theorem 3.1. Motivated by this, we next consider conditions which together with (2.9) imply (2.8).

Theorem 2.2. Let $\{Y_{ni}\}$ be a component process and I_i a rate function as in Theorem 2.1 for $i \in \{1, \ldots, m\}$. Assume that upper bound (2.9) is satisfied for every closed set $F_i \subseteq \mathbf{R}^{d_i}$ and that the restriction of I_i to $\mathcal{D}(I_i)$ is continuous. Then condition (2.8) is satisfied for every $\alpha > 0$ if and only if

$$(2.11) \qquad \lim_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in (\operatorname{cl} \mathcal{D}(I_i))^c) = -\infty.$$

The following facts show that the conditions of Theorem 2.2 are satisfied in various general cases. The above list of existing results shows that upper bounds (2.9) hold for many interesting processes. The continuity requirement for the rate function I_i is satisfied if $d_i = 1$ and I_i is convex. This follows easily from the lower semicontinuity and the convexity of I_i. For a multidimensional convex rate function I_i, the requirement is satisfied if $\mathcal{D}(I_i)$ is open. We refer to Rockafellar (1970), Theorem 10.1. Finally, limit (2.11) automatically holds if upper bounds (2.9) are satisfied and $\mathcal{D}(I_i)$ is open. Namely, we then have the stronger estimate

$$(2.12) \qquad \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in \mathcal{D}(I_i)^c) \leq -\inf\{I_i(x_i) \mid x_i \in \mathcal{D}(I_i)^c\} = -\infty.$$

We end the section with a partial converse of Theorem 2.1.

Theorem 2.3. Let the processes $\{Y_{n1}\}, \ldots, \{Y_{nm}\}$ and $\{Y_n\}$ be as in Theorem 2.1 and $I_1, \ldots, I_m$ rate functions on $\mathbf{R}^{d_1}, \ldots, \mathbf{R}^{d_m}$ respectively. Let I be as in (2.7) and $i_1, i_2 \in \{1, \ldots, m\}$ two different indices. Assume that $\{Y_{ni_1}/n\}$ satisfies the large deviations principle with the rate function I_{i_1}. Assume further that (1.5) fails for $i = i_1$ and that (2.8) fails for $i = i_2$ for some $\alpha > 0$. Then there exists a closed set $F \subseteq \mathbf{R}^d$ such that

$$(2.13) \qquad \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F) > -\inf\{I(x) \mid x \in F\}.$$

3. Refinements in the framework of the Gärtner-Ellis theorem

We specify and extend in this section results of Section 2 in the framework of the Gärtner-Ellis theorem. A background for the theorem is given in Section 3.1. General cases where the component processes are one-dimensional are studied in Section 3.2. The results will be further applied in Section 3.3 to the case where the components of $\{Y_n\}$ are appropriate increments of a one-dimensional random walk. In Section 3.4, we consider refinements for random walks.

The process $\{Y_n\}$ to be studied will be a composition as in (2.4). The components $Y_{n1}, \ldots, Y_{nm}$ will always be independent for every $n \in \mathbf{N}$. We make use of the notations of Section 2 except that the rate functions will be specific.

3.1. Background for the Gärtner-Ellis theorem

We give in this section a brief description of the Gärtner-Ellis theorem and connect it with our setup. For more information, we refer to Ellis (1984), de Acosta (1985) and Sections 2.3, 4.5 and 6.1 of Dembo and Zeitouni (1998).

We need here a new concept. Let k be a positive integer and $f : \mathbf{R}^k \to \mathbf{R} \cup \{\pm\infty\}$ an arbitrary function. The *Fenchel-Legendre transform* f^* of f is defined by

$$f^*(y) = \sup\{< \gamma, y > -f(\gamma) \mid \gamma \in \mathbf{R}^k\} \tag{3.1}$$

for $y \in \mathbf{R}^k$ where $< \gamma, y >$ denotes the Euclidean inner product of γ and y.

To describe the Gärtner-Ellis theorem, let $\{Y_{ni}\}$ be as in Theorem 2.1 for $i \in \{1, \ldots, m\}$. Write

$$\Lambda_i(\lambda_i) = \limsup_{n\to\infty} n^{-1} \log \mathbf{E}_n\{e^{<\lambda_i, Y_{ni}>}\} \in \mathbf{R} \cup \{\pm\infty\} \tag{3.2}$$

for $\lambda_i \in \mathbf{R}^{d_i}$ where $\mathbf{E}_n$ denotes the expectation operator on $(\Omega_n, S_n, \mathbf{P}_n)$. Then Λ_i is convex and Λ_i^* a convex rate function. The Gärtner-Ellis theorem gives sufficient conditions for the large deviations principle for the sequence $\{Y_{ni}/n\}$ with the rate function Λ_i^*. Upper bound (2.9) holds for every compact set $F_i \subseteq \mathbf{R}^{d_i}$ without any assumptions. It holds for every closed set F_i if $0 \in \operatorname{int} \mathcal{D}(\Lambda_i)$. The large deviations lower bounds for the open sets are guaranteed under suitable additional conditions on limit (3.2).

Let $\{Y_n\}$ be as in Theorem 2.1. We consider throughout Section 3 the special case where the rate function associated with the component sequence $\{Y_{ni}/n\}$ is Λ_i^* for $i = 1, \ldots, m$. The rate function I for the sequence $\{Y_n/n\}$ will be as in (2.7) so that

$$I(x) = \Lambda_1^*(x_1) + \cdots + \Lambda_m^*(x_m) \tag{3.3}$$

for $x = (x_1, \ldots, x_m) \in \mathbf{R}^d$. Another natural candidate would be Λ^* where

$$\Lambda(\lambda) = \limsup_{n\to\infty} n^{-1} \log \mathbf{E}_n\{e^{<\lambda, Y_n>}\} \tag{3.4}$$

for $\lambda \in \mathbf{R}^d$. It is again the rate function of the Gärtner-Ellis theorem. Because the vectors $Y_{n1}, \ldots, Y_{nm}$ are independent we always have

$$\Lambda^*(x) \geq \Lambda_1^*(x_1) + \cdots + \Lambda_m^*(x_m) = I(x) \tag{3.5}$$

for $x = (x_1, \ldots, x_m) \in \mathbf{R}^d$. As discussed above, upper bound (1.2) is satisfied with the rate function Λ^* for every compact set $F \subseteq \mathbf{R}^d$. Thus to have sharp upper bounds with the rate function I, we should generally have equality in (3.5). It is easy to see that $I = \Lambda^*$ in the case where (3.2) holds as the limit for every $x_i \in \mathbf{R}^{d_i}$ for $i = 1, \ldots, m$.

Consider now the case of Cramér's theorem, namely, assume that $\{Y_{ni}\}$ is a random walk for $i \in \{1, \ldots, m\}$. The rate function of the theorem is Λ_i^* where now Λ_i is simply the cumulant generating function of Y_{1i}. The sequence $\{Y_{ni}/n\}$ satisfies the large deviations lower bounds for the open sets with the rate function Λ_i^* without any further assumptions. The large deviations principle is satisfied if $\{Y_{ni}\}$ is one-dimensional. Finally, I of (3.3) equals Λ^* if $\{Y_{n1}\}, \ldots, \{Y_{nm}\}$ are independent random walks.

3.2. General cases with one-dimensional components

We assume in this section that the component processes $\{Y_{n1}\}, \ldots, \{Y_{nm}\}$ are one-dimensional so that $d_i = 1$ for each i and $m = d$. As earlier, assume that $d \geq 2$ and that $Y_{n1}, \ldots, Y_{nd}$ are independent for every $n \in \mathbf{N}$. The standard sufficient condition for upper bounds (1.2) for the closed sets with the rate function Λ^* requires that $0 \in \operatorname{int} \mathcal{D}(\Lambda)$. Extensions are included in Theorem 3.1 below.

Let Λ_i be as in (3.2) for $i = 1, \ldots, d$. We consider three cases (C1), (C2) and (C3).

(C1) Condition (1.5) is satisfied for every $i \in \{1, \ldots, d\}$.

(C2) Condition (2.11) is satisfied with $I_i = \Lambda_i^*$ for every $i \in \{1, \ldots, d\}$.

(C3) Both (1.5) and (2.11) with $I_i = \Lambda_i^*$ hold for every $i \in \{1, \ldots, d\} \setminus \{i_0\}$ where $i_0 \in \{1, \ldots, d\}$.

Theorem 3.1. Let the processes $\{Y_{n1}\}, \ldots, \{Y_{nd}\}$ be one-dimensional as described above and $\{Y_n\}$ as in (2.4). Let the functions $\Lambda_1, \ldots, \Lambda_d$ and I be as in (3.2) and (3.3). Assume that Λ_i is lower semicontinuous at the origin i.e. that

$$\liminf_{\lambda_i \to 0} \Lambda_i(\lambda_i) \geq 0 \tag{3.6}$$

for every $i \in \{1, \ldots, d\}$. Then upper bound (1.2) holds for every closed set $F \subseteq \mathbf{R}^d$ in each case (C1), (C2) and (C3).

We next consider the large deviations principle for the sequence $\{Y_n/n\}$ by starting with the large deviations principles for the component sequences. Recall from Section 1 that then lower bounds (1.3) are automatically satisfied with the rate function of (1.4). The following result shows that conditions (C1) - (C3) completely describe upper bounds (1.2) in the present case.

Theorem 3.2. Let the processes $\{Y_{n1}\}, \ldots, \{Y_{nd}\}$ and $\{Y_n\}$ and the functions $\Lambda_1, \ldots, \Lambda_d$ and I be as in Theorem 3.1. Assume that the sequence $\{Y_{ni}/n\}$ satisfies the large deviations principle with the rate function Λ_i^* for $i = 1, \ldots, d$. Then the sequence $\{Y_n/n\}$ satisfies the large deviations principle with the rate function I if and only if at least one of conditions (C1) - (C3) is satisfied. Further, there exists no large deviations principle for the sequence $\{Y_n/n\}$ in the case where every condition (C1) - (C3) fails.

3.3. Large deviations for consecutive increments of a random walk in R

We apply in this section results of Section 3.2 to the case where the components are consecutive increments of a one-dimensional random walk. This process gives an example where the existence and the form of the large deviations principle can be described by means of simple conditions.

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed real valued random variables on the probability space $(\Omega, S, \mathbf{P})$. Let $Z_n = \xi_1 + \cdots + \xi_n$ for $n \in \mathbf{N}$ and $Z_0 \equiv 0$. Thus $\{Z_n \mid n = 1, 2, \ldots\}$ is a random walk in $\mathbf{R}$. Let $d \geq 2$ be an integer and $0 = t_0 < t_1 < \ldots < t_{d-1} < t_d = 1$ fixed real numbers. We consider the process $\{Y_n\}$ defined by

$$Y_n = (Y_{n1}, \ldots, Y_{nd}) = (Z_{[nt_1]} - Z_{[nt_0]}, \ldots, Z_{[nt_d]} - Z_{[nt_{d-1}]}) \tag{3.7}$$

for $n \in \mathbf{N}$ ($[a]$ denotes the integer part of $a \in \mathbf{R}$). Obviously, $Y_{n1}, \ldots, Y_{nd}$ are independent for every $n \in \mathbf{N}$.

Large deviations for the above process are of their own interest. They also have strong connections with the sample path large deviations. See Section 5.1 in Dembo and Zeitouni (1998).

Let $i \in \{1, \ldots, d\}$. Recall the definitions of the functions Λ_i, Λ and I from Section 3.1. The functions Λ_i and Λ_i^* have simple forms in the present case. Denote by Γ the cumulant generating function of ξ. It is easy to see that

$$\Lambda_i(\lambda_i) = (t_i - t_{i-1})\,\Gamma(\lambda_i) \tag{3.8}$$

for every $\lambda_i \in \mathbf{R}$ and

$$\Lambda_i^*(x_i) = (t_i - t_{i-1})\,\Gamma^*(x_i/(t_i - t_{i-1})) \tag{3.9}$$

for every $x_i \in \mathbf{R}$. It is also obvious that $I = \Lambda^*$.

Let $\mathcal{C}$ be the smallest convex subset of $\mathbf{R}$ such that $\mathbf{P}(\xi \in \mathcal{C}) = 1$. It is clear that $\mathcal{C}$ is well defined. We need the following conditions.

(C4) $0 \in \operatorname{int} \mathcal{D}(\Gamma)$.

(C5) $\mathcal{C}$ is open.

(C6) $\mathcal{C} = (-\infty, 0]$ or $\mathcal{C} = [0, \infty)$.

Corollary 3.3. Let the process $\{Y_n\}$ be as in (3.7) and Λ as in (3.4). Then the sequence $\{Y_n/n\}$ satisfies the large deviations principle with the rate function Λ^* if and only if at least one of conditions (C4) - (C6) is satisfied. Further, there exists no large deviations principle for the sequence $\{Y_n/n\}$ in the case where every condition (C4) - (C6) fails.

To see that every condition (C4) - (C6) may fail, take $\mathbf{P}(\xi = k) = a/k^2$ for $k = 1, 2, \ldots$ where a is a normalizing constant. Then $\mathcal{D}(\Gamma) = (-\infty, 0]$ and $\mathcal{C} = [1, \infty)$.

3.4. Refinements for random walks

We derive in this section a refinement of Theorem 2.2 in the case where a component sequence is a random walk. We also study the case where $\{Y_n\}$ is a random walk with independent one-dimensional components.

Let $\{Y_{ni}\}$ be a component process for $i \in \{1, \ldots, m\}$ as in Theorem 2.2. A big part of the following result is covered by Theorem 2.4 of Bahadur and Zabell (1979).

Theorem 3.4. Let $\{Y_{ni}\}$ be a random walk in $\mathbf{R}^{d_i}$ and Λ_i the cumulant generating function of Y_{1i} for $i \in \{1, \ldots, m\}$. Then for every $n \in \mathbf{N}$,

$$\mathbf{P}_n(Y_{ni}/n \in (\operatorname{cl} \mathcal{D}(\Lambda_i^*))^c) = 0. \tag{3.10}$$

It is seen that $\{Y_{ni}\}$ satisfies condition (2.11) with $I_i = \Lambda_i^*$. The other conditions of Theorem 2.2 may fail. See Example 2.1 in Bahadur and Zabell (1979) and Example 2.1 in Dinwoodie (1991).

Consider now a case where $\{Y_n\}$ itself is a random walk.

Theorem 3.5. Let $\{Y_n\}$ be a random walk in $\mathbf{R}^d$ such that for every $n \in \mathbf{N}$, the one-dimensional components $Y_{n1}, \ldots, Y_{nd}$ of Y_n are independent. Let Λ be the cumulant generating function of Y_1. Then the sequence $\{Y_n/n\}$ satisfies the large deviations principle with the rate function Λ^*.

4. Proofs

Proof of Theorem 2.1. Let $F \subseteq \mathbf{R}^d$ be closed. We have to prove (1.2) where I is as in (2.7). Fix small $\varepsilon > 0$ and large α. Let $N = [\alpha/\varepsilon] + 1$. Define the set F_{ij} by

$$F_{ij} = \{x_i \in \mathbf{R}^{d_i} \mid I_i(x_i) \in [(j-1)\varepsilon, j\varepsilon]\} \tag{4.1}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, N$. Let $j_1, \ldots, j_m \in \{1, \ldots, N\}$ and

$$A_{j_1 \ldots j_m} = F \cap (F_{1j_1} \times \cdots \times F_{mj_m}). \tag{4.2}$$

We begin by proving that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in A_{j_1\ldots j_m}) \leq -\inf\{I(x) \mid x \in F\} + m\varepsilon. \tag{4.3}$$

We may assume that $A_{j_1\ldots j_m}$ is non-empty. By (2.8), we have for $j \geq 2$ and small $\delta > 0$,

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in F_{ij}) \leq -(j-1)\varepsilon + \delta \tag{4.4}$$

for every $i \in \{1, \ldots, m\}$. By letting δ tend to zero, we conclude that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in F_{ij}) \leq -(j-1)\varepsilon. \tag{4.5}$$

For $j = 1$, inequality (4.5) holds trivially. By (4.5) and our assumption of independence,

$$\begin{aligned} &\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in A_{j_1\ldots j_m}) \\ &\leq \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F_{1j_1} \times \cdots \times F_{mj_m}) \\ &\leq -\sum_{i=1}^{m} (j_i - 1)\varepsilon. \end{aligned} \tag{4.6}$$

Because $A_{j_1\ldots j_m}$ is non-empty we have by (2.7), (4.1) and (4.2),

$$\begin{aligned} \sum_{i=1}^{m} j_i\varepsilon &\geq \inf\{I(x) \mid x \in A_{j_1\ldots j_m}\} \\ &\geq \inf\{I(x) \mid x \in F\}. \end{aligned} \tag{4.7}$$

This and (4.6) imply (4.3).

Denote

$$H_i = \{(x_1, \ldots, x_m) \in \mathbf{R}^d \mid x_i \in \Psi_{I_i}(\alpha)^c\} \tag{4.8}$$

for $i = 1, \ldots, m$. By (2.8),

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in H_i) \le -\alpha. \tag{4.9}$$

It follows from the definitions of the sets F_{ij} and H_i that

$$F \subseteq \bigcup_{i=1}^{m} H_i \bigcup_{j_1,\ldots,j_m=1}^{N} A_{j_1\ldots j_m}. \tag{4.10}$$

By (4.3), (4.9), (4.10) and Lemma 1.2.15 of Dembo and Zeitouni (1998),

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F) \le \max\{-\inf\{I(x) \mid x \in F\} + m\varepsilon, -\alpha\}. \tag{4.11}$$

Upper bound (1.2) follows by letting ε tend to zero and α to infinity.

Proof of Theorem 2.2. Assume that (2.11) is satisfied. Let $\alpha > 0$. We have to prove (2.8). Denote

$$K_i = \{x_i \in \mathbf{R}^{d_i} \mid I_i(x_i) \in (\alpha, \infty)\}. \tag{4.12}$$

Because I_i is continuous on $\mathcal{D}(I_i)$ we have $I_i(x_i) \ge \alpha$ for every $x_i \in \operatorname{cl} K_i$. By (2.9),

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in \operatorname{cl} K_i) \le -\alpha. \tag{4.13}$$

For arbitrary $x_i \in \mathcal{D}(I_i)^c \cap \operatorname{cl} \mathcal{D}(I_i)$, x_i is a boundary point of $\mathcal{D}(I_i)$ and $I_i(x_i) = +\infty$. It follows from the lower semicontinuity of I_i that $x_i \in \operatorname{cl} K_i$. Hence,

$$\begin{aligned} \Psi_{I_i}(\alpha)^c \cap \operatorname{cl} \mathcal{D}(I_i) &= (K_i \cup \mathcal{D}(I_i)^c) \cap \operatorname{cl} \mathcal{D}(I_i) \\ &= (K_i \cap \operatorname{cl} \mathcal{D}(I_i)) \cup (\mathcal{D}(I_i)^c \cap \operatorname{cl} \mathcal{D}(I_i)) \subseteq \operatorname{cl} K_i. \end{aligned} \tag{4.14}$$

Thus

$$\begin{aligned} \Psi_{I_i}(\alpha)^c &= (\Psi_{I_i}(\alpha)^c \cap \operatorname{cl} \mathcal{D}(I_i)) \cup (\Psi_{I_i}(\alpha)^c \cap (\operatorname{cl} \mathcal{D}(I_i))^c) \\ &\subseteq (\operatorname{cl} K_i) \cup (\operatorname{cl} \mathcal{D}(I_i))^c. \end{aligned} \tag{4.15}$$

We obtain (2.8) by (2.11), (4.13) and Lemma 1.2.15 of Dembo and Zeitouni (1998).

Assume now that (2.11) fails. Then

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{ni}/n \in (\operatorname{cl} \mathcal{D}(I_i))^c) = -\beta \tag{4.16}$$

for some $\beta \in [0, \infty)$. Clearly, $(\mathrm{cl}\,\mathcal{D}(I_i))^c \subseteq \Psi_{I_i}(\alpha)^c$ for every $\alpha > 0$. It follows from (4.16) that (2.8) fails for every $\alpha > \beta$.

Proof of Theorem 2.3. By symmetry, we may assume that $i_1 = 1$ and $i_2 = 2$. We begin by associating appropriate sequences of closed sets with $\{Y_{n1}/n\}$ and $\{Y_{n2}/n\}$.

Consider first the sequence $\{Y_{n1}/n\}$. Because (1.5) fails we have

$$\lim_{M\to\infty} \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(| Y_{n1}/n |> M) = -\beta \tag{4.17}$$

for some $\beta \in [0, \infty)$. Let $\delta_1 > 0$ be arbitrary. Denote by $B(x_1, r)$ the open ball of $\mathbf{R}^{d_1}$ with the centre x_1 and the radius r. We show that there exist vectors $x_{11}, x_{12}, x_{13}, \ldots$ of $\mathbf{R}^{d_1}$ such that

$$\lim_{j\to\infty} | x_{1j} |= +\infty \tag{4.18}$$

and that for every $j \in \mathbf{N}$,

$$\inf\{I_1(x_1) \mid x_1 \in \mathrm{cl}\,B(x_{1j}, 1)\} \geq \beta - \delta_1 \tag{4.19}$$

and

$$\liminf_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{n1}/n \in \mathrm{cl}\,B(x_{1j}, 1)) \geq -\beta - \delta_1. \tag{4.20}$$

For given $M > 0$, we have by the large deviations upper bounds,

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(| Y_{n1}/n |\geq M) \leq -\inf\{I_1(x_1) \mid | x_1 |\geq M\}. \tag{4.21}$$

Combined with (4.17), it is seen that

$$\inf\{I_1(x_1) \mid | x_1 |\geq M\} \leq \beta. \tag{4.22}$$

This is true for every $M > 0$ so that we may find vectors $x_{11}, x_{12}, x_{13}, \ldots$ of $\mathbf{R}^{d_1}$ such that (4.18) is satisfied and

$$I_1(x_{1j}) \leq \beta + \delta_1 \tag{4.23}$$

for $j = 1, 2, \ldots$. By this and the large deviations lower bounds,

$$\begin{aligned} &\liminf_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{n1}/n \in \mathrm{cl}\,B(x_{1j}, 1)) \\ &\geq \liminf_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{n1}/n \in B(x_{1j}, 1)) \\ &\geq -I_1(x_{1j}) \geq -\beta - \delta_1 \end{aligned} \tag{4.24}$$

for $j = 1, 2, \ldots$. Thus (4.20) is satisfied. Consider condition (4.19) for the above sequence $x_{11}, x_{12}, x_{13}, \ldots$ satisfying (4.18) and (4.20). By the large deviations lower bounds,

$$\liminf_{n\to\infty} n^{-1} \log \mathbf{P}_n(\mid Y_{n1}/n \mid > M) \geq -\inf\{I_1(x_1) \mid\mid x_1 \mid > M\}. \tag{4.25}$$

By (4.17) and (4.25), we may fix $M_0 > 0$ such that

$$\inf\{I_1(x_1) \mid\mid x_1 \mid > M_0\} \geq \beta - \delta_1. \tag{4.26}$$

Thus $I_1(x_1) \geq \beta - \delta_1$ whenever $\mid x_1 \mid > M_0$. It is seen by (4.18) that (4.19) is satisfied for large j. Without a loss of generality, we may assume that it is satisfied for each j.

Consider now the sequence $\{Y_{n2}/n\}$. Let $\alpha > 0$ be such that (2.8) fails for $i = 2$. Fix $\delta_2 > 0$ such that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{n2}/n \in \Psi_{I_2}(\alpha)^c) \geq -\alpha + 3\delta_2. \tag{4.27}$$

For given $\varepsilon > 0$, denote

$$\Psi_{I_2}(\alpha, \varepsilon) = \bigcup_{x_2 \in \Psi_{I_2}(\alpha)} B(x_2, \varepsilon) \tag{4.28}$$

where $B(x_2, \varepsilon)$ is the open ball of $\mathbf{R}^{d_2}$ with the centre x_2 and the radius ε. We show that there exist positive real numbers $\varepsilon_1, \varepsilon_2, \ldots$ and positive integers $n_1 < n_2 < \ldots$ such that

$$\mathbf{P}_{n_k}(Y_{n_k 2}/n_k \in \Psi_{I_2}(\alpha, \varepsilon_k)^c) \geq e^{-n_k(\alpha - \delta_2)} \tag{4.29}$$

for $k = 1, 2, \ldots$. By (4.27), we may find $0 < n_1 < n_2 < \ldots$ such that

$$\mathbf{P}_{n_k}(Y_{n_k 2}/n_k \in \Psi_{I_2}(\alpha)^c) \geq e^{-n_k(\alpha - 2\delta_2)} \tag{4.30}$$

for $k = 1, 2, \ldots$. It is easy to see by the lower semicontinuity of I_2 that

$$\bigcap_{\varepsilon > 0} \Psi_{I_2}(\alpha, \varepsilon) = \Psi_{I_2}(\alpha). \tag{4.31}$$

It follows that

$$\mathbf{P}_{n_k}(Y_{n_k 2}/n_k \in \Psi_{I_2}(\alpha)^c) = \lim_{\varepsilon \to 0+} \mathbf{P}_{n_k}(Y_{n_k 2}/n_k \in \Psi_{I_2}(\alpha, \varepsilon)^c) \tag{4.32}$$

for $k = 1, 2, \ldots$. By (4.30), we may find $\varepsilon_1, \varepsilon_2, \ldots$ such that (4.29) holds.

We next put the closed balls from (4.20) and sets from (4.29) together. By (4.20) and (4.29), we may find positive integers $n_{k_1} < n_{k_2} < \ldots$ such that

$$\mathbf{P}_{n_{k_j}}(Y_{n_{k_j} 1}/n_{k_j} \in \operatorname{cl} B(x_{1j}, 1)) \geq e^{-n_{k_j}(\beta + 2\delta_1)} \tag{4.33}$$

and

$$\mathbf{P}_{n_{k_j}}(Y_{n_{k_j}2}/n_{k_j} \in \Psi_{I_2}(\alpha, \varepsilon_{k_j})^c) \geq e^{-n_{k_j}(\alpha-\delta_2)} \tag{4.34}$$

for $j = 1, 2, \ldots$. Choose

$$F = \bigcup_{j=1}^{\infty} (\operatorname{cl} B(x_{1j}, 1) \times \Psi_{I_2}(\alpha, \varepsilon_{k_j})^c \times \mathbf{R}^{d_3+\cdots+d_m}). \tag{4.35}$$

The members of union (4.35) are closed. It follows from (4.18) that also F is closed. By (4.33), (4.34) and the independence of the components,

$$\begin{aligned} &\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F) \\ &\geq \limsup_{j\to\infty} n_{k_j}^{-1} \log \mathbf{P}_{n_{k_j}}((Y_{n_{k_j}1}, Y_{n_{k_j}2})/n_{k_j} \in \operatorname{cl} B(x_{1j}, 1) \times \Psi_{I_2}(\alpha, \varepsilon_{k_j})^c) \\ &\geq -\beta - 2\delta_1 - \alpha + \delta_2. \end{aligned} \tag{4.36}$$

On the other hand, we have by (4.19) and the definition of $\Psi_{I_2}(\alpha)$,

$$\begin{aligned} &\inf\{I(x) \mid x \in F\} \\ &\geq \inf\{I_1(x_1) + I_2(x_2) \mid (x_1, x_2) \in \bigcup_{j=1}^{\infty} (\operatorname{cl} B(x_{1j}, 1) \times \Psi_{I_2}(\alpha, \varepsilon_{k_j})^c)\} \\ &\geq \beta - \delta_1 + \alpha. \end{aligned} \tag{4.37}$$

Combined with (4.36), we conclude that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F) \geq -\inf\{I(x) \mid x \in F\} - 3\delta_1 + \delta_2. \tag{4.38}$$

We obtain (2.13) by choosing δ_1 small enough. The proof of Theorem 2.3 is completed.

Before proving Theorem 3.1, we state a technical lemma.

Lemma 4.1. Let Λ_i be as in Theorem 3.1 for $i \in \{1, \ldots, d\}$. Assume that (3.6) is satisfied. If $\Lambda_i(\lambda_i)$ is finite for some $\lambda_i > 0$ then

$$\lim_{x_i\to\infty} \Lambda_i^*(x_i) = +\infty \tag{4.39}$$

and if $\Lambda_i(\lambda_i)$ equals $+\infty$ for every $\lambda_i > 0$ then

$$\lim_{x_i\to\infty} \Lambda_i^*(x_i) = 0. \tag{4.40}$$

Of course, the analogous results hold for the limits when x_i tends to $-\infty$. The proof of (4.39) is easy. Limit (4.40) is included in the proof of Theorem 3.1 in Nyrhinen (2001) and is not repeated here.

Proof of Theorem 3.1. Assume that (C1) is satisfied. Upper bound (1.2) holds for every compact F with the rate function Λ^* as discussed in Section 3.1. It holds for every closed F because by (C1), the sequence of the distributions of $\{Y_n/n\}$ is exponentially tight. See Lemma 1.2.18 of Dembo and Zeitouni (1998). By (3.5), upper bound (1.2) holds for every closed F with the rate function I.

Assume that (C2) is satisfied. We apply Theorems 2.1 and 2.2. Let $i \in \{1, \ldots, d\}$. By (3.6) and Theorem 3.1 of Nyrhinen (2001), upper bound (2.9) holds for every closed set $F_i \subseteq \mathbf{R}$ with the rate function Λ_i^*. As observed in Section 2, the restriction of Λ_i^* to $\mathcal{D}(\Lambda_i^*)$ is continuous because Λ_i^* is one-dimensional and convex. Condition (2.11) is satisfied by (C2). By Theorem 2.2, (2.8) holds for every $\alpha > 0$ with $I_i = \Lambda_i^*$. By Theorem 2.1, (1.2) holds with the rate function of (3.3) for every closed set $F \subseteq \mathbf{R}^d$.

Assume that (C3) is satisfied. By symmetry, it is sufficient to consider the case where $i_0 = d$. Thus we assume that (1.5) and (2.11) are satisfied for $i = 1, \ldots, d-1$. Let $F \subseteq \mathbf{R}^d$ be closed and I as in (3.3). We have to prove (1.2).

Fix small $\varepsilon > 0$ and large $\alpha > 0$. Let

$$F_1 = F \cap (\mathbf{R}^{d-1} \times [0, \infty)) \tag{4.41}$$

and

$$F_2 = F \cap (\mathbf{R}^{d-1} \times (-\infty, 0]). \tag{4.42}$$

Consider the set F_1. We prove that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F_1) \leq -\inf\{I(x) \mid x \in F\}. \tag{4.43}$$

As observed in the part of the proof where (C2) was assumed, upper bound (2.9) holds for every closed set $F_i \subseteq \mathbf{R}$ with $I_i = \Lambda_i^*$ for $i = 1, \ldots, d$. By Lemma 4.1, either (4.39) or (4.40) holds for the function Λ_d^*.

Consider first the case where (4.39) holds for Λ_d^*. Then by the large deviations upper bounds,

$$\lim_{x_d\to\infty} \limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_{nd}/n \geq x_d) = -\infty. \tag{4.44}$$

By (1.5) for $i = 1, \ldots, d-1$, (4.44) and Lemma 1.2.15 of Dembo and Zeitouni (1998), we may fix $y > 0$ such that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in ([-y,y]^{d-1} \times (-\infty, y])^c) \le -\alpha. \tag{4.45}$$

Recall that $F_1 \subseteq \mathbf{R}^{d-1} \times [0,\infty)$. For the compact set

$$F_0 = F_1 \cap ([-y,y]^{d-1} \times (-\infty, y]), \tag{4.46}$$

we have

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F_0) \le -\inf\{\Lambda^*(x) \mid x \in F_0\} \tag{4.47}$$

where Λ is as in (3.4). See Section 3.1. By using Lemma 1.2.15 of Dembo and Zeitouni (1998), we obtain by (3.5), (4.45) and (4.47),

$$\begin{aligned}\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F_1) &\le \max\{-\inf\{\Lambda^*(x) \mid x \in F_0\}, -\alpha\} \\ &\le \max\{-\inf\{I(x) \mid x \in F_0\}, -\alpha\} \\ &\le \max\{-\inf\{I(x) \mid x \in F\}, -\alpha\}.\end{aligned} \tag{4.48}$$

We obtain (4.43) by letting α tend to infinity.

Consider now (4.43) in the case where (4.40) holds for the function Λ_d^*. Fix $y > 0$ such that $\Lambda_d^*(x_d) \le \varepsilon$ for every $x_d \ge y$. Write $F_1 = F_1' \cup F_1''$ where

$$F_1' = F_1 \cap (\mathbf{R}^{d-1} \times [0, y]) \tag{4.49}$$

and

$$F_1'' = F_1 \cap (\mathbf{R}^{d-1} \times [y, \infty)). \tag{4.50}$$

It is seen similarly to the case where (4.39) holds that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}_n(Y_n/n \in F_1') \le -\inf\{I(x) \mid x \in F\}. \tag{4.51}$$

Consider the set F_1''. Recall from (4.1) the definitions of the sets F_{ij} and that now $I_i = \Lambda_i^*$ and $m = d$. Let also $N = [\alpha/\varepsilon] + 1$ be as in the proof of Theorem 2.1. For $j_1, \ldots, j_{d-1} \in \{1, \ldots, N\}$, denote

$$B_{j_1 \ldots j_{d-1}} = F_1'' \cap (F_{1j_1} \times \cdots \times F_{d-1, j_{d-1}} \times [y, \infty)). \tag{4.52}$$

We prove that

$$\limsup_{n\to\infty} n^{-1}\log \mathbf{P}_n(Y_n/n \in B_{j_1\dots j_{d-1}}) \le -\inf\{I(x) \mid x \in F\} + d\varepsilon. \tag{4.53}$$

We may assume that $B_{j_1\dots j_{d-1}}$ is non-empty. It follows from (C3) and Theorem 2.2 that (2.8) holds for every $\alpha > 0$ and every $i \in \{1,\dots,d-1\}$. See the part of the proof where (C2) was assumed. We conclude as in (4.5) and (4.6) that

$$\begin{aligned}\limsup_{n\to\infty} n^{-1}\log \mathbf{P}_n(Y_n/n \in B_{j_1\dots j_{d-1}}) \\ \le \limsup_{n\to\infty} n^{-1}\log \mathbf{P}_n((Y_{n1},\dots,Y_{n,d-1})/n \in F_{1j_1}\times\cdots\times F_{d-1,j_{d-1}}) \\ \le -\sum_{i=1}^{d-1}(j_i-1)\varepsilon.\end{aligned} \tag{4.54}$$

Recall that $\Lambda_d^*(x_d) \le \varepsilon$ for every $x_d \ge y$ and that I has representation (3.3). Because $B_{j_1\dots j_{d-1}}$ is non-empty we see as in (4.7) that

$$\begin{aligned}\sum_{i=1}^{d-1} j_i\varepsilon \ge \inf\{I(x) \mid x \in B_{j_1\dots j_{d-1}}\} - \varepsilon \\ \ge \inf\{I(x) \mid x \in F\} - \varepsilon.\end{aligned} \tag{4.55}$$

We obtain (4.53) by combining this with (4.54).

Let $i \in \{1,\dots,d-1\}$ and H_i as in (4.8). As observed above, (2.8) holds for every $\alpha > 0$. Thus we have estimates (4.9) for the sets $H_1,\dots,H_{d-1}$. Clearly,

$$F_1'' \subseteq \bigcup_{i=1}^{d-1} H_i \bigcup_{j_1,\dots,j_{d-1}=1}^{N} B_{j_1\dots j_{d-1}}. \tag{4.56}$$

By (4.9), (4.53), (4.56) and Lemma 1.2.15 of Dembo and Zeitouni (1998),

$$\limsup_{n\to\infty} n^{-1}\log \mathbf{P}_n(Y_n/n \in F_1'') \le \max\{-\inf\{I(x) \mid x \in F\} + d\varepsilon, -\alpha\}. \tag{4.57}$$

By letting ε tend to zero and α to infinity, we conclude that

$$\limsup_{n\to\infty} n^{-1}\log \mathbf{P}_n(Y_n/n \in F_1'') \le -\inf\{I(x) \mid x \in F\}. \tag{4.58}$$

Because $F_1 = F_1' \cup F_1''$ we obtain (4.43) by (4.51), (4.58) and Lemma 1.2.15 of Dembo and Zeitouni (1998).

By symmetry, (4.43) holds when F_1 is replaced by F_2. Because $F = F_1 \cup F_2$ we conclude as above that (4.43) also holds when F_1 is replaced by F.

Proof of Theorem 3.2. The large deviations lower bounds are satisfied with the rate function I as observed in Section 3.2. By Theorem 3.1 of Nyrhinen (2001), condition (3.6) is satisfied for every $i \in \{1, \ldots, d\}$. Thus by Theorem 3.1 of the present paper, every condition (C1) - (C3) implies the required large deviations principle.

Assume now that every condition (C1) - (C3) fails. We first show that there exists a closed set $F \subseteq \mathbf{R}^d$ such that (1.2) fails with I of (3.3). Obviously, we may pick up an index i_1 such that (1.5) fails, and another index i_2 such that (2.11) fails with $I_{i_2} = \Lambda^*_{i_2}$. This pair of indices is appropriate to apply Theorem 2.3. Clearly, the sequence $\{Y_{ni_1}/n\}$ satisfies the conditions of the theorem with $I_{i_1} = \Lambda^*_{i_1}$. By our assumptions, upper bounds (2.9) hold for the sequence $\{Y_{ni_2}/n\}$ with the rate function $\Lambda^*_{i_2}$. By convexity, $\Lambda^*_{i_2}$ satisfies the continuity requirement of Theorem 2.2. Because (2.11) fails we conclude by Theorem 2.2 that (2.8) fails for $i = i_2$ for some $\alpha > 0$. By Theorem 2.3, upper bound (1.2) fails for some closed set $F \subseteq \mathbf{R}^d$.

To see that there exists no large deviations principle for the sequence $\{Y_n/n\}$ in the case where every condition (C1) - (C3) fails, recall from Section 1 that upper bound (1.2) holds for every compact F and lower bound (1.3) for every open G with the rate function I of (3.3). By the first part of the proof and by the uniqueness theorem, there exists no large deviations principle for the sequence $\{Y_n/n\}$ in the case where every condition (C1) - (C3) fails. See Lemma 4.1.4 of Dembo and Zeitouni (1998) and the discussion thereafter. The proof of Theorem 3.2 is completed.

We next state two lemmata to be used in the proof of Corollary 3.3.

Lemma 4.2. Let $\{Y_{ni}\}$ as in (3.7) for $i \in \{1, \ldots, d\}$. Then $\{Y_{ni}/n\}$ satisfies the large deviations principle with the rate function Λ^*_i.

Lemma 4.3. Let $\mathcal{C}$ and Γ be as in Corollary 3.3. Then

$$\mathcal{C} = \mathcal{D}(\Gamma^*). \tag{4.59}$$

Proof of Lemma 4.2. Every cumulant generating function is lower semicontinuous. By (3.8), so is Λ_i. By Theorem 3.1 of Nyrhinen (2001), upper bounds (2.9) are satisfied with $I_i = \Lambda^*_i$.

Consider the lower bounds. Let $x_i \in \mathbf{R}$ and $\varepsilon > 0$ be arbitrary. Denote $y = x_i/(t_i - t_{i-1})$ and $\varepsilon' = \varepsilon/(2(t_i - t_{i-1}))$. The variable Y_{ni} is a sum of $[nt_i] - [nt_{i-1}]$ independent and identically distributed random variables. By the lower bounds of Cramér's theorem,

$$\liminf_{n\to\infty} ([nt_i] - [nt_{i-1}])^{-1} \log \mathbf{P}(Y_{ni}/([nt_i] - [nt_{i-1}]) \in B(y, \varepsilon')) \geq -\Gamma^*(y) \tag{4.60}$$

where $B(y,\varepsilon')$ is the open ball of $\mathbf{R}$ with the centre y and the radius ε'. Consequently,

$$\liminf_{n\to\infty} n^{-1}\log \mathbf{P}(Y_{ni}/n \in B((t_i - t_{i-1})y,\varepsilon)) \geq -(t_i - t_{i-1})\,\Gamma^*(y). \tag{4.61}$$

By (3.9),

$$\liminf_{n\to\infty} n^{-1}\log \mathbf{P}(Y_{ni}/n \in B(x_i,\varepsilon)) \geq -\Lambda_i^*(x_i). \tag{4.62}$$

This is true for every $x_i \in \mathbf{R}$ and $\varepsilon > 0$. It follows that $\{Y_{ni}/n\}$ satisfies the large deviations lower bounds with the rate function Λ_i^*. See Section 1.2 in Dembo and Zeitouni (1998).

Proof of Lemma 4.3. We first show that

$$\mathcal{C}^c \subseteq \mathcal{D}(\Gamma^*)^c. \tag{4.63}$$

By the convexity of $\mathcal{C}$,

$$\mathcal{C}^c = J_1 \cup J_2 \tag{4.64}$$

where J_1 and J_2 are intervals. Clearly, J_k is either semi-infinite or empty for $k = 1,2$. It follows from Corollary 2.2.19 and Theorem 6.1.3 of Dembo and Zeitouni (1998) that

$$\lim_{n\to\infty} n^{-1}\log \mathbf{P}_n(Z_n/n \in J_k) = -\inf\{\Gamma^*(y) \mid y \in J_k\} \tag{4.65}$$

for $k = 1,2$ (by convention, infimum over the empty set equals $+\infty$). It is clear by convexity that

$$\mathbf{P}(Z_n/n \in \mathcal{C}) = 1 \tag{4.66}$$

for every $n \in \mathbf{N}$. Thus by (4.64), the left hand side of (4.65) equals $-\infty$ so that $\Gamma^*(y) = +\infty$ for every $y \in J_1 \cup J_2$. This implies (4.63).

To complete the proof, it suffices to show that

$$\mathcal{C} \subseteq \mathcal{D}(\Gamma^*). \tag{4.67}$$

Recall that also $\mathcal{D}(\Gamma^*)$ is convex. Write

$$\mathcal{D}(\Gamma^*)^c = J_1' \cup J_2' \tag{4.68}$$

as in (4.64) where J_1' and J_2' are intervals. Let $p_k = \mathbf{P}(\xi \in J_k')$ for $k = 1,2$. By the convexity of J_1' and J_2',

$$\mathbf{P}(Z_n/n \in J_k') \geq \mathbf{P}(\xi_j \in J_k' \text{ for } j = 1,\ldots,n) = p_k^n \tag{4.69}$$

for $k = 1, 2$. It is seen as in (4.65) that

$$-\infty = -\inf\{\Gamma^*(y) \mid y \in J_k'\} = \lim_{n\to\infty} n^{-1} \log \mathbf{P}_n(Z_n/n \in J_k') \geq -\log p_k \tag{4.70}$$

for $k = 1, 2$. It follows that $p_1 = p_2 = 0$. Hence, $\mathbf{P}(\xi \in J_1' \cup J_2') = 0$ so that $\mathbf{P}(\xi \in \mathcal{D}(\Gamma^*)) = 1$. By the definition of $\mathcal{C}$, (4.67) holds.

Proof of Corollary 3.3. Consider sufficiency of conditions (C4) - (C6). By Theorem 3.2 and Lemma 4.2, we have to prove that each of conditions (C4) - (C6) implies some of conditions (C1) - (C3).

Assume that (C4) is satisfied. We show that (C1) holds. By (3.8), $0 \in \operatorname{int} \mathcal{D}(\Lambda_i)$ for $i = 1, \ldots, d$. It is well known that this implies exponential tightness (1.5). See for example the proof of Theorem 2.3.6 of Dembo and Zeitouni (1998). Thus (C1) holds.

Assume that (C5) is satisfied. We show that (C2) holds. By (3.9) and Lemma 4.3, $\mathcal{D}(\Lambda_i^*)$ is open for each i. By Lemma 4.2, this implies (2.11) with $I_i = \Lambda_i^*$. See (2.12). Thus (C2) holds.

Assume finally that (C6) is satisfied. We show that (C2) also holds in this case. Suppose that $\mathcal{C} = [0, \infty)$ (the case where $\mathcal{C} = (-\infty, 0]$ is symmetric). Then $\mathbf{P}(\xi < 0) = 0$. It follows that $\mathbf{P}(Y_{ni} < 0) = 0$ for every $n \in \mathbf{N}$ and $i = 1, \ldots, d$. This implies (2.11) because by (3.9) and Lemma 4.3, $(\operatorname{cl} \mathcal{D}(\Lambda_i^*))^c = (-\infty, 0)$ for each i. Thus (C2) holds.

Assume now that every condition (C4) - (C6) fails. We have to show that $\{Y_n/n\}$ satisfies no large deviations principle. By Theorem 3.2 and Lemma 4.2, we have to prove that every condition (C1) - (C3) fails. To this end, it is sufficient to show that (1.5) fails for every i and that (2.11) fails for some i with $I_i = \Lambda_i^*$.

We begin by describing the set $\mathcal{C}$ in detail. Because condition (C4) is not satisfied we have $\Gamma(\gamma) = +\infty$ for every $\gamma > 0$, say (the case where $\Gamma(\gamma) = +\infty$ for every $\gamma < 0$ is symmetric). By applying Lemma 4.1 to the random walk $\{Z_n\}$, we conclude that

$$\lim_{y\to\infty} \Gamma^*(y) = 0. \tag{4.71}$$

Condition (C5) is not satisfied so that by (4.71) and Lemma 4.3,

$$\mathcal{C} = [y_0, \infty) \tag{4.72}$$

for some $y_0 \in \mathbf{R}$. Finally, we have $y_0 \neq 0$ because condition (C6) is not satisfied.

Let $i \in \{1, \ldots, d\}$. We show that (1.5) fails. Let $M > 0$. By (3.9) and (4.71), $\Lambda_i^*(x_i)$ tends to zero when x_i tends to infinity. By Lemma 4.2,

$$\liminf_{n\to\infty} n^{-1} \log \mathbf{P}(Y_{ni}/n > M) \geq -\inf\{\Lambda_i^*(x_i) \mid x_i > M\}. \tag{4.73}$$

The right hand side of (4.73) equals zero. Thus (1.5) fails.

To complete the proof, we have to show that (2.11) fails for some i with $I_i = \Lambda_i^*$. Assume first that y_0 in (4.72) is negative. We prove that then (2.11) fails for $i = d$. Recall that $t_d = 1$ and $t_{d-1} \in (0,1)$. Obviously, it is possible to find an increasing sequence $n_1, n_2, \ldots$ of positive integers such that

$$[n_k t_d] - [n_k t_{d-1}] = n_k - [n_k t_{d-1}] > n_k(1 - t_{d-1}) \tag{4.74}$$

for every $k \in \mathbf{N}$. By (3.9), (4.72) and Lemma 4.3,

$$\mathcal{D}(\Lambda_d^*) = \operatorname{cl} \mathcal{D}(\Lambda_d^*) = [(1 - t_{d-1})y_0, \infty). \tag{4.75}$$

By the definition of $\mathcal{C}$, $\mathbf{P}(\xi = y_0) > 0$. On the event $\{\xi_1 = y_0, \ldots, \xi_{n_k} = y_0\}$, we have by (4.74),

$$Y_{n_k d} = ([n_k t_d] - [n_k t_{d-1}])y_0 < n_k(1 - t_{d-1})y_0. \tag{4.76}$$

Hence by (4.75),

$$\mathbf{P}(Y_{n_k d}/n_k \in (\operatorname{cl} \mathcal{D}(\Lambda_d^*))^c) \geq \mathbf{P}(\xi_1 = y_0, \ldots, \xi_{n_k} = y_0) = \mathbf{P}(\xi = y_0)^{n_k} \tag{4.77}$$

for every $k \in \mathbf{N}$. Thus (2.11) fails for $i = d$.

Consider finally the case where y_0 in (4.72) is positive. We show that then (2.11) fails for $i = 1$. Corresponding to (4.74), we now find positive integers $n_1, n_2, \ldots$ such that

$$[n_k t_1] - [n_k t_0] = [n_k t_1] < n_k t_1 \tag{4.78}$$

for every $k \in \mathbf{N}$. On the event $\{\xi_1 = y_0, \ldots, \xi_{n_k} = y_0\}$, we have

$$Y_{n_k 1} = [n_k t_1] y_0 < n_k t_1 y_0. \tag{4.79}$$

It is seen as in (4.77) that (2.11) fails for $i = 1$.

Proof of Theorem 3.4. We may consider here a single probability space $(\Omega, S, \mathbf{P})$ instead of a sequence $\{(\Omega_n, S_n, \mathbf{P}_n) \mid n \in \mathbf{N}\}$. Fix $n_0 \in \mathbf{N}$. We have to prove that

$$\mathbf{P}(Y_{n_0 i}/n_0 \in (\operatorname{cl} \mathcal{D}(\Lambda_i^*))^c) = 0. \tag{4.80}$$

Consider an open ball $B(y_i, r) \subseteq (\operatorname{cl} \mathcal{D}(\Lambda_i^*))^c$ where $y_i \in \mathbf{R}^{d_i}$ and $r > 0$. Let $Y_{0i} \equiv 0$. Because $B(y_i, r)$ is convex we have for every $k \in \mathbf{N}$,

$$\mathbf{P}(Y_{kn_0, i}/(kn_0) \in B(y_i, r)) \tag{4.81}$$

$$\geq \mathbf{P}((Y_{jn_0,i} - Y_{(j-1)n_0,i})/n_0 \in B(y_i, r) \text{ for } j = 1, \ldots, k)$$
$$= \mathbf{P}(Y_{n_0 i}/n_0 \in B(y_i, r))^k.$$

It follows that

$$\limsup_{n\to\infty} n^{-1} \log \mathbf{P}(Y_{ni}/n \in B(y_i, r)) \geq n_0^{-1} \log \mathbf{P}(Y_{n_0 i}/n_0 \in B(y_i, r)). \tag{4.82}$$

By Theorem 6.1.3 of Dembo and Zeitouni (1998),

$$\lim_{n\to\infty} n^{-1} \log \mathbf{P}(Y_{ni}/n \in B(y_i, r)) = -\inf\{\Lambda_i^*(x_i) \mid x_i \in B(y_i, r)\} = -\infty. \tag{4.83}$$

Combined with (4.82), we conclude that $\mathbf{P}(Y_{n_0 i}/n_0 \in B(y_i, r)) = 0$. The open set $(\mathrm{cl}\,\mathcal{D}(\Lambda_i^*))^c$ is a countable union of open balls like $B(y_i, r)$. This implies (4.80).

Proof of Theorem 3.5. We apply Theorem 3.2. As outlined in Section 3.1, the one-dimensional component sequence $\{Y_{ni}/n\}$ satisfies the large deviations principle with the rate function Λ_i^* for each i. By Theorem 3.4, condition (C2) is satisfied. By Theorem 3.2, $\{Y_n/n\}$ satisfies the large deviations principle with the rate function I of (3.3). This completes the proof because $I = \Lambda^*$.

References

De Acosta, A. (1985). Upper bounds for large deviations of dependent random vectors. Z. Wahrsch. Verw. Geb. **69** 551-565.

Bahadur, R.R. and Zabell, S.L. (1979). Large deviations of the sample mean in general vector spaces. Ann. Probab. **7** 587-621.

Dembo, A. and Zeitouni, O. (1998). Large deviations techniques and applications. Second edition. Springer, New York.

Dinwoodie, I.H. (1991). A note on the upper bound for i.i.d. large deviations. Ann. Probab. **19** 1732-1736.

Ellis, R.S. (1984). Large deviations for a general class of random vectors. Ann. Probab. **12** 1-12.

Nyrhinen, H. (2001). Large deviations and the Gärtner-Ellis theorem for the sequences of random variables. Reports of the Department of Mathematics 287. University of Helsinki.

Slaby, M. (1988). On the upper bound for large deviations of sums of i.i.d. random vectors. Ann. Probab. **16** 978-990.

Rockafellar, R.T. (1970). Convex analysis. Princeton University Press.