On Cobb–Douglas dynamical exchange economies

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Abstract

If the system is in equilibrium, then the prices prevailing the market exactly clear this market and hence diminish total excess demand (individual demand minus supply) to nothing. However, prices are not per se in equilibrium. The initial offer may be out of equilibrium by most diverse reasons. Such an offer, nevertheless, should generate pressure for a price change. In a fair economy tendency towards equilibrium should arise. As total excess demand is non-zero at initial prices, they would go up if excess demand is positive and go down otherwise. It now depends on agent's behaviour whether the prices will approach the equilibrium as time evolves. Understanding the process of price forming (often called tâtonnement, after Walras) is the subject of vast literature. Recently, principles of stochastic equilibrium attracted considerable interest. We discuss merits of the new theory and possibilities for future developments.

1 Introduction

In this report some ideas of two papers [5] and [11] are worked out in an attempt to understand the nature of the dynamical process towards the so-called Arrow–Debreu equilibrium, with emphases on modelling uncertainties in agent's behaviour. Market equilibrium comes into being due to the compatibility of the demands for and supplies of goods (commodities). The quantities demanded decrease with increasing prices and the quantities supplied increase. At certain point one should expect some stabilization of prices, in theory at least. This is a familiar story (see e.g. [12], chapter 19, for a comprehensive introduction to the subject). The idea, nevertheless, must be formalized and given a mathematical shape. To this end, we start with elementary examples taken from practice, making them gradually more sophisticated. At the end, we intend to feather an agent whose behaviour depends on random parameters.

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From [5] the general concept of *equilibrium relative to adjustments* will be taken over, see section 2. A special model for an agent will serve as a touchstone for these ideas. Namely, the so-called generalized Cobb–Douglas agent whose individual demands and adjustments are explicitly given in the form (3.1) and (2.17), see section 3 or [11, section A4], for more details. As applied to this special model, the mathematical formulation of our problem becomes completely transparent. It allows for a straightforward application of ready techniques and results from Markov chain theory, as well as the results of [7] concerning the so-called *random difference equations*. Our objective is thus quite moderate: we only intend to discuss some of the possibilities to contribute to modelling uncertainties like in [5] and [11] (we also want to mention recent papers in the area of stationary Markov equilibria [4] and [2] though there is only quite remote relationship to the present context). Our selection of topics leads us to the field of the probability theory and reflects our own interests, rather biased from the traditional approach as pursued e.g. in [1] or [9]. See also the mathematical treatise [10] on stability of linear dynamical systems of economy (with extended bibliography starting from the seminal work by Timbergen, Samuelson, etc.)

As is usually understood, the equilibrium prices of the commodities reduce total excess demand to zero. But in many cases this picture may appear too idealistic, even as applied to completely fair exchange economies. The initial offer may be out of equilibrium by most diverse reasons. Such an offer, however, should generate pressure for a price change. In a fair economy tendency towards equilibrium should arise. As total excess demand is non-zero at initial prices, they would go up if excess demand is positive and go down otherwise. It now depends on the behaviour of agents whether the prices will approach the equilibrium as time evolves. An agent in question is assumed to be completely characterized by his initial endowment and his individual demands reflecting his preferences. Those are factors determining price development in time. Unlike [11], it will be assumed throughout that agent's individual demands may vary in time, for tastes and preferences may undergo all kind of influences. But even in the case of a conservative agent with constant behaviour the question has to be asked whether the equilibrium is attainable, see section 4.3 for more details. Simplicity of our model – the so-called generalized Cobb–Douglas model (hereafter abbreviated to CD model) – allows us to cast this in terms of the theory of homogeneous Markov chains and to associate in this way our question with ergodicity of the underlying chain. It will be seen in particular, under which circumstances we do have exponentially fast convergence toward equilibrium. In the generalized CD model, described in section 3, the individual demands may be given in the matrix form with the entry at the k^{th} row and j^{th} column representing the share of the agent's initial wealth on the k^{th} commodity that he is willing to spend on the j^{th} one. In terms of the underlying Markov chains, this is exactly a transition matrix. Plainly, the matters become much more complicated with time varying individual demands, for in this case we will enter the field of non-homogeneous Markov chains where the answer to our question is much more complicated, see section 4.4 for more details. We have to deal now with the sequence of preference matrices. The n^{th} matrix in this sequence has the entry at the $(k, j)^{th}$ place that is to be interpreted as the share during the n^{th} period of the agent's initial wealth on the k^{th} commodity that he is going to spend in the next period on the j^{th} commodity.

The concluding section 5 is devoted to the main subject of the present paper. Here we argue like in the papers [5] and [11] admitting randomness of individual demands. The idea behind is that one cannot be completely sure about agent's preferences and thus there always remains some uncertainty. More precisely, the sequence of preference matrices consists now of i.i.d. copies of a certain random matrix. It is preferable, by technical reasons, to begin with the independence assumption and to keep the probability distribution identical, since there is ready mathematical results to be applied (section 5.1 following Vervaat (1979) is confined to one dimensional case; general results due to Kesten (1973) are used in section 5.2, see also [8] or [3])).

Unlike the present report, the seminal paper [11] is concerned with large economic systems evolving increasing number of economic agents. For the sake of simplicity of exposition we restrict our analysis to a single economic agent, though most of the theory may straightforwardly be extended to finite set of agents, especially part concerning the Cobb–Douglas model. The reason is explained by the fact that *an economy comprised of several CD agents acts like a single CD agent with integrated characteristics.* This is statement (A4.4) in [11, appendix A4], called the first *closure property* of the CD model. See also the second closure property, statement (A4.6), that we shall touche upon at the end of section 4.2.

2 Towards equilibrium

Consider an exchange economy in which certain commodities are purchased, $\ell + 1$ in number. One of these commodities, say $\ell + 1^{st}$, is viewed as a *numeraire* and it's price is always taken unity. That is, if we let p^j denote the price of the j^{th} commodity, then $p^{\ell+1} = 1$. Suppose that on each of these commodities, say j^{th} one, the demand δ^j and the supply σ^j of an individual economic agent both depend on its price p^j, in such a way that if we draw two corresponding curves they intersect at a certain price coordinate (suppose for simplicity, that this happens only at one site to ensure uniqueness of the solution like in Figure 1). Moreover, with the growing price we think of increase in supply and of decrease in demand, for instance linearly, as is depicted in our figure. Imagine the following concrete situation.

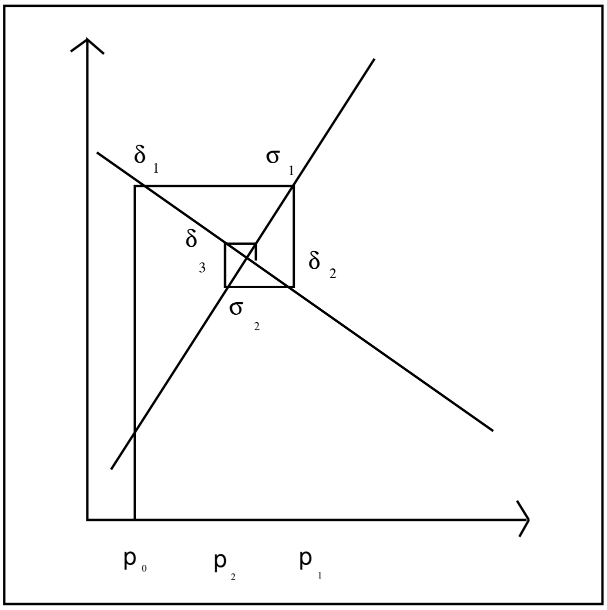

Figure 1: convergence

Example 2.1. Each day agent's order on a certain commodity is influenced by its yesterday's price (for simplicity, our attention is focused awhile on a single commodity). If the price is high, he is going to decrease his purchase, while if price is low he is going to increase his purchase. In this example both the quantity demanded and the quantity supplied are available on a daily basis. The same applies to the price of a commodity as well. Let the current day be the n^{th} one. As we said, agent's demand of to-day, δ_n, depends on yesterday's price, p_{n-1}. The dependence is linear

$$\delta_n = -a\, p_{n-1} + a^\circ$$

with positive constants a and a°. The supply for to-day, σ_n, is a linear function of the current price, p_n: with some positive constants b and b°

$$\sigma_n = b\, p_n + b^\circ.$$

The current price p_n stabilizes at the value which exactly clears the market. It thus solves the equation $\delta_n = \sigma_n$. Consequently, the following recurrent price development occurs:

$$p_n = -A\, p_{n-1} + A^\circ \tag{2.1}$$

where the constants A and A° are easily calculated: $A = a/b$ and $A^\circ = (a^\circ - b^\circ)/b$. To initialize the recurrence procedure start with some initial price p_0 to obtain δ_1, the demand in period 1. Next find σ_1 in accordance with the market clearing principle $\delta_1 = \sigma_1$. Given the value of σ_1, determine the market price in period 1, i.e. p_1. Now repeat the procedure starting with p_1. Determine δ_2, then σ_2, etc. We see the resulted cobweb-like graph in our figure. Here we deliberately depicted the case $0 < A < 1$, for in this case the price sequence converges as $n \to \infty$ to an *equilibrium* price. Clearly, when $A = 1$ the price fluctuates, and when $A > 1$ the price grows unboundedly. This special so-called *cobweb* model will serve us as a starting point for justifying more sophisticated models. The parameters (a, a°) and (b, b°) were kept constant for ease of the graphical exposition, but when agent's policy is time sensitive, these parameters also get daily index. Linear dependencies too may appear unsuitable and more appropriate adjustment functions may be sought. Lastly, in section 5 the parameters are viewed as certain *random variables*. Subsection 5.1 is devoted to the scalar linear relationship of type (2.1) with random parameters (the discussion there is based on Vervaat's results, see [15], or [6, section 8.4]). In subsection 5.2 higher dimensional extensions are discussed.

In conclusion, a brief remark on an inverse counterpart of our cobweb model in which to-day's demand depends on the current price, while to-day's supply depends on yesterday's price. We are not going to dwell here upon this, for if the dependencies are again linear, this would cause plain inversion of our picture (the matters are not always that straightforward, of course; for instance, in the generalized CD model of section 3 the preference matrices have to be inverted). Nevertheless, this inverse model is of equal importance. Think for instance, of farmers that decide how much of a certain crop they wish to produce next year on the basis of this year's price. If the price is high, they increase production, while if price is low they decrease the production. Clearly, in this case the quantity supplied depends on the price prevailing the market in the past year, while the quantity demanded depends on the current price. More cases on price forming by the supply and demand alike the cases of this example can be found in [12], chapter 20; see also the next chapter 21 for a subtle discussion of applications to agriculture.

For simplicity, we have in the preceding example treated just one commodity. In general, however, discussed will be exchange economies with finite number of commodities, just as in the beginning of this section. Let us fix some more notations and terminology. To distinguish between the commodities, we have labelled them from 1 to $\ell + 1$. The symbols presenting the quantities supplied and the quantities demanded got upper indices, i.e. σ^j and δ^j for the j^{th} commodity. The same applies to the corresponding price p^j. Clearly, both the demand δ^j and the supply σ^j are subject to price fluctuations. This fact will be expressed by writing $\delta^j = \delta^j(p)$ and $\sigma^j = \sigma^j(p)$ with the help of two functions $\delta^j(\cdot)$ and $\sigma^j(\cdot)$ of a vectorvalued argument p that represents the vector of prices on all commodities: $p = (p^1, \dots, p^{\ell+1})$. These functions are usually referred to as the *structure functions*. If certain prices p prevail the market, then both the demand δ^j and the supply σ^j get certain values that clearly amount to $p^j\delta^j$ and $p^j\sigma^j$, respectively.

Market equilibrium we are concerned with comes into being due to the compatibility of the demands and supplies. We need therefore to examine the so-called *excess demands* on each of the commodities defined as the differences

$$\zeta^j \doteq \delta^j - \sigma^j.$$

Let $\zeta^j(p)$ be the corresponding structure function at the price vector p. At this price the value of the excess demand $\zeta^j(p)$ again amounts to $p^j\zeta^j(p)$. Let awhile the numeraire out of picture and look at the situation in which the market is completely cleared in the sense that the supply of and demand for each commodity are equalized. We say then that the prices prevailing in this situation are the *equilibrium prices*, cf. e.g. [1, section 2.5]. Denoting this special price on the j^{th} commodity by $\underline{p}^j$, and the corresponding vector by $\underline{p}$, we thus see that the equilibrium is sought as the root of the system of equation

$$\zeta^j(p) = 0, \quad j = 1, \dots, \ell, \tag{2.2}$$

cf. e.g. [10], formula (21) on p. 122. Plainly, at the equilibrium price the excess demands are diminished in value to nothing, i.e. $\underline{p}^j\zeta^j(\underline{p}) = 0$ for all $j = 1, \dots, \ell$.

Throughout we will always deal with *Walrasian* economies meaning that an exchange economy in question will always obey *Walras' law* defined as the identity

$$\sum_{1\le j\le \ell+1} p^j\zeta^j(p) = 0,$$

cf. e.g. [1], assumption 3 on p. 21, or [10], identity (40) on p. 127. This determines the excess demand on the numeraire

$$\zeta^{\ell+1}(p) = -\sum_{1\le j\le \ell} p^j\zeta^j(p).$$

In accordance to this law, we get at the equilibrium price that also the excess demand for the numeraire diminishes in value, i.e. $\zeta^{\ell+1}(\underline{p}) = 0$, cf. e.g. [1], assumption 3 on p. 21.

Let us now consider a simple example of the equilibrium price calculation. In this example the standard Cobb–Douglas (CD) model is introduced.

Example 2.2. We are going to feature a behaviour of an economic agent that reveals his tastes and preferences by maximizing a special *utility function.* This is a function of $\ell+1$ variables, say $x = (x^1, \dots, x^{\ell+1})$, defined as follows:

$$U(x) \doteq \prod_{1\le j\le \ell+1} (x^j)^{a^j} \tag{2.3}$$

where a^j are certain non-negative parameters summing up to one, i.e. $a^1 + \dots + a^{\ell+1} = 1$ (these parameters will shortly get a clear interpretation). The function U so defined is called the *Cobb–Douglas utility*, see e.g. [14, section 7.5].

Let $e^1, \dots, e^{\ell+1}$ be the *initial endowments* by the agent on the corresponding commodities. Denote by

$$w^j(e;p) \doteq p^j e^j, \quad w(e;p) \doteq \sum_{1\le j\le \ell+1} p^j e^j, \quad \varpi^j(e;p) \doteq \frac{w^j(e;p)}{w(e;p)} \tag{2.4}$$

the *value* w^j of the particular endowment e^j, the *wealth* w required for the endowment e and the *share* ϖ^j of w allocated to e^j. Presumed is that certain prices $p^1, \dots, p^\ell$ prevail.

To form his demands the agent maximizes the CD utility (2.3) with respect to the arguments $x^1, \dots, x^{\ell+1}$. Prior to this, one has to take into account the budget constraint that the consumption expenditure cannot exceed one's wealth w. Moreover, when all commodities are assumed wanted, there is no reason to keep here inequality and in that case the constraint consists of equating the total expenditure to the wealth w. Formally, one extra equation has to be satisfied: $p^1x^1 + \dots + p^{\ell+1}x^{\ell+1} = w$. The problem of maximizing (2.3) under this equality constrain is solved by the Lagrange multiplier method (see e.g. [10, section 7.4]). The particular values of the arguments $x^1, \dots, x^{\ell+1}$ at which the solution is attained are by definition agent's demands, denoted as usual by $\delta^1, \dots, \delta^{\ell+1}$. By simple algebraic arguments like in [14, p. 111] one arrives at the following expression for the structure function of agent's demand on the j^{th} commodity:

$$\delta^j(p) = \frac{a^j}{p^j} \sum_{1\le k\le \ell+1} p^k e^k = \frac{a^j}{p^j} w(e;p). \tag{2.5}$$

Clearly, it's value at price p^j amounts to

$$p^j \delta^j(p) = a^j w(e;p). \tag{2.6}$$

Furthermore, in this example the excess demand $\zeta^j = \delta^j - \sigma^j$ for the j^{th} commodity is given by the following structure function

$$\zeta^j(p) = \frac{a^j}{p^j} w(e;p) - e^j.$$

The first term comes from the above form of the demands δ^j and the second from the identification of the supplies with the initial endowments, i.e.

$$\sigma^j \equiv e^j \tag{2.7}$$

identically. Let us look now for a particular price vector that clears the market and consists of the equilibrium prices, by definition. This is then the root of the system of linear equations

$$\zeta^j(p) = 0,\ j = 1, \dots, \ell + 1$$

(cf. (2.2)) that in terms of (2.4) can be written explicitly as follows:

$$w^j(e;p) = a^j\, w(e;p) \ \text{ or } \ a^j = \varpi^j(e;p). \tag{2.8}$$

One of these equations is redundant and we get to solve ℓ equations with respect to ℓ unknowns $p^1, \dots, p^\ell$. This yields the special prices $\underline{p}^j$ that comprise the vector $\underline{p}$ of the equilibrium prices. The two step procedure consists in first solving the system of ℓ equations

$$\begin{aligned} w^1 =& a^1(w^1 + \cdots + w^\ell) + a^1\, e^{\ell+1} \\ &\dots \\ w^\ell =& a^\ell(w^1 + \cdots + w^\ell) + a^\ell\, e^{\ell+1} \end{aligned} \tag{2.9}$$

and then using the roots $w^1, \dots, w^\ell$ to calculate equilibrium prices $\underline{p}^j = w^j/e^j$. It is easily verified that the roots are $w^j = e^{\ell+1}\, a^j/a^{\ell+1}$. Therefore

$$\underline{p}^j = \frac{e^{\ell+1}}{e^j}\, \frac{a^j}{a^{\ell+1}}. \tag{2.10}$$

The matters become somewhat more complicated in section 3 that handles the generalized CD model.

We close this example by a note about the meaning of the parameters $a^1, \dots, a^{\ell+1}$. In view of (2.8), they satisfy $a^j = \varpi^j(e;\underline{p})$ and therefore represent the *share* of the initial wealth that the agent is willing to allocate to commodity j, presuming the equilibrium prices prevail the market.

The model in the preceding example is static in the sense that no time factor occur, in contrast with the cobweb model of example 2.1 (a dynamical version of the standard CD model will be introduced in the next example). In the static situation we are able to define equilibrium prices that exactly clear the market. In real life situation, however, prices are time dependent and not *per se* in equilibrium. Instead, it is often the case that fair trading in the market causes the tendency towards the equilibrium as time evolves (the process is often referred to as a *tâtonnement* after Walras himself; cf. e.g. [1, section 11.2]). Throughout, all changes in prices, or in demands and supplies, will be assumed to occur periodically, just as in example 2.1 Both the demands and supplies within the n^{th} period are thus presented by corresponding structure functions indexed by n, i.e. $\delta_n^j(\cdot)$ and $\sigma_n^j(\cdot)$.

Furthermore, we will assume throughout that *there is a certain adjustment procedure determining prices on all commodities at the consequent date on the basis of the current prices*, by starting at a certain initial price offer p_0. That is the vector comprising initial prices on all commodities $p_0^1, \dots, p_0^\ell$ and $p_0^{\ell+1} = 1$. The devise will differ from case to case, depending on specific models for demands and supplies. This is our adjustment hypothesis, comparable with that of [5]; see in particular section 4 of the latter paper where some restrictions and advantages are pointed out. Formally, the price forming may be described by means of a certain *adjustment function* h as follows:

$$p_{n+1}^j = h(p_n^1, \dots, p_n^\ell)$$

with certain dependence $h = h_{\delta,\sigma}$ to be pointed out separately in each particular case. Notice the simplicity of the situation in example 2.1 with the linear relationship (2.1). The aforementioned dependence on the demands and supplies is easy to trace via the constants involved. As for the standard CD model of example 2.2, we will first make it time dependent and then let it obey a certain adjustment hypothesis that will yield the daily prices (2.22) of the same form as prices (2.10) in the static case.

Example 2.3. We turn to a dynamic counterpart of the previous CD model that captures agent's behaviour in the situation when his demands are placed on a periodical basis, say daily. Staying with a standard CD agent, we assume the demands coming into being by daily maximization of the utility function of the same type as in the beginning of this example, but then provided by a daily index: at date n say,

$$U_n(x) \doteq \prod_{1 \le j \le \ell+1} (x^j)^{a_n^j} \tag{2.11}$$

with $a_n^1 + \cdots + a_n^{\ell+1} = 1$. Time dependence occurs via the daily parameters a_n^j that will get the meaning of the share of the total investment that the agent allocates at date n to commodity j, see the remark at the end of this section. The price development in time is described as follows. The procedure starts from the initial offer p_0^j and the initial endowment e_0^j on the j^{th} commodity. This requires investment that amounts to

$$w_0 = \sum_{1 \le k \le \ell+1} p_0^k \, e_0^k. \tag{2.12}$$

Regarding the CD agent, it will be always assumed that agent's wealth stays unaltered during all trading periods to come, because all future purchase is founded on the initial endowment. The trading consists of redistributing from date to date this initial endowment w_0 over the $\ell+1$ commodities, hence no infusion or withdrawal of funds takes place. In the option pricing theory concerning securities markets, such a trading strategy is called *self-financing*. Let us use the same term here as well. Let us also drop the superfluous subindex and write w always. More precisely, this condition can be described as follows. Suppose we are given not only the initial endowments $e_0 = (e_0^1, \dots, e_0^{\ell+1})$ but also agent's daily basket e_n that encloses at each date n quantity e_n^j of commodity j. If we consequently extend (2.12) and define agent's wealth at each date n as

$$w_n = \sum_{1 \leq k \leq \ell+1} p_n^k \, e_n^k,$$

then our condition of unalterable wealth is expressed as follows:

$$w \equiv w_0 = w_1 = \cdots \tag{2.13}$$

The necessity of this restriction will be pointed out in due course.

When passing from date n to the next date $n+1$, the redistribution of the wealth w is conducted according to agent's preference to maximize the utility U_n, as is pointed out earlier. In parallel to (2.5) and (2.6), agent's demand for commodity j at date n is

$$\delta_n^j = \frac{a_n^j}{p_n^j} \, w \tag{2.14}$$

whose value at price p_n^j amounts to

$$p_n^j \delta_n^j(p) = a_n^j \, w. \tag{2.15}$$

As for the supplies, we depart from the identity (2.7) as follows. Suppose that each day the supply takes account of agent's purchase so that like in the static case, i.e. like in (2.7), the identity

$$\sigma_n^j \equiv e_n^j \tag{2.16}$$

holds at each date n. Here e_n^j is again the j^{th} component of agent's daily basket e_n. For more details and generalizations, see section 3.

The present dynamical version of the CD model is supposed to obey a special adjustment hypothesis. Namely, the adjustments are supposed to be specified by the linear relationship

$$p_{n+1}^j \doteq c_n^j \, p_n^j, \quad j = 1, \dots, \ell, \tag{2.17}$$

with $c_n^j = \delta_n^j / e_{n+1}^j$. Due to (2.16) we can replace e_{n+1}^j in the denominator by σ_{n+1}^j to see that (2.17) is equivalent to equating the values of the current demands to the values of the supplies at the consequent date, i.e.

$$p_{n+1}^j \, \sigma_{n+1}^j = p_n^j \, \delta_n^j, \quad j = 1, \dots, \ell, \tag{2.18}$$

the identities that may be regarded as a substitute for the identities $\underline{p}^j \zeta^j(\underline{p}) = 0$ we had in the static case. Thus by (2.15) the new prices are calculated by the formula

$$p_{n+1}^j = \frac{a_n^j \, w}{e_{n+1}^j} \tag{2.19}$$

that also can be written in the form $w_{n+1}^j = a_n^j \, w$ or $a_n^j = \varpi_{n+1}^j$. Here the notations similar to (2.4) is used, i.e.

$$w_n^j = p_n^j \, e_n^j \;\text{ and }\; \varpi_n^j = \frac{w_n^j}{w}. \tag{2.20}$$

At this point we can exploit our self-financing hypothesis to bring (2.19) into another form (2.22) close to its static counterpart (2.10). By assumption (2.13) we have, in particular,

$$w = w_{n+1} = w_{n+1}^1 + \dots + w_{n+1}^\ell + e_{n+1}^{\ell+1}.$$

So, we arrive at the system of ℓ equations of the exactly same form as in (2.9), but with the substitutions leading to the system

$$\begin{aligned} w_{n+1}^1 =& a_n^1 (w_{n+1}^1 + \dots + w_{n+1}^\ell) + a_n^1 \, e_{n+1}^{\ell+1} \\ \dots& \\ w_{n+1}^\ell =& a_n^\ell (w_{n+1}^1 + \dots + w_{n+1}^\ell) + a_n^\ell \, e_{n+1}^{\ell+1} \end{aligned} \tag{2.21}$$

that possesses the roots $w_{n+1}^j = e_{n+1}^{\ell+1} \, a_n^j / a_n^{\ell+1}$ similar to that of (2.9). Thus, the conclusion similar to (2.10) can be drawn:

$$p_{n+1}^j = \frac{e_{n+1}^{\ell+1}}{e_{n+1}^j} \, \frac{a_n^j}{a_n^{\ell+1}}. \tag{2.22}$$

Obviously, in the case of a conservative agent with the same preferences each day $a_n^j \equiv a^j$ and the same basket $e_n^j \equiv e^j$ we descend back to the static situation of example 2.2. Moreover, each day formula (2.22) yields same prices, equal to the *equilibrium* prices of the static situation with the utility function (2.3) and the initial endowment $e^1, \dots, e^{\ell+1}$. In general, at date $n+1$ the prices p_{n+1}^j calculated according to (2.22) coincide with the *equilibrium* prices that would prevail in static situation, if agent's utility were (2.11) and the initial endowment were $e_{n+1}^1, \dots, e_{n+1}^{\ell+1}$. Clearly, these prices tend to approach certain limits if and only if agent's daily behaviour tends to stabilize as time evolves (see section 4.1 for further discussion of this example).

Through the identities $a_n^j = \varpi_{n+1}^j$ mentioned above (just after formula (2.19)) the coefficients a_n^j get the meaning of the *share* of agent's wealth w that he is willing to allocate to commodity j at the consequent date. These coefficients and the supply $\sigma_{n+1}^j = e_{n+1}^j$ at the next date forecast the prices via the principle (2.18). As applied to the present *standard* CD agent, this principle does not require any historical data to adjust prices to agent's daily requirements and tastes. The above comparison to the static situation exhibits a likeness to equilibrium price forming.

We intend to apply the principle (2.18) in the next section to the generalized version of the CD model. This will bring about considerable technical and conceptual complexity.

3 Generalized CD agent

We begin the present section with the subsection on the static model for the so-called generalized CD agent, for more details see Nummelin [11, Appendix A2]. It's dynamical version will be introduced and discussed in the separate subsection 3.2.

3.1 Static model

It is said that an economic agent is of the *generalized* Cobb–Douglas type if its individual demands on all the commodities are formed as follows:

$$\delta^j = \frac{1}{p^j} \sum_{1 \le k \le \ell+1} p^k e^k \, a^{kj} \tag{3.1}$$

while the supplies remain same as in example 2.2 and satisfy the identities (2.7). The quantities e^j are again agent's initial endowments. The parameter matrix with nonnegative entries $\{a^{kj}\}$ is assumed *stochastic* in the sense that

$$\sum_{1 \le j \le \ell+1} a^{kj} = 1$$

for all $k = 1, \dots, \ell+1$. It is convenient to express this in matrix notations. Form the $\ell+1$ dimensional parameter matrix

$$A = \begin{pmatrix} a^{11} & \cdots & a^{1\ell+1} \\ & \cdots & \\ a^{\ell+11} & \cdots & a^{\ell+1\ell+1} \end{pmatrix} \tag{3.2}$$

and the $\ell + 1$ dimensional column vector $\mathbb{I}$ with unity in each position. Our assumption turns into $\mathbb{I} = A\,\mathbb{I}$. By comparing (2.5) and (3.1) we thus see that the former focuses on the special case of the matrix A with constant entries along each row, namely $a^{kj} \equiv a^j$ for all $k = 1, \dots, \ell + 1$.

By definition, the equilibrium prices clear the market which in the present case means that at these prices the value of demands

$$p^j \delta^j = \sum_{1 \le k \le \ell+1} w^k(e;p)\, a^{kj}$$

(where $w^k(e;p) = p^k e^k$ as in (2.4)) becomes equal to that of supplies

$$p^j \sigma^j = w^j(e;p).$$

Hence the equilibrium prices $\underline{p}^1, \dots, \underline{p}^\ell$ are sought as the roots of the following system of equations

$$w^j(e;p) = \sum_{1 \le k \le \ell+1} w^k(e;p)\, a^{kj}, \quad j = 1, \dots, \ell + 1 \tag{3.3}$$

one of which is in fact redundant because summing up both sides and taking into consideration that A is stochastic, we end up with the trivial identity $w(e;p) = w(e;p)$, cf. (2.4). So, the procedure for determining the equilibrium prices is now clear: first solve the system of equations

$$\begin{aligned} w^1 =& w^1 a^{11} + \dots + w^\ell a^{\ell 1} + e^{\ell+1} a^{\ell+1\,1} \\ & \dots \\ w^\ell =& w^1 a^{1\ell} + \dots + w^\ell a^{\ell\ell} + e^{\ell+1} a^{\ell+1\,\ell} \end{aligned} \tag{3.4}$$

that are exactly the first ℓ of equations (3.3). Having the roots $w^1, \dots, w^\ell$ in disposal, calculate the equilibrium prices according to

$$\underline{p}^j = \frac{w^j}{e^j}. \tag{3.5}$$

In the special case of the standard CD model system (3.4) simplifies to the earlier system (2.9) whose solution is easily obtained, see (2.10). But in general the matters are somewhat more complicated and one has to recourse to classical algebraic results in order to understand under which conditions do the unique solution obtains. In fact, the answer has to be sought within the Perron–Frobenius theory for non-negative matrices, see e.g. Seneta (1981), theorem 4.1 on p. 119 that requires an *irreducible* parameter matrix A. We leave here technicalities (postponed till the more specialized section 4) and focus our attention in the next subsection on a dynamical modelling of agent's behaviour.

Note meanwhile that since the equilibrium prices satisfy (3.3) or, in the matrix notations

$$\varpi(e; \underline{p}) = \varpi(e; \underline{p})\, A \tag{3.6}$$

with the row vector $\varpi = (\varpi^1, \ldots, \varpi^{\ell+1})$ defined according to (2.4), the particular entry a^{kj} in A may be interpreted as the *share* of the initial wealth allocated to commodity k that the agent is willing to reallocate to commodity j, presuming the equilibrium prices prevail the market. For this reason, we will sometimes call A the *preference matrix.*

3.2 Dynamical model

The generalized CD model of the preceding subsection becomes time dependent if one starts from the initial offer p_0^j and the initial endowment e_0^j on the j^{th} commodity and proceeds in parallel to example 2.3. The investment (2.12), for instance, requires no subindex, since agent's trading strategy is again assumed *self-financing.* Thus his wealth stays unaltered and amounts $w = w_0$ always, moreover the identities (2.13) are supposed to hold. When passing from the n^{th} period to the next $n+1^{st}$ one, the redistribution of the wealth w is conducted according to agent's preferences that is again described by means of certain matrix A_n whose entries a_n^{kj} represent at date n the share of the wealth w on the k^{th} commodity, that the agent is willing to reallocate to the j^{th} one during the following period. Thus agent's trading strategy is completely determined by sequence of stochastic preference matrices $A_0, A_1, \ldots$, each satisfying $\mathbb{I} = A_n\, \mathbb{I}$.

To describe price development in the present case, we act as before applying the adjustment procedure of the form (2.17) with the coefficients $c_n^j = \delta_n^j / e_{n+1}^j$, where $e_n^1, \ldots, e_n^{\ell+1}$ is again agent's daily basket, while his daily demands are now defined by

$$\delta_n^j = \frac{1}{p_n^j} \sum_{1 \le k \le \ell+1} p_n^k e_n^k\, a_n^{kj}.$$

This is, as we know, the same as to apply (2.18) which equates the value of the demand δ_n^j for commodity j at date n to the value of the supply $\sigma_{n+1}^j \equiv e_{n+1}^j$ of the same commodity j but at the consequent date $n+1$. The former value amounts to

$$p_n^j \delta_n^j = \sum_{1 \le k \le \ell+1} w_n^k\, a_n^{kj}$$

and the latter to

$$p_{n+1}^j \sigma_{n+1}^j = w_{n+1}^j.$$

Here the notations (2.20) are used. The resulted equations are

$$w_{n+1}^{j} = \sum_{1 \le k \le \ell+1} w_n^k \, a_n^{kj} \tag{3.7}$$

with $j = 1, \dots, \ell+1$, that clearly may be viewed as a dynamical counterpart of (3.3). In virtue of the self-financing hypothesis we are able to cast (3.7) in the matrix form alike (3.6). To this end, divide both sides by w to get

$$\varpi_{n+1}^{j} = \sum_{1 \le k \le \ell+1} \varpi_n^k \, a_n^{kj} \tag{3.8}$$

with $j = 1, \dots, \ell+1$. Then present (3.8) as the following dynamical version of the matrix identity (3.6):

$$\varpi_{n+1} = \varpi_n \, A_n \tag{3.9}$$

with $\varpi_n = (\varpi_n^1, \dots, \varpi_n^{\ell+1})$.

The system of new equations is solved recurrently: given the current distribution of the wealth $(w_n^1, \dots, w_n^\ell)$ (and $w_n^{\ell+1} = e_n^{\ell+1}$ as well, since the price of the numeraire equals 1 always) we get the consequent distribution $(w_{n+1}^1, \dots, w_{n+1}^\ell)$ by

$$w_{n+1}^{j} = \sum_{1 \le k \le \ell} w_n^k \, a_n^{kj} + e_n^{\ell+1} \, a_n^{\ell+1\, j}, \quad j = 1, \dots, \ell, \tag{3.10}$$

cf. (3.4). Next, having w_{n+1}^j in our disposal we calculate the prices according to

$$p_{n+1}^{j} = \frac{w_{n+1}^{j}}{e_{n+1}^{j}}. \tag{3.11}$$

Recall that in the special case of the ordinary CD model the prices p_{n+1}^j were easily determined via (2.22) in terms of the local characteristics of agent's behaviour, namely in terms of his basket $e_{n+1}^1, \dots, e_{n+1}^{\ell+1}$ and his preferences $a_n^1, \dots, a_n^{\ell+1}$. In general, however, the iterations are needed that make prices p_{n+1}^j dependent on agent's past behaviour, as well. Our principal question whether the prices so defined achieve a certain equilibrium as time evolves would then receive an affirmative answer only under certain restrictions on this behaviour. The purpose of the next section is to provide an exact mathematical formulation of necessary conditions.

4 Stability of the dynamical CD model

4.1 The standard CD model

As was said above, in the dynamical CD model the prices at date $n+1$ are calculated according to the ratio (3.11). Therefore they tend to settle down to a condition of equilibrium if both the numerator and denominator converge in their own right as time evolves. The denominators form the sequence of daily baskets that has to be assumed to converge towards a certain limit as time increases unboundedly. We do assume this throughout. Next, we need to impose adequate restrictions on the sequence of preference matrices $A_0, A_1 \dots$ so as to guarantee the convergence of the numerators, as well. This is a challenging problem even in the case of a conservative agent with constant preferences $A \equiv A_0 = A_1 \dots$. As we will see in section 4.3, however, comprehensive answers to related questions are readily available thanks to clear links to the well-developed theory of homogeneous Markov chains. The general, not necessarily homogeneous case will be touched upon in section 4.4.

It will be convenient for the present purposes to utilize matrix notations and matrix calculus like in (3.6) and (3.9). The required iterations of the latter leads from $\varpi_1 = \varpi_0\, A_0$ to $\varpi_2 = \varpi_0\, A_0 A_1$ and so forth. Hence at the n^{th} stage we arrive at

$$\varpi_{n+1} = \varpi_0\, A_0 A_1 \cdots A_n \tag{4.1}$$

and the desired limiting transition as $n \to \infty$ means treating the product of the increasing number of preference matrices. Recall that each of these preference matrices is stochastic $\mathbb{I} = A_n\, \mathbb{I}$, therefore so is their product $\mathbb{I} = A_0 A_1 \cdots A_n\, \mathbb{I}$. As we have mentioned in the preceding paragraph, the matters settle easier in the special homogeneous case, since one has to deal with the limiting behaviour as $n \to \infty$ of the n^{th} power of the common preference matrix, i.e. the matrix A^n. In conclusion, we want to bring forth another typical case that is simply understood despite of it's inhomogeneity.

Example 2.3 (continuation). We are going now to examine our dynamical version of the standard CD model from a different point of view in order to lighten the simplification brought by the pricing formula (2.22), i.e. by the dependence only on agent's local characteristics.

Let at each date n agent's preferences be characterized as in (2.11) by positive parameters $a_n^1, \dots, a_n^{\ell+1}$ so that $a_n^1 + \cdots + a_n^{\ell+1} = 1$. Form a row vector with these components, say $a_n = (a_n^1, \dots, a_n^{\ell+1})$ with

$$a_n \mathbb{I} = a_n^1 + \cdots + a_n^{\ell+1} = 1, \tag{4.2}$$

the property obvious since $\mathbb{I}$ is the column with unity in each position. In this situation agent's preference matrix at date n is formed by means of the tensor product of the column vector $\mathbb{I}$ and the row vector a_n that by definition results in a $\ell + 1$ dimensional square matrix with the entry at the $(k,j)^{th}$ place equal to the product of the k^{th} component of $\mathbb{I}$ and the j^{th} component of a_n. That is

$$A_n = \mathbb{I}\, a_n = \begin{pmatrix} a_n^1 & \cdots & a_n^{\ell+1} \\ & \cdots & \\ a_n^1 & \cdots & a_n^{\ell+1} \end{pmatrix}$$

and the product of such matrices is simply $A_0 A_1 \cdots A_n = A_n$. To verify this apply (4.2) repeatedly. The relation (4.1) simplifies to

$$\varpi_{n+1} = \varpi_0\, A_n$$

and this explains the local dependence pointed out earlier. Plainly, the sequence of distributions ϖ_n stabilizes if and only if the sequence of parameter vectors $a_n = (a_n^1, \ldots, a_n^{\ell+1})$ does. Suppose namely the convergence of each component towards a certain limit: $\varpi_{n+1}^j = a_n^j \to \varpi^j$ (the first identity is known to us as a consequence of (2.19)). In terms of the row vector $\varpi = (\varpi^1, \ldots, \varpi^{\ell+1})$ we thus have $A_n \to \mathbb{I}\,\varpi$, the expression that recurs at the end of section 4.3. Note finally that if agent's daily basket $e_n^1, \ldots, e_n^{\ell+1}$ stabilizes as well and the limit is defined by $e_n^j \to e^j$, then in accordance to (2.22) the daily prices tend to settle down to a condition of equilibrium and

$$p_n^j \to \frac{e^{\ell+1}}{e^j}\, \frac{\varpi^j}{\varpi^{\ell+1}}.$$

4.2 Irreducible preference matrices

Let us turn back to the generalized CD model in it's static setup as in section 3.1. We have already mentioned the fact of existence and uniqueness of the equilibrium prices $\underline{p}^j = w^j/e^j$, provided existence and uniqueness of the roots w^j to the system of equations (3.4). In matrix terms, the row vector $\varpi = (\varpi^1, \ldots, \varpi^{\ell+1})$ has been required that does satisfy the so-called *equilibrium equations*

$$\varpi = \varpi\, A, \tag{4.3}$$

cf. (3.6). The desired row vector ϖ with positive entries summing up to unity, is usually called the *equilibrium distribution.*

The preference matrix A is called *irreducible* if there is no permutation which, when applied to it's rows and columns in the same way, changes A to a certain block matrix of the form

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \tag{4.4}$$

with certain square matrices A_{11} and A_{22} along the diagonal; 0 denotes here a (not necessarily square) matrix of zero elements. Otherwise, if A is *reducible* – it is of the form (4.4) – then the zero sub-matrix is preserved through all powers of the matrix. So all the powers are reducible, as well.

A hypothetical permutation leading to (4.4) would mean suitable relabelling of the commodities. The commodities that get the labels corresponding to A_{11} form a closed set. Recall that each particular element of a preference matrix, say $(k, j)^{th}$ one, represents the *share* of the initial wealth allocated to commodity k that the agent is willing to reallocate to commodity j. Thus, in the case (4.4) no wealth allocated to aforementioned closed set is intended to be reallocated outside. Broadly speaking, in this case it occurs forming of a separate sub-market. In this connection, interesting questions may arise e.g. under which circumstances clustering of the commodities take place and how the resulting sub-markets interact. The answers may be sought within the structural analysis of general non-negative matrices giving classification of the states of a Markov chain, like in [13], section 1.2. Nevertheless we leave these questions here, as this would lead us too far afield. Instead, we confine our attention to irreducible preference matrices. Those are important ingredients in the Perron–Frobenius theory, for any matrix A in question can be in principle reduced to a block matrix of the lower triangular form with solely irreducible matrices along the diagonal. On account of this, any eigenvalue of A is an eigenvalue of one of these irreducible matrices.

Thus, let us consider exclusively agents whose preference matrices are irreducible. Then the famous Perron–Frobenius theorem holds (this theorem provides the mathematical background for econometrical treatises like in [9, Appendix], [1, Appendix A] or [10, chapter 4]; for our purposes we prefer to refer again to [13, chapter 1]). It tells us that any non-negative irreducible matrix possesses a real positive eigenvalue λ_1 (called the *Perron–Frobenius eigenvalue*) such that no other eigenvalue exceeds it in absolute value. That is $\lambda_1 \geq |\lambda|$ for any eigenvalue $\lambda \neq \lambda_1$. Furthermore, one can associate with λ_1 strictly positive left (row) and right (column) eigenvectors (called the *Perron–Frobenius eigenvectors*; let us denote them by l and r, respectively), that are unique to constant factors. For more details, see [13], theorems 1.1 and 1.5, as well as corollary 1. The latter asserts that the Perron–Frobenius eigenvalue λ_1 lies between the smallest row sum and the largest row sum. The same applies to the column sums.

As we already know, a preference matrix A is stochastic and therefore satisfies $\mathbb{I} = A\,\mathbb{I}$ where $\mathbb{I}$ is a column vector with unity in each position. Hence, 1 is an eigenvalue and $\mathbb{I}$ a corresponding column eigenvector. Since all row sums are equal to unity, and since the Perron–Frobenius eigenvalue λ_1 lies between the smallest and the largest, we conclude that the above eigenvalue 1 is in fact the Perron–Frobenius eigenvalue and $\mathbb{I}$ a corresponding right Perron–Frobenius eigenvector. Let the positive row vector ϖ, normalized so that $\varpi \mathbb{I} = 1$, be the corresponding left Perron–Frobenius eigenvector. Then it satisfies $\varpi = \varpi A$ that is exactly our equilibrium equations (4.3). Therefore the above left Perron–Frobenius eigenvector is the desired equilibrium distribution.

For explicit construction of this equilibrium distribution and for explicit calculations of equilibrium prices according to (3.5) we need to apply some matrix notations and matrix calculus. Since the $\ell + 1^{st}$ component, the *numeraire*, plays a special rôle, it is handy to subdivide A in blocks in the following manner:

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{4.5}$$

where a is an ℓ dimensional upper left corner submatrix, b an ℓ dimensional row vector and so forth. This allows us to rewrite the system of equations (3.4) in the following compact form: $w = w\,a + e^{\ell+1}\,b$ where w is the ℓ dimensional row vector consisting of the roots to the system (3.4), i.e. $w = (w^1, \dots, w^\ell)$. The solution is now obvious

$$w = e^{\ell+1}\,b\,(\mathrm{I} - a)^{-1}. \tag{4.6}$$

Using the components of this row vector as the numerators, we can calculate by means of (3.5) the equilibrium prices $\underline{p}^j$. It is now easy to find also the equilibrium distribution for the matrix (4.5). We will show, namely, that the row vector $(b\,(\mathrm{I} - a)^{-1}, 1)$ is a left Perron–Frobenius eigenvector since

$$(b\,(\mathrm{I} - a)^{-1}, 1) = (b\,(\mathrm{I} - a)^{-1}, 1) \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

This is verified by the following arguments. Firstly $b\,(\mathrm{I} - a)^{-1}a + b = b\,(\mathrm{I} - a)^{-1}$ and secondly $b\,(\mathrm{I} - a)^{-1}c + d = 1$ because by the usual formula for calculating determinants for block matrices $\det(\mathrm{I} - A) = 1 - d - b\,(\mathrm{I} - a)^{-1}c$ which equals to 0 since A is stochastic and $(\mathrm{I} - A)\,\mathbb{I} = 0$. Thus the row vector $(b\,(\mathrm{I} - a)^{-1}, 1)$ is indeed a left Perron–Frobenius eigenvector and it yields the equilibrium distribution after suitable normalization. Recall that the components of the equilibrium distribution have to sum up to unity. Thus in the present case the equilibrium distribution is

$$\frac{(b\,(\mathrm{I} - a)^{-1}, 1)}{1 + b\,(\mathrm{I} - a)^{-1}\mathbb{I}}. \tag{4.7}$$

As was already pointed out, the present static model for the generalized CD agent is taken over from the paper [11] in which the random equilibrium theory is developed. The source of randomness in the generalized CD model is in fact a random preference matrix A with a finite expectation $\mathrm{E}A < \infty$. Obviously, the expectation $\mathrm{E}A$ preserves the stochastic nature: $\mathbb{I} = \mathrm{E}A\,\mathbb{I}$ and its Perron–Frobenius eigenvalue equals 1. The corresponding left Perron–Frobenius eigenvector is easily found by taking the expectation of both sides in (4.5) and by repeating the above arguments. This yields the equilibrium distribution for EA equal to

$$\varpi = \frac{(\mathrm{E}b\,(\mathrm{I}-\mathrm{E}a)^{-1}, 1)}{1+\mathrm{E}b\,(\mathrm{I}-\mathrm{E}a)^{-1}\mathbb{I}} \tag{4.8}$$

cf. (4.7). It is equally simple in the present case (of a CD agent with the deterministic preference matrix EA) to determine the equilibrium prices. Indeed, parallel to the earlier considerations form the vector

$$\overline{w} = e^{\ell+1}\,\mathrm{E}b\,(\mathrm{I}-\mathrm{E}a)^{-1} \tag{4.9}$$

(cf. (4.6)) and determine equilibrium prices

$$\underline{\overline{p}}^j = \frac{\overline{w}^j}{e^j} \tag{4.10}$$

where the numerators are the components of $\overline{w}$. In [11] the term *expected equilibrium prices* is used, cf. appendix A4, statement (A4.8). Note, however, that normally E$w \neq \overline{w}$, hence the expected equilibrium prices $\underline{\overline{p}}^j$ differ from Ep^j as the term might mislead. In [11, appendix A4] the phenomenon discussed in the present paragraph is called (second) *closure property* of the random CD model.

4.3 Homogeneous preferences

If a CD agent has a single preference matrix A and reveals it each consecutive day, then the matrix identity (3.9) turns into

$$\varpi_{n+1} = \varpi_n\,A$$

whose iterations leads to

$$\varpi_{n+1} = \varpi_0\,A^n. \tag{4.11}$$

Hence the power matrix A^n is the subject of our main concern in this section. For convenience, let us introduce the special notation $a_{kj}^{(n)}$ for its $(k,j)^{th}$ element. Like in the previous section, assume A irreducible. If, in contrary, A would be reducible, i.e. representable in the form (4.4), then all its powers were reducible, as well. This remark leads to the following alternative definition: *a non-negative matrix A is irreducible if one can associate with every pair (k,j) of its index set a positive integer $m \equiv m(k,j)$ so that $a_{kj}^{(m)} > 0$,* cf. [13], definition 1.6.

The class of irreducible non-negative matrices breaks into subclasses of different periods. The necessity of classifying commodities (hence grouping into subclasses) arises when a preference matrix is not strictly positive. The occurrence of zero elements reflects unwillingness of the agent to reallocate some portion of the initial endowment from one commodity to another. In this regard, it is relevant to trace the location of zero's as time evolves by examining the powers A^n exclusively on their empty positions, thus disregarding non-zero elements. The aforementioned concept of the period is of the same nature. It is related to each individual commodity and inherits therefore it's index. Fix an index j and suppose that $a_{jj}^{(m)} > 0$ for at least one integer $m \geq 1$. The greatest common divisor of the whole variety of such integers is called the *period* of the commodity j. Denote it by $d(j)$ and note that if $a_{jj} > 0$, then $d(j) = 1$. The subclasses of irreducible non-negative matrices, mentioned above, are formed by unifying the commodities of common period. The so-called *aperiodic* (or aciclic) subclass, for instance, unites the commodities of period 1. Likewise are formed the *periodic* subclasses of period $d > 1$. A prominent rôle of the aperiodic subclass is substantiated by theorem 1.4 in [13] that states, in particular, that the irreducible aperiodic class of matrices is equivalent to the class of primitive matrices. The latter is defined as follows: a non-negative matrix A is called *primitive* if there exists a positive integer m so that $A^m > 0$, cf. [13, definition 1.1].

The primitivity assumption brings fourth one simplification to the Perron–Frobenius theory, rather significant, however. Indeed, under this assumption the Perron–Frobenius eigenvalue λ_1 strictly exceeds all other eigenvalues, i.e. $\lambda_1 > |\lambda|$ for any eigenvalue $\lambda \neq \lambda_1$, cf. [13], theorem 1.1. Along with the maximal eigenvalue λ_1, let us single out the next one λ_2 by first ordering all distinct eigenvalues $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_k|$ (here $k \leq \ell + 1$ due to possible multiplicities) so that if two or more consecutive eigenvalues are equal in absolute value, then their multiplicities are in descend order. In the case $|\lambda_2| \geq |\lambda_3|$, for instance, the multiplicity m_2 of λ_2 is at least as high as that of λ_3. With this convention, theorem 1.2 in [13] applies. This theorem refines the statement about the convergence of powers of a primitive matrix A. It tells us that as $n \to \infty$ we have

$$(A/\lambda_1)^n \to r\, l \tag{4.12}$$

where l and r are the left and right Perron–Frobenius eigenvectors normalized so that $l\, r = 1$ (as usual, a row multiplied by a column yields a scalar, while a column multiplied by a row, as in (4.12), yields a matrix). Moreover

$$A^n = \lambda_1^n\, r\, l + \mathcal{O}(n^{m_2 - 1} |\lambda_2|^n) \tag{4.13}$$

elementwise, provided $\lambda_2 \neq 0$. Of course, it may happen that $\lambda_2 = 0$, for instance,when a matrix is of the same form as (3.2). But this case is elementary: the second term $\mathcal{O}$ vanishes and $A^n = \lambda_1^n\, r\, l$.

Analogous results are available regarding irreducible periodic matrices, as well. We shall not dwell upon this here and only mention the following extension of (4.12): if d is the period, then

$$\frac{1}{d}[(A/\lambda_1)^n + \cdots + (A/\lambda_1)^{n+d-1}] \to r\, l \tag{4.14}$$

and the limit is again approached geometrically fast and uniformly for all elements.

Let us turn back to a primitive matrix A that is moreover stochastic and therefore represents a preference matrix. As we know, in this particular case $\lambda_1 = 1$, $r = \mathbb{I}$ and $l = \varpi$, so that all we need is to restate aforementioned results. In this way we arrive at the desired conclusion: *if a preference matrix A is primitive, then $A^n \to \mathbb{I}\,\varpi$.* The limit is approached geometrically fast and uniformly for all elements. In view of (4.11) this entails *ergodicity* in the sense that

$$\varpi_n \to \varpi \tag{4.15}$$

independently on the initial distribution ϖ_0, since $\varpi_0\,\mathbb{I} = 1$. The limit thus obtained satisfy the previous equilibrium equation (4.3) and therefore has been referred to as the equilibrium distribution.

Note at this juncture that if the initial distribution is the equilibrium distribution, i.e. $\varpi_0 = \varpi$, then $\varpi_0 = \varpi_1 = \varpi_2 = \ldots$. This phenomenon is called *stationarity.*

4.4 Inhomogeneous preferences

In this section a CD agent is allowed at each date to choose anew his preference matrix from a variety of stochastic matrices. If A_n is his choice at date n, the wealth distribution is updated according to (3.9). Iterating the latter we arrived at identity (4.1) that involves the (forward) product of stochastic matrices $A_0 \cdots A_n$. We want to stipulate conditions on the entire sequence $A_0, A_1, \ldots$ of preference matrices that guarantee *ergodicity* in the sense of (4.15). The aim remains the same as in the previous section, the problem is harder, however. There are various possibilities to proceed, nevertheless we intend at this instant to quote one particular statement with reference to [13], theorem 4.14 (the reader may consult this book also for a number of other possibilities; there exists vast literature on this subject). This statement concerns the sequence $A_0, A_1, \ldots$ that converges elementwise to a certain limit A which is irreducible. Then *ergodicity obtains and the limit in (4.15) is the unique equilibrium distribution vector of A.*

5 Uncertainties in agent's behaviour

In the present section a CD agent is supposed to reveal his daily preferences as independent copies of a certain preference matrix A. Then $A_0, A_1, \ldots$ is the sequence of i.i.d. random matrices. For simplicity of exposition, we will treat separately the scalar case that will be brought in relationship to our cobweb model of example 2.1.

5.1 Linear model with random parameters

Suppose recurrent price forming according to the following linear relationship

$$p_n = \alpha_n \, p_{n-1} + \beta_n \tag{5.1}$$

with $n = 1, 2, \ldots$. The iteration starts from the initial offer p_0 that is a certain random variable. Step by step the sequence of i.i.d. pair of random variables is involved

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2) \ldots \tag{5.2}$$

that is statistically independent of the initial offer p_0. Compare (5.1) to (2.1) to substantiate present setup with the basic model for price forming, the so-called cobweb model (cf. also [6, section 8.4] for other application areas, e.g. ARCH modelling of stochastic volatility in securities markets or perpetuity analysis in insurance). Clearly, by allowing time dependent parameters (A_n, A_n°) one can form the dynamic modification to the cobweb model, consequently these parameters may be viewed as i.i.d. random variables.

Initially we have $p_1 = \alpha_1 p_0 + \beta_1$, then $p_2 = \alpha_2 \alpha_1 p_0 + \alpha_2 \beta_1 + \beta_2$ and so fourth. After n steps we arrive at

$$p_n = \alpha_n \cdots \alpha_1 p_0 + \alpha_n \cdots \alpha_2 \, \beta_1 + \cdots + \alpha_n \beta_{n-1} + \beta_n. \tag{5.3}$$

The sequence of prices $p_0, p_1, \ldots$ so obtained constitute time series that we are going to examine on the desired stationarity property by applying the following results of Vervaat (1979). Let the sequence (5.2) consist of independent copies of the pair of random variables (α, β) that satisfy the moment conditions

$$-\infty < \mathrm{E} \log |\alpha| < 0 \;\text{ and }\; \mathrm{E} \log^+ |\beta| < \infty.$$

Then the procedure (5.1), starting from p_0, *converges in distribution to a certain limiting random variable* $\underline{p}$ *independent of the pair of random variables* (α, β), symbolically

$$p_n \xrightarrow{d} \underline{p}. \tag{5.4}$$

The limit is unique in distribution and

$$\underline{p} \stackrel{d}{=} \beta_1 + \alpha_1 \beta_2 + \alpha_1 \alpha_2 \beta_3 + \cdots \tag{5.5}$$

where the right-hand side converges absolutely with probability 1.

Let us make few comments on this result. The display (5.4) is meant to state only that the probability distribution of the random variable p_n tends as $n \to \infty$ to the probability distribution of the random variable $\underline{p}$, the convergence in weak sense. Likewise, (5.5) means that the limiting distribution coincides with the distribution of the right-hand side. This can be used, for instance, on calculating the limiting moments. It is more convenient, however, to do this via the relationship $\underline{p} \stackrel{d}{=} \alpha\, \underline{p} + \beta$ (this is [6, formula (8.47)]) by taking into consideration that $\underline{p}$ is independent of (α, β). Take, for instance, the expectation of both sides, assuming existence of expectation for both random variables α and β. Moreover, assume $\mathrm{E}|\alpha| < 1$. We get $\mathrm{E}\underline{p} = \mathrm{E}\beta/(1 - \mathrm{E}\alpha)$. That is, the prices converge in the mean and

$$\lim_{n\to\infty} \mathrm{E}p_n = \mathrm{E}\underline{p} = \frac{\mathrm{E}\beta}{1 - \mathrm{E}\alpha}. \tag{5.6}$$

In elementary terms, Vervaat's prove is based on the following arguments. First, it is verified that for each $n \geq 1$

$$(p_0, (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)) \stackrel{d}{=} (p_0, (\alpha_n, \beta_n), \dots, (\alpha_1, \beta_1)). \tag{5.7}$$

This makes the sum on the right hand-side of (5.3) equal in distribution to the sum with the same first term $p_0\alpha_1 \cdots \alpha_n$ plus renewed terms $\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2\beta_3 + \cdots + \alpha_1 \cdots \alpha_{n-1}\, \beta_n$. Hence

$$p_n \stackrel{d}{=} p_0\alpha_1 \cdots \alpha_n + \beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2\beta_3 + \cdots + \alpha_1 \cdots \alpha_{n-1}\, \beta_n \tag{5.8}$$

(at this point the necessity emerges of equations in distribution $\stackrel{d}{=}$). Then, it is observed that in the latter display the first term involving the initial offer vanishes as $n \to \infty$ and cannot contribute to the limit. Finally, the principal terms are verified to tend to desired limiting expression (5.5) in the sense of absolute convergence with probability 1.

In conclusion, a remark on stationarity of time series of prices $p_0, p_1, \dots$. If the initial offer p_0 is made equal in distribution to the equilibrium price $\underline{p}$, then it is not hard to verify by the same devise (5.7) (cf. [6, theorem 8.4.3(c)]) that time shifting do not alter finite dimensional joint distributions.

5.2 Random preference matrix

The present section treats a CD agent with daily preferences $A_0, A_1, \dots$ that is the sequence of i.i.d. random matrices. Some of results of the previous section are available also in matrix setting and our intention is to quote them here and then to apply for our purposes.

Let us extend the linear relationship (5.1) to the sequences of square matrices $\alpha_1, \alpha_2, \dots$ and of row vectors of same dimension $x_0, x_1, \dots$ and $\beta_1, \beta_2, \dots$ so that

$$x_{n+1} = x_n\, \alpha_n + \beta_n. \tag{5.9}$$

The meaning of these symbols in terms of the dynamical CD model will become clear shortly, as (5.9) will be associated with (3.4) (we confine our attention to this particular application, though various other applications are thinkable, e.g. to a model for interindustry analysis of the Leontief type, cf. [1], [9] or [10]). The iteration starts again from a certain initial random vector x_0 that is statistically independent of the pair (α, β) of a random matrix α and a random vector β. The sequence $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \dots$ involved in (5.9) consists of independent copies of (α, β).

The n-fold iteration leads to the same result as in (5.3):

$$x_{n+1} = x_0 \alpha_0 \cdots \alpha_n + \beta_0 \alpha_1 \cdots \alpha_n + \cdots + \beta_{n-1} \alpha_n + \beta_n. \tag{5.10}$$

As before, there arises the question about stationarity of multiple time series $x_0, x_1, \dots$ so obtained. To get the answer, let us require the moment conditions

$$-\infty < \mathrm{E} \log ||\alpha|| < 0 \;\text{ and }\; \mathrm{E} \log^+ |\beta| < \infty \tag{5.11}$$

where $|\cdot|$ is the usual Euclidean norm of vectors and $||\cdot||$ the operator norm $||\alpha|| = \sup_{|x|=1} |x\alpha|$ (the condition on α is a bit superfluous; it can be relaxed with recourse to the so-called top Lyapunov index, cf. [7]).

Under the conditions (5.11) we can proceed exactly as in the scalar case with the aid of the same device (5.7) that straightforwardly extends to multidimensional case. So we have

$$(x_0, (\alpha_0, \beta_0), \dots, (\alpha_n, \beta_n)) \stackrel{d}{=} (x_0, (\alpha_n, \beta_n), \dots, (\alpha_0, \beta_0)). \tag{5.12}$$

In view of (5.12) we can modify (5.10) in the form of an equality in distribution

$$x_{n+1} \stackrel{d}{=} x_0 \alpha_0 \cdots \alpha_n + \beta_0 + \beta_1 \alpha_0 + \beta_2 \alpha_1 \alpha_0 + \cdots + \beta_n \alpha_{n-1} \cdots \alpha_0, \tag{5.13}$$

cf. (5.8). It can be shown that the first term involving the initial vector x_0 vanishes as $n \to \infty$ and does not contribute to the limit. In fact, the conditions (5.11) guarantee that $||\alpha_0 \cdots \alpha_n|| \to 0$ exponentially fast. Next, the convergence with probability 1 is shown of the sum of n other terms in (5.13) (see [7] for both arguments). Summarizing, these arguments allows us to conclude that there is a limiting random variable $\underline{x}$, i.e.

$$x_n \xrightarrow{d} \underline{x}. \tag{5.14}$$

The limit is unique in distribution and

$$\underline{x} \stackrel{d}{=} \beta_0 + \beta_1\alpha_0 + \beta_2\alpha_1\alpha_0 + \cdots \tag{5.15}$$

where the right-hand side converges in the Euclidean norm with probability 1.

This yields, in particular, the convergence in the mean like in the previous scalar case (compare (5.6) with (5.16) below), provided that the expectations of α and β exist, moreover $\mathrm{E}||\alpha|| < 1$. Indeed, due to independence of the vectors and matrices in the limiting expression (5.15), the expectation of this expression is the convergent sum $\mathrm{E}\beta(\mathrm{I} + \mathrm{E}\alpha + (\mathrm{E}\alpha)^2 + \cdots)$ that converges to $\mathrm{E}\beta\,(\mathrm{I} - \mathrm{E}\alpha)^{-1}$. So, we have

$$\lim_{n\to\infty} \mathrm{E}x_n = \mathrm{E}\underline{x} = \mathrm{E}\beta\,(\mathrm{I} - \mathrm{E}\alpha)^{-1}, \tag{5.16}$$

like in (5.6).

The general results just referred to are intended for application to the dynamical model of section 3.2. This model has been described in terms of the sequence $A_0, A_1, \ldots$ of agent's preference matrices that are assumed to be stochastic, satisfying $\mathbb{I} = A_n\,\mathbb{I}$. In the remainder of this section we will be interested in a different situation in which the matrices $A_0, A_1, \ldots$ are assumed to be independent copies of a certain random non-negative matrix A with a finite expectation $\mathrm{E}A < \infty$. We shall assume that the random matrix A is *stochastic in the mean.* In other words, the expectation is subject to the following restriction: $\mathbb{I} = \mathrm{E}A\,\mathbb{I}$. Moreover, we assume $\mathrm{E}A$ to be primitive, so that its Perron–Frobenius eigenvalue equals 1. Let the corresponding left (row) eigenvector be denoted by ϖ and be normalized so that $\varpi\,\mathbb{I} = 1$. As we know, it is the equilibrium distribution for $\mathrm{E}A$ and satisfies $\varpi = \varpi\,\mathrm{E}A$. See (4.8) for the explicit expression for ϖ in terms of the expectations $\mathrm{E}a$ and $\mathrm{E}b$ where a is an ℓ dimensional upper left corner submatrix in the subdivision (4.5) of A and b an ℓ dimensional row vector. We shell subdivide likewise the observed sequence $A_0, A_1, \ldots,$ so the n^{th} sample A_n will have a_n as the upper left corner submatrix, b_n as the lower left corner row vector and so forth.

For complete characterization of agent's behaviour, we also need to take into consideration his initial endowments $e_0 = (e_0^1, \ldots, e_0^{\ell+1})$ and consequent daily baskets $e_n = (e_n^1, \ldots, e_n^{\ell+1})$ consisting of quantity e_n^j of commodity j at date n. In contrast with agent's preference matrices, the sequence of his baskets is supposed to be deterministic. Moreover, we suppose that this sequence stabilizes towards a certain $\ell + 1$ dimensional limiting vector $e = (e^1, \ldots, e^{\ell+1})$, i.e. $e_n^j \to e^j$ for any commodity j.

Our purpose is to examine the asymptotic behaviour as $n \to \infty$ of daily prices $p_n^j = w_n^j/e_n^j$ on each commodity $j = 1, \ldots \ell$ (cf. (3.11); the price of the numeraire is fixed $p_n^{\ell+1} = 1$ always). It suffices thus to know how the distribution of agent's wealth $(w_n^1, \dots, w_n^{\ell+1})$ behaves as $n \to \infty$. So far we only know that the last component corresponding to the numeraire stabilizes, $w_n^{\ell+1} = e_n^{\ell+1} \to e^{\ell+1}$. The first ℓ components are random variables defined recurrently by the system of equations (3.10). According to present subdivision of the observed matrix A_n, on the right-hand side of latter equations a_n^{kj} for $k, j = 1, \dots, \ell$ present the corresponding entries to the submatrix a_n, while $a_n^{\ell j}$ for $j = 1, \dots, \ell$ present the corresponding entries to the row vector b_n. Hence, in matrix notations the system of equations (3.10) becomes

$$w_{n+1} = w_n\, a_n + e_n^{\ell+1}\, b_n$$

where $w_n = (w_n^1, \dots, w_n^\ell)$ is an ℓ dimensional row vector. This equation is of the form (5.9), so we can apply the previous general results. Let $e_n^{\ell+1} \to e^{\ell+1}$ and let the conditions (5.11) hold with α and β substituted by a and b. Then we can deduce from (5.14) and (5.15) that

$$w_n \xrightarrow{d} \underline{w} \tag{5.17}$$

where $\underline{w} \stackrel{d}{=} e^{\ell+1}(b_0 + b_1 a_0 + b_2 a_1 a_0 + \cdots)$.

In conclusion, we want to deduce from this general result the convergence in the mean of the distribution of agent's wealth to the row vector $(\overline{w}, e^{\ell+1})$ where $\overline{w}$ is given by (4.9), i.e.

$$\lim_{n\to\infty} \mathrm{E}(w_n^1, \dots, w_n^{\ell+1}) = e^{\ell+1}(\mathrm{E}b\,(\mathrm{I} - \mathrm{E}a)^{-1}, 1). \tag{5.18}$$

To this end, note first that $\mathrm{E}w_n \to \mathrm{E}\underline{w}$ by (5.17) and then calculate $\mathrm{E}\underline{w}$ by taking into consideration i.i.d. nature of observations. Like in (5.16) we get $\mathrm{E}\underline{w} = e^{\ell+1}\mathrm{E}b\,(\mathrm{I} - \mathrm{E}a)^{-1}$ which proves (5.18).

In view of the results discussed in the concluding paragraph of section 4.2, the convergence in the mean (5.18) in conjunction with the stability assumption on agent's daily baskets $e_n = (e_n^1, \dots, e_n^{\ell+1})$ (it has been assumed that the sequence of agent's daily baskets stabilizes towards a certain $\ell + 1$ dimensional limiting vector $e = (e^1, \dots, e^{\ell+1})$), allow us to draw the following conclusion. Due to (4.9) and (4.10), the limiting behaviour of the expectation of daily prices $\mathrm{E}p_n^j = \mathrm{E}w_n^j / e_n^j$ on commodities $j = 1, \dots \ell$ is described by

$$\mathrm{E}p_n^j \to \overline{\underline{p}}^j. \tag{5.19}$$

In other words, the convergence in the mean obtains towards the equilibrium prices $\overline{\underline{p}}^j$ that would come into being in the case of a generalized CD agent with the initial endowment e and with the preference matrix $\mathrm{E}A$. In resemblance of Nummelin's terminology in [11], this might be called *asymptotic closure property*.

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