# NONLINEAR POTENTIAL THEORY ON METRIC SPACES

Juha Kinnunen

Institute of Mathematics, P.O. Box 1100,
FIN-02015 Helsinki University of Technology, Finland
juha.kinnunen@hut.fi

and

Olli Martio

Department of Mathematics, P.O. Box 4,
FIN-00014 University of Helsinki, Finland
olli.martio@helsinki.fi

Abstract. We study nonlinear potential theory on a metric measure space equipped with a doubling measure and supporting a Poincaré inequality. Minimizers, superminimizers and the obstacle problem for the $p$-Dirichlet integral play an important role in the theory. We prove lower semicontinuity of superminimizers and continuity of the solution to the obstacle problem with a continuous obstacle. We also show that the limit of a increasing sequence of superminimizers is a superminimizer provied it is bounded above. Our proofs are based on direct methods of Calculus of Variations and on De Giorgi type estimates. Moreover, we consider superharmonic functions and study their relations to superminimizers.

## 1. Introduction

Let $(X, d)$ be a metric space equipped with a Borel measure $\mu$. In the metric measure space a concept of an upper gradient serves as a substitute for the Sobolev gradient. Suppose that $1 \le p < \infty$ and let $u$ be a real-valued function on $X$. A non-negative Borel measurable function $g$ on $X$ is said to be a $p$-weak upper gradient of $u$ if for $p$-almost every rectifiable paths $\gamma$ joining points $x$ and $y$ in $X$ we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds. \tag{1.1}$$

This means that (1.1) holds for all paths in $X$ except of a family of paths which is of $p$-modulus zero. The Sobolev space on a metric measure space, called the Newtonian space $N^{1,p}(X)$, can be defined as an collection of equivalence classes of $p$-integrable functions with $p$-integrable upper gradients. Essentially, this is an

1991 *Mathematics Subject Classification.* 46E35, 31C15.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX extension of the absolute continuity property on almost all lines of Sobolev functions to metric spaces. For this approach on a metric measure space see [Sh2]; a slightly different but equivalent approach is used in [C].

Newtonian spaces enable us to study variational integrals on a metric measure space and nonlinear potential theoretic models can be built on minimizers of variational integrals

$$\int g_u^p \, d\mu, \tag{1.2}$$

where $g_u$ denotes the minimal $p$-weak upper gradient of $u$. Indeed, in [KS] it was shown that the minimizers, and even quasiminimizers, of (1.2) satisfy the Harnack inequality, maximum principle and are locally Hölder continuous under relatively mild conditions. For recent developments in analysis on metric spaces we refer to [B], [BBS], [BMS], [C], [Ha], [HaK], [He], [HeK], [Sh1], [Sh2].

The objective of this paper is to study nonlinear potential theory on a metric measure space. Due to the notion of the upper gradient, it is not clear how to employ partial differential equations. Our approach is based on variational integrals (1.2) only and therefore new arguments are needed. In contrast with the potential theory of minimizers of $p$-Dirichlet integral in the Euclidean case, which is nonlinear only when $p \neq 2$, our theory is nonlinear for all $p > 1$. The reason for this is that the operation of taking the upper gradient is not linear. Classical harmonic functions are replaced by continuous minimizers of (1.2). Superminimizers and the obstacle problem play an important role in the theory. We prove lower semicontinuity of superminimizers and continuity of the solution to the obstacle problem with a continuous obstacle. We also show that the limit of a increasing sequence of superminimizers is a superminimizer provied it is bounded above. Our proofs are based on direct methods of Calculus of Variations and on De Giorgi type estimates, see [Gia], [Giu]. Traditionally these estimates have been used to prove local Hölder continuity and the Harnack inequality for minimizers of variational integrals. We use them to handle semicontinuity problems associated with super- and subminimizers. We close the paper by considering superharmonic functions and studying their relations to superminimizers.

In a forthcoming paper the authors study further properties of nonlinear potential theory on metric measure spaces and show that much of the theory can be extended to quasiminimizers. Methods like the Poisson modification and the Perron method seem to be available also in the metric setting and they open possibilities to study more refined aspects of nonlinear potential theory on metric measure spaces.

## 2. Newtonian spaces

Let $X$ be a metric measure space such that closed and bounded sets are compact and let $\mu$ be a Borel measure on $X$. Throughout the paper we assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. It follows that $\mu$ is a *regular Borel measure*, that is, the measure of every $\mu$-measurable set is the infimum of measures of all open sets containing the given set. On the other hand, the measure of a $\mu$-measurable set is the supremum of all compact sets contained into the given set. These are useful properties which enable us to pass from statements for open or compact sets to arbitrary $\mu$-measurable sets. Later we impose further requirements on the space and on the measure.

**2.1. Upper gradients.** Let $u$ be a real-valued function on $X$. A non-negative Borel measurable function $g$ on $X$ is said to be an *upper gradient* of $u$ if for all rectifiable paths $\gamma$ joining points $x$ and $y$ in $X$ we have

$$|u(x) - u(y)| \leq \int_\gamma g\, ds. \tag{2.2}$$

See [C], [He], [HeK] and [Sh1] for a discussion on upper gradients.

Let $1 \leq p < \infty$. The *p-modulus* of a family of paths $\Gamma$ in $X$ is the number

$$\inf_\rho \int_X \rho^p \, d\mu,$$

where the infimum is taken over all non-negative Borel measurable functions $\rho$ such that for all rectifiable paths $\gamma$ which belong to $\Gamma$ we have

$$\int_\gamma \rho\, ds \geq 1.$$

It is known that the $p$-modulus is an outer measure on the collection of all paths in $X$.

A property is said to hold for *p-almost all* paths, if the set of paths for which the property fails is of zero $p$-modulus. If (2.2) holds for $p$-almost all paths $\gamma$, then $g$ is said to be a *p-weak upper gradient* of $u$.

**2.3. Newtonian spaces.** Let $1 \leq p < \infty$. We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all $p$-integrable functions $u$ on $X$ that have a $p$-integrable $p$-weak upper gradient $g$ on $X$. This space is equipped with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all $p$-weak upper gradients of $u$.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$\|u - v\|_{\widetilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of the Newtonian spaces we refer to [Sh1]. Cheeger [C] gives an alternative definition which leads to the same space when $1 < p < \infty$, see [Sh1].

We recall here some facts for the future reference.

(i) The functions in $\widetilde{N}^{1,p}(X)$ are defined outside a path family of $p$-modulus zero. This implies that functions in $\widetilde{N}^{1,p}(X)$ cannot be arbitrarily changed on sets of measure zero. However, if $u, v \in \widetilde{N}^{1,p}(X)$ such that $u = v$ $\mu$-almost everywhere, then $u$ and $v$ belong to the same equivalence class in $N^{1,p}(X)$.

(ii) The normed space $(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)})$ is a Banach space.

(iii) If $1 < p < \infty$, every function $u$ that has a $p$-integrable $p$-weak upper gradient has a minimal $p$-integrable $p$-weak upper gradient in $X$, denoted $g_u$, in the sense that if $g$ is another $p$-weak upper gradient of $u$, then $g_u \leq g$ $\mu$-almost everywhere in $X$.

(iv) For every $c \in \mathbf{R}$ the minimal $p$-weak upper gradient $g_u = 0$ $\mu$-almost everywhere on the set $\{x \in X : u(x) = c\}$. This implies that if $u, v \in N^{1,p}(X)$, then $\min(u, v) \in N^{1,p}(X)$ and $\max(u, v) \in N^{1,p}(X)$. Moreover, if $g_u$ and $g_v$ are the minimal $p$-weak upper gradients of $u$ and $v$, respectively, then $g_u\chi_E + g_v\chi_{X\setminus E}$, with $E = \{x \in X : u(x) \geq v(x)\}$, is a $p$-weak upper gradient of $\min(u, v)$.

(v) If $u \in N^{1,p}(X)$ and $v$ is a bounded Lipschitz continuous function, then $uv \in N^{1,p}(X)$ and $g_{uv} \leq |u|g_v + |v|g_u$ $\mu$-almost everywhere.

(vi) Suppose that $u_i$, $i = 1, 2, \ldots$, is a sequence of functions in $N^{1,p}(X)$ such that

$$\sup_i \|u_i\|_{N^{1,p}(X)} < \infty.$$

Then there is a subsequence $u_{i_j}$ and $u \in N^{1,p}(X)$ such that $u_{i_j} \to u$ weakly in $L^p(X)$ and

$$\|g\|_{L^p(X)} \leq \liminf_{j\to\infty} \|g_{u_{i_j}}\|_{L^p(X)}.$$

(vii) The functions in $N^{1,p}(X)$ are *absolutely continuous on $p$-almost every path*, which means that $u \circ \gamma$ is absolutely continuous on $[0, \operatorname{length}(\gamma)]$ for $p$-almost every rectifiable arc-length parametrized path $\gamma$ in $X$.

We emphasize that these properties hold without any additional assumptions on the measure and on the space. In particular, the topological assumptions made in the beginning of this section are not needed for these facts.

Next we prove an estimate for the $p$-weak upper gradient of a certain function which will be used as a test function later. This estimate does not seem to follow from (v) and hence more careful analysis is needed.

**2.4. Lemma.** *Suppose that $u, v \in N^{1,p}(X)$, $\eta$ is a Lipschitz continuous function in $X$ with $0 \leq \eta \leq 1$ and $v \geq u$ $\mu$-almost everywhere in $X$. Let $g_u$, $g_v$ and $g_\eta$ be the $p$-weak upper gradients of $u$, $v$ and $\eta$ respectively. Define $w = u + \eta(v - u)$. Then*

$$g_w \leq (1 - \eta)g_u + \eta g_v + (v - u)g_\eta \tag{2.5}$$

*$\mu$-almost everywhere in $X$.*

*Proof.* Let $\gamma$ be the arc-length parametrization of a rectifiable path on which the functions $u$, $v$ and $\eta$ are absolutely continuous. Define $h : [0, \operatorname{length}(\gamma)] \to [0, \infty)$,

$$h(s) = (u \circ \gamma)(s) + (\eta \circ \gamma)(s)((v \circ \gamma)(s) - (u \circ \gamma)(s)).$$

Then $h$ is absolutely continuous and for almost every $s \in [0, \operatorname{length}(\gamma)]$, with respect to the one-dimensional Lebesgue measure, and we have

$$\begin{aligned} h'(s) &= (u \circ \gamma)'(s)(1 - (\eta \circ \gamma)(s)) \\ &\quad + (\eta \circ \gamma)'(s)((v \circ \gamma)(s) - (u \circ \gamma)(s)) + (\eta \circ \gamma)(s)(v \circ \gamma)'(s). \end{aligned}$$

Since $|(u \circ \gamma)'(s)| \leq g_u(\gamma(s))$, $|(v \circ \gamma)'(s)| \leq g_v(\gamma(s))$ and $|(\eta \circ \gamma)'(s)| \leq g_\eta(\gamma(s))$ for almost every $s \in [0, \operatorname{length}(\gamma)]$, we obtain

$$\begin{aligned} |h'(s)| &\leq g_u(\gamma(s))(1 - \eta(\gamma(s))) \\ &\quad + g_\eta(\gamma(s))(v(\gamma(s)) - u(\gamma(s))) + \eta(\gamma(s))g_v(\gamma(s)) \end{aligned}$$

for almost every $s \in [0, \operatorname{length}(\gamma)]$. This proves (2.5). □

**2.6. Capacity.** The *p-capacity* of a set $E \subset X$ is defined by

$$\mathrm{C}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, whose restriction to $E$ is bounded below by 1. The discussion in [KM] can easily be adapted to show that the capacity is an outer measure, see also [Sh1]. A property is said to hold *p-quasieverywhere*, if it holds except on a set of $p$-capacity zero. A function is *p-quasicontinuous*, if there is an open set of arbirarily small $p$-capacity such that the function is continuous when restricted to the complement of the set.

The $p$-capacity is the natural measure for exceptional sets of Sobolev functions. Every function in $\widetilde{N}^{1,p}(X)$ is defined $p$-quasieverywhere. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ $\mu$-almost everywhere, then $u = v$ $p$-quasieverywhere. In particular, this implies that $u$ and $v$ belong to the same equivalence class in $N^{1,p}(X)$.

Suppose that $E \subset X$. Then $\Gamma_E$ denotes the family of all rectifiable paths that intersect the set $E$. It is possible to show that $\mathrm{C}_p(E) = 0$ implies that the $p$-modulus of $\Gamma_E$ is zero, see [Sh1]. From this it follows that if $u \in N^{1,p}(X)$ and $v$ is a measurable function on $X$ such that $v = u$ $p$-quasieverywhere, then $v$ belongs to the same equivalence class in $N^{1,p}(X)$.

**2.7. Newtonian spaces with zero boundary values.** In order to be able to compare the boundary values of Sobolev functions we need a notion of Sobolev spaces with zero boundary values in a metric measure space. Let $E$ be an arbitrary subset of $X$. Following the method of [KKM], we define $\widetilde{N}^{1,p}_0(E)$ to be the set of functions $u\colon E \to [-\infty, \infty]$ for which there exists a function $\widetilde{u} \in \widetilde{N}^{1,p}(X)$ such that $\widetilde{u} = u$ $\mu$-almost everywhere in $E$ and

$$\mathrm{C}_p(\{x \in X \setminus E\colon \widetilde{u}(x) \neq 0\}) = 0.$$

Next we define an equivalence relation on $\widetilde{N}^{1,p}_0(E)$ by requiring that $u \sim v$ if $u = v$ $\mu$-almost everywhere on $E$. Finally we let $N^{1,p}_0(E) = \widetilde{N}^{1,p}_0(E)/\sim$, equipped with the norm

$$\|u\|_{N^{1,p}_0(E)} = \|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)},$$

be the *Newtonian space with zero boundary values.* The norm is unambiguously defined by [Sh1] and the obtained space is a Banach space.

**2.8. Local Newtonian spaces.** We are mainly interested in the local properties of minimizers of variational integrals and hence we introduce the notion of a local Newtonian space. Let $\Omega$ be an open subset of $X$. We say that a subset $A$ of $\Omega$ is *compactly contained* in $\Omega$, abbreviated $A \subset\subset \Omega$, if the closure of $A$ is a compact subset of $\Omega$.

We say that $u$ belongs the the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(A)$ for every measurable set $A \subset\subset \Omega$. If $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ with $1 < p < \infty$, then $u$ has a minimal $p$-weak upper gradient $g_u$ in $\Omega$ in the following sense: If $\Omega' \subset\subset \Omega$ is an open set and $g$ is the minimal upper gradient of $u$ in $\Omega'$, then $g_u = g$ $\mu$-almost everywhere in $\Omega$.

**2.9. Doubling property.** The measure $\mu$ is *doubling*, if there is a constant $c_d \ge 1$ so that

$$\mu(B(z,2r)) \le c_d \mu(B(z,r))$$

for every open ball $B(z,r)$ in $X$. The constant $c_d$ is called the *doubling constant* of $\mu$. Note that by the doubling property, if $B(y,R)$ is a ball in $X$, $z \in B(y,R)$ and $0 < r \le R < \infty$, then

$$\frac{\mu(B(z,r))}{\mu(B(y,R))} \ge c\Big(\frac{r}{R}\Big)^Q$$

for some $c$ and $Q$ depending only on the doubling constant.

A metric space $X$ is said to be *doubling* if there is a constant $c < \infty$ such that every ball $B(x,r)$, $z \in X$, $r > 0$, can be covered by at most $c < \infty$ number of balls with the radii $r/2$. If $X$ is equipped with a doubling measure, then $X$ is doubling.

If the space is doubling, then bounded subsets of $X$ are *totally bounded*, which means that they can be covered by finitely many balls of arbitrarily small radii. If $X$ is also complete, then closed and bounded subsets of $X$ are compact, since compactness is equivalent to being closed and bounded. This implies that $\mu$ regular, as observed in the beginning of this section.

**2.10. Poincaré inequalities.** Let $1 \le q < \infty$. The space $X$ is said to *support a weak* $(1,q)$*-Poincaré inequality* if there are constants $c > 0$ and $\tau \ge 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \le cr \Big( \fint_{B(z,\tau r)} g^q \, d\mu \Big)^{1/q} \tag{2.11}$$

for all balls $B(z,r)$ in $X$, for all integrable functions $u$ in $B(z,\tau r)$ and for all $q$-weak upper gradients $g$ of $u$. The word weak refers to the possibility that $\tau > 1$. If $\tau = 1$, the space is said to support a $(1,q)$*-Poincaré inequality.* In a doubling measure space a weak $(1,q)$-Poincaré inequality implies a weak $(t,q)$-Poincaré inequality for some $t > q$ possibly with a different $\tau$, see [HaK]. More precisely, there are $t > q$, $c > 0$ and $\tau' \ge 1$ such that

$$\Big( \fint_{B(z,r)} |u - u_{B(z,r)}|^t \, d\mu \Big)^{1/t} \le cr \Big( \fint_{B(z,\tau' r)} g^q \, d\mu \Big)^{1/q}, \tag{2.12}$$

for all balls $B(z,r)$ in $X$, for all integrable functions $u$ in $B(z,r)$ and for all $p$-weak upper gradients $g$ of $u$.

In particular, if $X$ supports a weak $(1,q)$-Poincaré inequality for some $q$ with $1 < q < p$, then by Hölder's inequality it supports a weak $(1,p)$-Poincaré inequality and hence [HaK] implies that

$$\Big( \fint_{B(z,r)} |u - u_{B(z,r)}|^p \, d\mu \Big)^{1/p} \le cr \Big( \fint_{B(z,\tau' r)} g^p \, d\mu \Big)^{1/p} \tag{2.13}$$

for all balls $B(z,r)$ in $X$, for all integrable functions $u$ in $B(z,r)$ and for all $p$-weak upper gradients $g$ of $u$.

**2.14. General setup.** We make the following rather standard assumptions.

*From now on we assume, without further notice, that the complete metric measure space $X$ is equipped with a doubling Borel measure for which the measure of* *every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak* $(1,q)$*-Poincaré inequality for some* $q$ *with* $1 < q < p$.

We recall a few useful properties of the Newtonian spaces which hold under these assumptions.

(i) Closed and bounded sets are compact and $\mu$ is a regular Borel measure.
(ii) The different definitions of Sobolev spaces yield the same space.
(iii) The standard Sobolev embeddings hold in the Newtonian space.
(iv) Lipschitz continuous functions are dense in the Newtonian space $N^{1,p}(X)$ and Lipschitz continuous functions which vanish in the complement of $E$ are dense in $N_0^{1,p}(E)$.
(v) Every function in $N^{1,p}(X)$ is $p$-quasicontinuous.
(vi) If $u \in N^{1,p}(X)$, then

$$\lim_{r\to 0} \fint_{B(x,r)} u\,d\mu = u^*(x)$$

exists $p$-quasieverywhere and defines a $p$-quasicontinous function.

See [Sh1] for the properties (i)-(v) and for (vi) we refer to [KL]. The assumption on Poincaré inequality is also needed in the regularity theory for quasiminimizers of variational integrals on metric spaces, see [KS].

## 3. The obstacle problem

Let $1 < p < \infty$. Suppose that $\Omega \subset X$ is an open set, let $\vartheta \in N^{1,p}(\Omega)$ and $\psi\colon \Omega \to [-\infty,\infty]$ be an arbitrary function. We define

$$\mathcal{K}_{\psi,\vartheta}(\Omega) = \{v \in N^{1,p}(\Omega) \colon v - \vartheta \in N_0^{1,p}(\Omega), v \ge \psi \ \mu\text{-a.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ is a *solution of the* $\mathcal{K}_{\psi,\vartheta}(\Omega)$*-obstacle problem with the obstacle* $\psi$ *and the boundary values* $\vartheta$, if

$$\int_\Omega g_u^p\,d\mu \le \int_\Omega g_v^p\,d\mu \tag{3.1}$$

for every $v \in \mathcal{K}_{\psi,\vartheta}(\Omega)$. Here $g_u$ and $g_v$ are the minimal $p$-weak upper gradients of $u$ and $v$ in $\Omega$, respectively. In other words, the solution of the obstacle problem minimizes the $p$-Dirichlet energy among all functions which lie above the obstacle function $\psi$ and have the boundary values $\vartheta$. If $\psi = -\infty$, then the obstacle does not hinder and a solution of the $\mathcal{K}_{-\infty,\vartheta}(\Omega)$-obstacle problem is said to be a *minimizer with boundary values* $\vartheta$ *in* $\Omega$. In other words, it is a solution of the Dirichlet problem with the boundary values $\vartheta$ in $\Omega$.

Using the direct methods in calculus of variations it is possible to prove that the obstacle problem has a unique solution under very mild conditions.

**3.2. Theorem.** *Suppose that* $\Omega \subset X$ *is a bounded open set such that* $\mu(X\setminus\Omega) > 0$. *If* $\mathcal{K}_{\psi,\vartheta}(\Omega) \ne \emptyset$, *there is a unique solution of the* $\mathcal{K}_{\psi,\vartheta}(\Omega)$*-obstacle problem.*

*Proof.* Set

$$I = \inf \int_\Omega g_v^p\,d\mu,$$

where the infimum is taken over all $v \in \mathcal{K}_{\psi,\vartheta}(\Omega)$. Since $\mathcal{K}_{\psi,\vartheta}(\Omega) \neq \emptyset$, we observe that $0 \leq I < \infty$. Let $u_i \in \mathcal{K}_{\psi,\vartheta}(\Omega)$, $i = 1, 2, \dots$, be a minimizing sequence such that

$$I = \lim_{i\to\infty} \int_\Omega g_{u_i}^p \, d\mu.$$

In particular, this implies that the sequence $g_{u_i}$, $i = 1, 2, \dots$, is bounded in $L^p(\Omega)$. Since $\mu(X \setminus \Omega) > 0$ and $\Omega$ is bounded, the Poincaré inequality (Lemma 2.1 in [KS]) implies that

$$\int_\Omega |u_i - \vartheta|^p \, d\mu \leq c \int_\Omega |g_{u_i - \vartheta}|^p \, d\mu \leq c \int_\Omega g_{u_i}^p \, d\mu + c \int_\Omega g_\vartheta^p \, d\mu.$$

from which it follows that

$$\sup_i \|u_i - \vartheta\|_{N_0^{1,p}(\Omega)} < \infty.$$

Since the sequence $u_i - \vartheta$, $i = 1, 2, \dots$, is uniformly bounded in $N_0^{1,p}(\Omega)$, there is a subsequence $u_{i_j}$ and $u \in N^{1,p}(\Omega)$ such that $u - \vartheta \in N_0^{1,p}(\Omega)$, $u_{i_j} \to u$ weakly in $L^p(\Omega)$ and

$$\int_\Omega g_u^p \, d\mu \leq \liminf_{j\to\infty} \int_\Omega g_{u_{i_j}}^p \, d\mu \leq I,$$

see 2.3 (vi). Hence $u$ is the desired minimizer.

We show that $u \in \mathcal{K}_{\psi,\vartheta}(\Omega)$. We already know that $u - \vartheta \in N_0^{1,p}(\Omega)$ so that it is enough to prove that $u \geq \psi$ $\mu$-almost everywhere in $\Omega$. The Mazur lemma implies that a sequence $v_j$, $j = 1, 2, \dots$, of convex combinations of $u_{i_j}$ converges strongly to $u$ in $L^p(\Omega)$. Passing to a subsequence we may assume that the sequence $v_j$ converges to $u$ $\mu$-almost everywhere in $\Omega$. Since $u_{i_j} \geq \psi$ $\mu$-almost everywhere in $\Omega$, we conclude that $v_j \geq \psi$ $\mu$-almost everywhere in $\Omega$ for every $j = 1, 2, \dots$ and consequently $u \geq \psi$ $\mu$-almost everywhere in $\Omega$.

To prove the uniqueness, suppose that $u_1, u_2 \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ are solutions. As in Theorem 7.14 of [C] we conclude that $g_{u_1 - u_2} = 0$ $\mu$-almost everywhere in $\Omega$ and the uniqueness follows from the Poincaré inequality. $\square$

We are mostly interested in local properties of solutions of the obstacle problem and we are keen on defining solutions locally without any requirements on the boundary values. To this end, let $\Omega \subset X$ be open. A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *minimizer* in $\Omega$, if it is a solution of the $\mathcal{K}_{-\infty,u}(\Omega')$-obstacle problem for every open set $\Omega' \subset\subset \Omega$. On the other hand, a function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *superminimizer* in $\Omega$, if it is a solution of the $\mathcal{K}_{u,u}(\Omega')$-obstacle problem for every open set $\Omega' \subset\subset \Omega$.

A solution of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem is a superminimizer, but the converse is not true in general. However, if both $u$ and $-u$ are superminimizers in $\Omega$, then $u$ is a minimizer in $\Omega$.

We observe that if $u$ is a superminimizer, then $\alpha u$ and $u + \beta$ are superminimizers when $\alpha \geq 0$ and $\beta \in \mathbf{R}$. However, the sum of superminimizers is not a superminimizer in general.

We prove a useful result for superminimizers which corresponds to the classical result that the minimum of two supersolutions is a supersolution.

**3.3. Lemma.** *If $u_1$ and $u_2$ are superminimizers in $\Omega$, then* $\min(u_1, u_2)$ *is a superminimizer in $\Omega$.*

*Proof.* Let $u = \min(u_1, u_2)$. Since $N^{1,p}_{\mathrm{loc}}(\Omega)$ is a lattice we observe that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Let $\Omega' \subset\subset \Omega$ be an open set and $v \in N^{1,p}(\Omega')$ such that $u - v \in N^{1,p}(\Omega')$ and $v \ge u$ $\mu$-almost everywhere in $\Omega'$. Set $A = \{x \in \Omega' : u_1(x) > u_2(x)\}$, $B = \{x \in \Omega' : v(x) \ge u_1(x)\}$ and $w = \max(\min(u_1, v), u_2)$. Then $w - u_2 \in N^{1,p}_0(\Omega')$, $w \ge u_2$ $\mu$-almost everywhere in $\Omega'$ and $w = u_2$ $\mu$-almost everywhere in $\Omega' \setminus A$. By the superminimizing property of $u_2$ in $\Omega'$ we obtain

$$\begin{aligned}\int_A g_{u_2}^p \, d\mu + \int_{\Omega'\setminus A} g_{u_2}^p \, d\mu &= \int_{\Omega'} g_{u_2}^p \, d\mu \le \int_{\Omega'} g_w^p \, d\mu \\ &= \int_A g_w^p \, d\mu + \int_{\Omega'\setminus A} g_w^p \, d\mu = \int_A g_w^p \, d\mu + \int_{\Omega'\setminus A} g_{u_2}^p \, d\mu.\end{aligned}$$

From this it follows that

$$\int_A g_{u_2}^p \, d\mu \le \int_A g_w^p \, d\mu \le \int_{A\cap B} g_{u_1}^p \, d\mu + \int_{A\setminus B} g_v^p \, d\mu.$$

This implies that

$$\begin{aligned}\int_{\Omega'} g_u^p \, d\mu &\le \int_A g_{u_2}^p \, d\mu + \int_{\Omega'\setminus A} g_{u_1}^p \, d\mu \\ &\le \int_{A\cap B} g_{u_1}^p \, d\mu + \int_{A\setminus B} g_v^p \, d\mu + \int_{\Omega'\setminus A} g_{u_1}^p \, d\mu \\ &= \int_B g_{u_1}^p \, d\mu + \int_{A\setminus B} g_v^p \, d\mu.\end{aligned}$$

Since $u_1 - \max(u_1, v) \in N^{1,p}(\Omega')$, $\max(u_1, v) \ge u_1$ $\mu$-almost everywhere in $\Omega'$, $\max(u_1, v) = u_1$ $\mu$-almost everywhere in $\Omega' \setminus B$ and $u_1$ is a superminimizer, we obtain

$$\begin{aligned}\int_B g_{u_1}^p \, d\mu + \int_{\Omega'\setminus B} g_{u_1}^p \, d\mu &= \int_{\Omega'} g_{u_1}^p \, d\mu \\ &\le \int_{\Omega'} g_{\max(u_1,v)}^p \, d\mu \le \int_B g_v^p \, d\mu + \int_{\Omega'\setminus B} g_{u_1}^p \, d\mu.\end{aligned}$$

This implies that

$$\int_B g_{u_1}^p \, d\mu \le \int_B g_v^p \, d\mu.$$

From this we conclude that

$$\int_{\Omega'} g_u^p \, d\mu \le \int_B g_{u_1}^p \, d\mu + \int_{A\setminus B} g_v^p \, d\mu = \int_{\Omega'} g_v^p \, d\mu$$

and hence $u$ is a superminimizer in $\Omega$. □

Next we present a preliminary version of the comparison principle.

**3.4. Lemma.** *Suppose that $u$ is the solution of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem and that $\Omega' \subset \Omega$ is open. If $v \in N^{1,p}(\Omega')$ is a minimizer with the boundary values $v$ in $\Omega'$ such that* $\min(u-v,0) \in N_0^{1,p}(\Omega')$. *Then $u \geq v$ $\mu$-almost everywhere in $\Omega'$.*

*Proof.* We observe that $\max(u,v)-u = \max(v-u,0) = -\min(u-v,0) \in N_0^{1,p}(\Omega')$ and clearly $\max(u,v) \geq u$ $\mu$-almost everywhere in $\Omega'$. By the superminimizing property of $u$, we have

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_{\max(u,v)}^p \, d\mu \leq \int_A g_v^p \, d\mu + \int_{\Omega' \setminus A} g_u^p \, d\mu,$$

where $A = \{x \in \Omega' \colon u(x) < v(x)\}$. This implies that

$$\int_A g_u^p \, d\mu \leq \int_A g_v^p \, d\mu.$$

From this we conclude that

$$\int_{\Omega'} g_{\min(u,v)}^p \, d\mu \leq \int_A g_u^p \, d\mu + \int_{\Omega' \setminus A} g_v^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu.$$

Finally we observe that $\min(u,v) - v = \min(u-v,0) \in N_0^{1,p}(\Omega')$ and that $v$ is the unique minimizer with boundary values $v$ in $\Omega'$. It follows that $\min(u,v) = v$ $\mu$-almost everywhere in $\Omega'$ and hence $u \geq v$ $\mu$-almost everywhere in $\Omega'$. □

The following lemma shows that the solution of the obstacle problem is essentially the smallest superminimizer which lies above the obstacle.

**3.5. Lemma.** *Suppose that $u$ is a solution to the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem. If $v$ is a superminimizer in $\Omega$ such that $v \in \mathcal{K}_{\psi,u}(\Omega')$ for some open $\Omega' \subset\subset \Omega$, then $v \geq u$ $\mu$-almost everywhere in $\Omega'$.*

*Proof.* By Lemma 3.3 the function $\min(u,v)$ is a superminimizer in $\Omega$, clearly $\min(u,v) - u \in N_0^{1,p}(\Omega')$ and $u \geq \min(u,v)$ $\mu$-almost everywhere in $\Omega'$. By the superminimizing property of $\min(u,v)$ in $\Omega'$, we have

$$\int_{\Omega'} g_{\min(u,v)}^p \, d\mu \leq \int_{\Omega'} g_u^p \, d\mu.$$

Since the solution of the obstacle problem is unique, we conclude that $\min(u,v) = u$ $\mu$-almost everywhere in $\Omega'$ and consequently $v \geq u$ $\mu$-almost everywhere in $\Omega'$. □

## 4. Regularity of superminimizers

We are interested in the regularity of superminimizers. Minimizers satisfy Harnack's inequality and are locally Hölder continuous after redefinition on a set of measure zero, see [KS]. In contrast with minimizers, superminimizers are not continuous in general. However, it is possible to obtain some regularity results. Our method is essentially the same as in [KS]. We begin by recalling the definition of the De Giorgi class.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ belongs to the *De Giorgi class* $DG_p(\Omega, k_0)$, $k_0 \in \mathbf{R}$, if there is a constant $c < \infty$ such that for all $k \geq k_0$, $z \in \Omega$ and $0 < \rho < R$ for which $B(z, R) \subset\subset \Omega$ the function $u$ satisfies the Caccioppoli type estimate

$$\int_{A_z(k,\rho)} g_u^p \, d\mu \leq c(R-\rho)^{-p} \int_{A_z(k,R)} (u-k)^p \, d\mu$$

where $A_z(k, r) = \{x \in B(z, r) \colon u(x) > k\}$ and $g_u$ is the minimal $p$-weak upper gradient of $u$ in $\Omega$. If the inequality above holds for all $k \in \mathbf{R}$, then we simply write $u \in DG_p(\Omega)$.

**4.1. Lemma.** *Suppose that $u$ is a superminimizer in $\Omega$. Then $-u$ belongs to $DG_p(\Omega)$.*

*Proof.* Let $B(z, R) \subset\subset \Omega$ and $0 < \rho < R$. We show that there is a constant $c < \infty$ such that

$$\int_{A_\rho} g_u^p \, d\mu \leq c(R-\rho)^{-p} \int_{A_R} (k-u)^p \, d\mu \tag{4.2}$$

where $A_r = \{x \in B(z, r) \colon u(x) < k\}$. Let $\eta$ be a $2/(R-\rho)$-Lipschitz cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(z, \rho)$ and $\eta = 0$ on $X \setminus B(z, R)$. We define a test function as

$$v = u + \eta \max(k-u, 0).$$

Then $u - v \in N^{1,p}_0(B(z, R))$, $v \geq u$ $\mu$-almost everywhere in $B(z, R)$ and $v = u$ $\mu$-almost everywhere in $B(z, R) \setminus A_R$. This implies that $v \in \mathcal{K}_{u,u}(B(z, R))$ and the superminimizing property of $u$ gives

$$\int_{A_\rho} g_u^p \, d\mu \leq \int_{A_R} g_u^p \, d\mu \leq \int_{A_R} g_v^p \, d\mu.$$

On the other hand, we observe that $v = (1-\eta)(u-k) + k$ on $A_R$ and hence $g_v \leq (k-u)g_\eta + (1-\eta)g_u$ $\mu$-almost everywhere on $A_R$. Since $g_\eta \leq 2/(R-\rho)$ and $\eta = 1$ on $B(z, \rho)$, we obtain

$$\begin{aligned}
\int_{A_\rho} g_u^p \, d\mu &\leq c \int_{A_R} \left((k-u)^p g_\eta^p + (1-\eta)^p g_u^p\right) d\mu \\
&\leq c(R-\rho)^{-p} \int_{A_R} (k-u)^p \, d\mu + c \int_{A_R \setminus A_\rho} g_u^p \, d\mu.
\end{aligned}$$

Adding the term $c \int_{A_\rho} g_u^p \, d\mu$ to the both sides we arrive at the inequality

$$\int_{A_\rho} g_u^p \, d\mu \leq \theta \int_{A_R} g_u^p \, d\mu + c(R-\rho)^{-p} \int_{A_R} (k-u)^p \, d\mu,$$

where $\theta = c/(c+1) < 1$. For $0 < \rho < r \leq R$ inequality above can be rewritten as

$$\int_{A_\rho} g_u^p \, d\mu \leq \theta \int_{A_r} g_u^p \, d\mu + c(r-\rho)^{-p} \int_{A_R} (k-u)^p \, d\mu.$$

Using a simple iteration argument, see Lemma 5.1 in [Gia], we obtain (4.2) from the previous inequality. □

**4.3. Lemma.** *Let $\psi\colon \Omega \to [-\infty, \infty]$ such that $k_0 = \operatorname{ess\,sup}_\Omega \psi < \infty$. Suppose that $u$ is the solution of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem. Then $u$ belongs to the class $DG_p(\Omega, k_0)$.*

*Proof.* Let $B(z,R) \subset\subset \Omega$ and $0 < \rho < R$. Set

$$v = u - \eta \max(u-k, 0)$$

where $k \geq k_0$ and $\eta$ is a cutoff function as in the proof of Lemma 4.1.

Let $x \in \Omega$. If $u(x) \leq k$, then $v(x) = u(x) \geq \psi(x)$. On the other hand, if $u(x) > k$, then

$$\begin{aligned} v(x) &= u(x) - \eta(x)(u(x)-k) \\ &= (1-\eta(x))(u(x)-k) + k \geq k_0 = \operatorname*{ess\,sup}_{\Omega} \psi \geq \psi(x). \end{aligned}$$

From this we conclude that $v \geq \psi$ $\mu$-almost everywhere in $\Omega$ and that $v$ belongs to the class $\mathcal{K}_{\psi,\vartheta}(\Omega)$. Denote $A_r = A_z(k,r)$. Since $v - u \in N_0^{1,p}(B(z,R))$ and since $g_v \leq (u-k)g_\eta + (1-\eta)g_u$ $\mu$-almost everywhere in $A_R$, a similar reasoning as in the proof of Lemma 4.1 gives

$$\int_{A_\rho} g_u^p \, d\mu \leq c(R-\rho)^{-p} \int_{A_R} (u-k)^p \, d\mu$$

for every $k \geq k_0$ as required. $\square$

The solution of an obstacle problem need not be locally bounded above. However, if the obstacle is bounded above, then the solution is locally bounded above. We denote $u_+ = \max(u, 0)$.

**4.4. Theorem.** *Suppose that $u$ is the solution of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem. If $k_0 = \operatorname{ess\,sup}_\Omega \psi$ and $k \geq k_0$, then*

$$\operatorname*{ess\,sup}_{B(z,R/2)} u \leq k + c\Big(\fint_{B(z,R)} (u-k)_+^p \, d\mu\Big)^{1/p}$$

*whenever $B(z,R) \subset\subset \Omega$. The constant $c$ depends only on $p$, the doubling constant and the constants in* (2.11).

The proof is based on Lemma 4.3 and the De Giorgi method. It is a straightforward adaptation of the argument which proves Theorem 4.2 in [KS] and we leave the details to the interested reader.

*4.5. Remark.* It is useful to observe, that the claim of Theorem 4.4 holds for every exponent $q > 0$. To be more precise, for every $q > 0$ there is constant $c$ such that

$$\operatorname*{ess\,sup}_{B(z,R/2)} u \leq k + c\Big(\fint_{B(z,R)} (u-k)_+^q \, d\mu\Big)^{1/q}.$$

This follows from Remark 4.4 (2) in [KS].

Next we derive a local lower bound for superminimizers. We denote $u_- = -\min(u, 0)$

**4.6. Theorem.** *Suppose that $u$ is a superminimizer in $\Omega$. Then*

$$\operatorname*{ess\,inf}_{B(z,R)} u \geq -c\Big(\fint_{B(z,2R)} u_-^p \, d\mu\Big)^{1/p}$$

*whenever $B(z,2R) \subset\subset \Omega$. In particular, the function $u$ is bounded from below on every set $\Omega' \subset\subset \Omega$.*

*Proof.* Lemma 4.1 implies that $-u \in DG_p(\Omega)$ and hence we can apply Theorem 4.2 in [KS] to conclude that for every $k \in \mathbf{R}$ we have

$$-\operatorname*{ess\,inf}_{B(z,R)} u = \operatorname*{ess\,sup}_{B(z,R)}(-u) \leq -k + c\Big(\fint_{B(z,2R)} (k-u)_+^p \, d\mu\Big)^{1/p}$$

and choosing $k = 0$ we arrive at the desired conclusion. □

The next result, called the weak Harnack inequality, follows immediately from Theorem 7.1 in [KS] by Lemma 4.1.

**4.7. Lemma.** *Suppose that $u \geq 0$ is a superminimizer in an open set $\Omega \subset X$. Then for every ball $B(z,R)$ with $B(z,5R) \subset \Omega$ we have*

$$\Big(\fint_{B(z,R)} u^\sigma \, d\mu\Big)^{1/\sigma} \leq c \operatorname*{ess\,inf}_{B(z,3R)} u,$$

*where $c < \infty$ and $\sigma > 0$ depend on the data in the same way as in Theorem 4.4.*

## 5. Pointwise behaviour of superminimizers

Next we prove that a superminimizer has a lower semicontinuous representative. We denote

$$\operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \operatorname*{ess\,inf}_{B(x,r)} u.$$

**5.1. Theorem.** *Suppose that $u$ is a superminimizer in $\Omega$. Then the function $u^*\colon \Omega \to [-\infty, \infty]$ defined by*

$$u^*(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y) \tag{5.2}$$

*is a lower semicontinuous function in $\Omega$ and it belongs to the same equivalence class as $u$ in $N^{1,p}_{\mathrm{loc}}(\Omega)$.*

*Proof.* First we note that the function $u^*$ defined by (5.2) is automatically lower semicontinuous as an increasing limit of lower semicontinuous functions. Hence it suffices to show that $u$ and $u^*$ belong to the same equivalence class in $N^{1,p}_{\mathrm{loc}}(\Omega)$.

Let $\Omega' \subset\subset \Omega$ be an open set. Suppose first that $M = \operatorname{ess\,sup}_{\Omega'} u + 1 < \infty$. Fix $x \in \Omega'$ and let $B(x,2R) \subset\subset \Omega'$. Denote $m(r) = \operatorname{ess\,inf}_{B(x,r)} u$, where $0 < r \leq 2R$. Theorem 4.6 implies that $m(R) > -\infty$. For every $0 < 5r \leq R$ the function $u - m(5r)$ is a superminimizer in $\Omega$ and $u - m(5r) \geq 0$ $\mu$-almost everywhere in $B(x,5r)$. Hence from Lemma 4.7 we obtain for $0 < r < R/5$ that

$$\begin{aligned} m(r) - m(5r) &\geq m(3r) - m(5r) \geq c\Big(\fint_{B(x,r)} (u - m(5r))^\sigma \, d\mu\Big)^{1/\sigma} \\ &\geq c(M - m(5r))^{(\sigma-1)/\sigma} \Big(\fint_{B(x,r)} (u - m(5r)) \, d\mu\Big)^{1/\sigma}, \end{aligned}$$

where $0 < \sigma < 1$. This implies that

$$0 \le \fint_{B(x,r)} u \, d\mu - m(5r) \le c(M - m(5r))^{1-\sigma}(m(r) - m(5r))^{\sigma}.$$

Since $m(R) > -\infty$, the right hand side tends to zero as $r \to 0$ and hence

$$\operatorname*{ess\,lim\,inf}_{y \to x} u(y) = \lim_{r \to 0} \fint_{B(x,r)} u \, d\mu. \tag{5.3}$$

On the other hand, it follows from the Lebesgue differentiation theorem for Sobolev functions in [KL] that

$$\lim_{r \to 0} \fint_{B(x,r)} u \, d\mu = u(x)$$

for every $x \in \Omega' \setminus A$ where $\mathrm{C}_p(A) = 0$. This implies that $u = u^*$ $p$-quasieverywhere in $\Omega'$ and that they represent the same function in $N^{1,p}(\Omega')$.

The general case, when $u$ is possibly not bounded above, follows by truncation. By Lemma 3.3 the function $u_k = \min(u, k)$ is a superminimizer in $\Omega$ for each $k = 1, 2, \dots$ and by the previous argument $u_k^*$ is a lower semicontinuous representative of $u_k$ in $N^{1,p}_{\mathrm{loc}}(\Omega)$. Moreover, we have

$$u^*(x) = \lim_{k \to \infty} u_k^*(x)$$

for every $x \in \Omega$. Let $\Omega' \subset\subset \Omega$ be an open set. Since $u_k = u_k^*$ in $\Omega' \setminus A_k$ with $\mathrm{C}_p(A_k) = 0$, we conclude that $u^* = u$ in $\Omega' \setminus A$ where $A = \bigcup_{k=1}^{\infty} A_k$ and

$$\mathrm{C}_p(A) = \mathrm{C}_p\Big(\bigcup_{k=1}^{\infty} A_k\Big) \le \sum_{k=1}^{\infty} \mathrm{C}_p(A_k) = 0.$$

Thus $u^* = u$ $p$-quasieverywhere in $\Omega'$ and hence they represent the same function in $N^{1,p}(\Omega')$ as required. □

*5.4. Remark.* As a byproduct of the proof of Theorem 5.1, we obtain the fact that if $u^*$ is the lower semicontinuous representative defined by (5.2) of a superminimizer which is locally bounded above, then

$$u^*(x) = \lim_{r \to 0} \fint_{B(x,r)} u^* \, d\mu$$

for every $x \in \Omega$.

Then we prove that the solution of the obstacle problem is continuous provided the obstacle is continuous.

**5.5. Theorem.** *Suppose that $\psi \colon \Omega \to [-\infty, \infty)$ is continuous. Then the solution $u$ of the $\mathcal{K}_{\psi,\vartheta}(\Omega)$-obstacle problem is continuous. Moreover, the function $u$ is a minimizer in the open set $\{x \in \Omega \colon u(x) > \psi(x)\}$.*

*Proof.* We recall that the solution of the obstacle problem is a superminimizer. Let $u$ be the lower semicontinuous representative of the solution given by Theorem 5.1. By (5.2) we observe that it is enough to show that

$$\operatorname*{ess\,lim\,sup}_{y \to x} u(y) \le u(x) \tag{5.6}$$

for every $x \in \Omega$. To this end, let $x \in \Omega$ and $\varepsilon > 0$. Then by (5.2) and the fact that $u \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ we have

$$u(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y) \geq \operatorname*{ess\,lim\,inf}_{y \to x} \psi(y) = \psi(x)$$

for every $x \in \Omega$ since $\psi$ is continuous.

We observe that $\operatorname{ess\,sup}_{B(x,R)} \psi < \infty$ for every $B(x,R) \subset\subset \Omega$ and from Theorem 4.4 we conclude that $\operatorname{ess\,sup}_{B(x,R)} u < \infty$ for every $B(x,R) \subset\subset \Omega$. Choose $B(x,R) \subset\subset \Omega$ such that

$$\sup_{B(x,R)} \psi \leq u(x) + \varepsilon \quad \text{and} \quad \inf_{B(x,R)} u > u(x) - \varepsilon.$$

This is possible since $u$ is lower semicontinuous and $\psi$ is continuous. Theorem 4.4 and Remark 4.5 imply that

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x,r/2)} (u - (u(x)+\varepsilon)) &\leq c \fint_{B(x,r)} (u - (u(x)+\varepsilon))_+ \, d\mu \\ &= c \fint_{B(x,r)} (u - \min(u, (u(x)+\varepsilon))) \, d\mu \end{aligned}$$

for every $0 < r < R$. Obviously

$$\min(u(y), u(x)+\varepsilon) \geq \min(\inf_{B(x,r)} u, u(x)+\varepsilon) \geq u(x) - \varepsilon$$

for every $y \in B(x,r)$. Hence

$$\begin{aligned} &\fint_{B(x,r)} (u - \min(u, (u(x)+\varepsilon))) \, d\mu \\ &\qquad \leq \fint_{B(x,r)} (u - (u(x)-\varepsilon)) \, d\mu = \fint_{B(x,r)} u \, d\mu - u(x) + \varepsilon. \end{aligned}$$

Since $u$ is locally bounded above, Remark 5.4 implies that

$$\lim_{r \to 0} \fint_{B(x,r)} u \, d\mu = u(x),$$

and the estimates above imply that

$$\operatorname*{ess\,lim\,sup}_{y \to x} u(y) \leq u(x) + c\varepsilon.$$

Inequality (5.6) follows by letting $\varepsilon \to 0$.

Then we prove that $u$ is a minimizer in the open set $A = \{x \in \Omega \colon u(x) > \psi(x)\}$. Let $\Omega' \subset\subset A$ be open and $v \in N^{1,p}(\Omega')$ such that $u - v \in N^{1,p}_0(\Omega')$. We show that

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu.$$

We define $v(x) = u(x)$ when $x \in A \setminus \Omega'$.

By [Sh2] there is a continuous function $\widetilde{v}$ in $\Omega'$ such that $u - \widetilde{v} \in N_0^{1,p}(\Omega')$ and

$$\int_{\Omega'} g_{\widetilde{v}}^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu + \varepsilon.$$

In fact, the function $u - v$ can be approximated by a Lipschitz continuous function with a compact support in $\Omega'$.

Since $u$, $\widetilde{v}$ and $\psi$ are continuous functions on the compact set $\overline{\Omega'}$ and since $u(x) > \psi(x)$ for every $x \in \overline{\Omega'}$, there is $t$, $0 < t < 1$, such that

$$w(x) = u(x) + t(\widetilde{v}(x) - u(x)) \ge \psi(x)$$

for every $x \in \overline{\Omega'}$. Since $w - u = t(\widetilde{v} - u) \in N_0^{1,p}(\Omega')$ and $w(x) \ge \psi(x)$ for every $x \in \Omega'$, we conclude that $w \in \mathcal{K}_{u,\psi}(\Omega')$ and hence by convexity we have

$$\begin{aligned} \int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_w^p \, d\mu &= \int_{\Omega'} g_{u+t(\widetilde{v}-u)}^p \, d\mu \\ \le \int_{\Omega'} ((1-t)g_u + t g_{\widetilde{v}})^p \, d\mu &\le (1-t) \int_{\Omega'} g_u^p \, d\mu + t \int_{\Omega'} g_{\widetilde{v}}^p \, d\mu. \end{aligned}$$

This implies that

$$t \int_{\Omega'} g_u^p \, d\mu \le t \int_{\Omega'} g_{\widetilde{v}}^p \, d\mu$$

and since $t > 0$ we obtain

$$\int_{\Omega'} g_u^p \, d\mu \le \int_{\Omega'} g_{\widetilde{v}}^p \, d\mu \le \int_{\Omega'} g_v^p \, d\mu + \varepsilon.$$

The claim follows, since $\varepsilon > 0$ was arbitrary. □

## 6. A convergence result for superminimizers

We are interested in limits of sequences of superminimizers. We show that the superminimizing property is preserved in increasing or local uniform convergence.

**6.1. Theorem.** *Suppose that $u_i$, $i = 1, 2, \dots$, is an increasing sequence of superminimizers in $\Omega$ and that $u = \lim_{i\to\infty} u_i$ is locally bounded from above in $\Omega$. Then $u$ is a superminimizer in $\Omega$.*

*Proof.* Lemma 4.1 shows that $-u_i \in DG_p(\Omega)$, $i = 1, 2, \dots$ This implies that for every $B(z, R) \subset\subset \Omega$ and $0 < \rho < R$ we have

$$\int_{A_z(k,\rho)} g_{u_i}^p \, d\mu \le c(R-\rho)^{-p} \int_{A_z(k,R)} (-u_i - k)^p \, d\mu,$$

where $A_z(k, r) = \{x \in B(z, r) \colon -u_i(x) > k\}$ and $k \in \mathbf{R}$. For fixed ball $B(z, R) \subset\subset \Omega$, we choose $k < -\sup_i \operatorname{ess\,sup}_{B(z,R)} u_i$. Then

$$\int_{B(z,\rho)} g_{u_i}^p \, d\mu \le c(R-\rho)^{-p} \int_{B(z,R)} (u_i - k)^p \, d\mu,$$

from which we conclude that the sequence $g_{u_i}$ is uniformly bounded in $L^p(\Omega')$. This implies that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Moreover, we may assume that $g_{u_i}$, $i = 1, 2, \dots$, converge weakly to $g_u$ in $L^p(\Omega')$, where $g_u$ is an upper gradient of $u$.

It remains to show the superminimizing property of $u$. To this end fix an open set $\Omega' \subset\subset \Omega$ and a function $v \in N^{1,p}(\Omega')$ such that $v \geq u$ $\mu$-almost everywhere in $\Omega'$ and $v - u \in N^{1,p}_0(\Omega')$. We define $v = u$ in $X \setminus \Omega'$. Let $\varepsilon > 0$ and choose an open set $\Omega''$ such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and

$$\int_{\Omega'' \setminus \Omega'} g_v^p \, d\mu < \varepsilon. \tag{6.2}$$

Let $\eta$ be a Lipschitz cutoff function such that $0 \leq \eta \leq 1$, $\eta = 0$ in $\Omega \setminus \Omega''$ and $\eta = 1$ in $\Omega'$. For each $i = 1, 2, \dots$ set

$$w_i = u_i + \eta(v - u_i).$$

Then $w_i \geq u_i$ $\mu$-almost everywhere in $\Omega''$, $w_i - u_i \in N^{1,p}_0(\Omega'')$ and by Lemma 2.4 we have

$$g_{w_i} \leq (1 - \eta) g_{u_i} + \eta g_v + (v - u_i) g_\eta \tag{6.3}$$

$\mu$-almost everywhere in $\Omega''$.

The superminimizing property of $u_i$ implies that

$$\begin{aligned}
\Big( \int_{\Omega''} g_{u_i}^p \, d\mu \Big)^{1/p} &\leq \Big( \int_{\Omega''} g_{w_i}^p \, d\mu \Big)^{1/p} \\
&\leq \Big( \int_{\Omega''} ((1 - \eta) g_{u_i} + \eta g_v)^p \, d\mu \Big)^{1/p} + \Big( \int_{\Omega''} (v - u_i)^p g_\eta^p \, d\mu \Big)^{1/p} \\
&= \alpha_i + \beta_i.
\end{aligned} \tag{6.4}$$

Then we use the elementary inequality: If $\alpha, \beta \geq 0$, then

$$(\alpha + \beta)^p \leq \alpha^p + p\beta(\alpha + \beta)^{p-1}.$$

Together with (6.4) this yields

$$\begin{aligned}
\int_{\Omega''} g_{u_i}^p \, d\mu &\leq \int_{\Omega''} ((1 - \eta) g_{u_i} + \eta g_v)^p \, d\mu + p\beta_i(\alpha_i + \beta_i)^{p-1} \\
&\leq \int_{\Omega''} (1 - \eta) g_{u_i}^p \, d\mu + \int_{\Omega''} \eta g_v^p \, d\mu + p\beta_i(\alpha_i + \beta_i)^{p-1},
\end{aligned}$$

where we also used convexity of the function $t \mapsto t^p$. This implies that

$$\begin{aligned}
\int_{\Omega'} g_{u_i}^p \, d\mu \leq \int_{\Omega''} \eta g_{u_i}^p \, d\mu &\leq \int_{\Omega''} \eta g_v^p \, d\mu + p\beta_i(\alpha_i + \beta_i)^{p-1} \\
&\leq \int_{\Omega'} g_v^p \, d\mu + \varepsilon + p\beta_i(\alpha_i + \beta_i)^{p-1},
\end{aligned} \tag{6.5}$$

where we used (6.2) in the last inequality.

Since $g_{u_i} \to g_u$ weakly in $L^p(\Omega')$ the lower semicontinuity of the $L^p$-norm implies that

$$\int_{\Omega'} g_u^p \, d\mu \leq \liminf_{i \to \infty} \int_{\Omega'} g_{u_i}^p \, d\mu. \tag{6.6}$$

Recall that $g_{\eta_i} = 0$ $\mu$-almost everywhere in $\Omega'$ and $v = u$ in $\Omega \setminus \Omega'$. Since $u_i \to u$ in $L^p_{\mathrm{loc}}(\Omega)$, we conclude that

$$\beta_i = \Big( \int_{\Omega'' \setminus \Omega'} (u - u_i)^p g_\eta^p \, d\mu \Big)^{1/p} \to 0$$

as $i \to \infty$. Since the sequence $\alpha_i$, $i = 1, 2, \dots$, is bounded, letting $i \to \infty$ in (6.5) and using (6.6), we obtain

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this is the required minimizing property of $u$. $\square$

*6.7. Remarks.* (1) Suppose that $u_i$, $i = 1, 2, \dots$, are as in the statement of Theorem 6.1 and instead of requiring that the sequence is bounded from above we assume that the limit function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then $u$ is a superminimizer.

(2) It is easy to modify the previous proof to cover the following result: If the sequence $u_i$ of minimzers or superminimizers converges locally uniformly in $\Omega$, then the limit function $u = \lim_{i \to \infty} u_i$ is a minimizer or a superminimizer in $\Omega$ respectively.

## 7. Superharmonic functions

In this section we define superharmonic functions and study their basic properties.

Let $\Omega$ be an open subset of $X$. A function $v \colon \Omega \to \mathbf{R}$ is *p-harmonic*, or simply *harmonic*, in $\Omega$ if it is a continuous minimizer in $\Omega$. Observe that every minimizer is locally Hölder continuous after redefinition on a set of measure zero by [KS].

Suppose that $\Omega' \subset\subset \Omega$ is open and $v \in N^{1,p}(\Omega')$. Let $h_{\Omega'}(v) = h(v)$ be the unique harmonic function in $\Omega'$ with $v - h(v) \in N^{1,p}_0(\Omega')$.

A function $u \colon \Omega \to (-\infty, \infty]$ is called *p-superharmonic*, or simply *superharmonic* in $\Omega$, if

(i) $u$ is lower semicontinuous in $\Omega$,
(ii) $u$ is not identically $\infty$ in any component of $\Omega$,
(iii) for every open $\Omega' \subset\subset \Omega$ the comparison principle holds: if $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ and $v \leq u$ on $\partial\Omega'$, then $h(v) \leq u$ in $\Omega'$.

A function $u$ is *subharmonic* in $\Omega$, if $-u$ is superharmonic in $\Omega$.

We begin with some elementary observations. If $u$ is superharmonic, then $\alpha u$ and $u + \beta$ are superharmonic when $\alpha \geq 0$ and $\beta \in \mathbf{R}$. Moreover, the minimum of two superharmonic functions is superharmonic. However, the sum of two superharmonic function is not superharmonic in general.

We observe that the class of superharmonic functions is closed under increasing or local uniform convergence.

**7.1. Lemma.** *Suppose that $u_i$, $i = 1, 2, \dots$, is a sequence of superharmonic functions in $\Omega$. If the sequence $u_i$ is either increasing or converges locally uniformly, then the limit function $u = \lim_{i\to\infty} u_i$ is superharmonic unless $u$ is identically $\infty$ in any component of $\Omega$.*

*Proof.* We only prove the case when the sequence is increasing. The other case is easier. It suffices to check the comparison principle (iii) in the definition. Fix an open set $\Omega' \subset\subset \Omega$ and let $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ with $v \le u$ on $\partial\Omega'$. Let $\varepsilon > 0$. Since $u_i$, $i = 1, 2, \dots$, are lower semicontinuous and $v$ is continuous, the sets

$$A_i = \{x \in \partial\Omega' \colon v(x) < u_i(x) + \varepsilon\},$$

$i = 1, 2, \dots$, are open in $\partial\Omega'$ and $A_1 \subset A_2 \subset \cdots$. Since $\partial\Omega' \subset \bigcup_{i=1}^{\infty} A_i$ and $\partial\Omega'$ is compact, there is $i_0$ such that $\partial\Omega' \subset A_i$ for $i \ge i_0$. Since $u_i + \varepsilon$, $i = 1, 2, \dots$, are superharmonic, by comparison principle we have $h(v) \le u_i + \varepsilon$, $i \ge i_0$, in $\Omega'$ and consequently $h(v) \le u + \varepsilon$ in $\Omega'$. This implies that $h(v) \le u$ in $\Omega'$ as required. □

Following version of the comparison principle is very useful in nonlinear potential theory.

**7.2. Theorem.** *Suppose that $u$ is superharmonic and that $v$ is subharmonic in $\Omega$. If*

$$\limsup_{y\to x} v(y) \le \liminf_{y\to x} u(y) \tag{7.3}$$

*for every $x \in \partial\Omega$, and if both sides of* (7.3) *are not simultaneously $\infty$ or $-\infty$, then $v \le u$ in $\Omega$*

*Proof.* Let $\varepsilon > 0$. Let $\Omega'$ be an open set such that $\Omega' \subset\subset \Omega$ and that $v < u + \varepsilon$ on $\partial\Omega'$. Since $v$ is upper semicontinuous in $\Omega$, there is an decreasing sequence $\varphi_i$, $i = 1, 2, \dots$, of Lipschitz continuous functions in $\overline{\Omega'}$ such that $v = \lim_{i\to\infty} \varphi_i$ everywhere in $\overline{\Omega'}$. Since $\partial\Omega'$ is compact and $u + \varepsilon$ is lower semicontinuous, there is $i_0$ such that $\varphi_i \le u + \varepsilon$ on $\partial\Omega'$ when $i \ge i_0$. Since $\varphi_i \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$, $i = 1, 2, \dots$, and $u + \varepsilon$ is superharmonic, by the comparison principle we conclude that $h(\varphi_i) \le u + \varepsilon$ in $\Omega'$ for $i \ge i_0$. On the other hand, $\varphi_i \ge v$ on $\partial\Omega'$ and since $v$ is subharmonic we have $h(\varphi_i) \ge v$ in $\Omega'$. This implies that $v \le h(\varphi_i) \le u + \varepsilon$ in $\Omega'$ when $i \ge i_0$ and by letting $\varepsilon \to 0$ we obtain $v \le u$ in $\Omega'$. The claim follows from this, since $\Omega$ can be exhausted by sets $\Omega' \subset\subset \Omega$ with the required properties. □

Next we study the connection of superminimizers and superharmonic functions.

**7.4. Proposition.** *If $u$ is a superminimizer in $\Omega$ such that*

$$u(x) = \operatorname*{ess\,lim\,inf}_{y\to x} u(y) \tag{7.5}$$

*for every $x \in \Omega$, then $u$ is superharmonic.*

*Proof.* The function $u$ is lower semicontinuous by (7.5). Since $u$ is lower semicontinuous, it is bounded from below on every $\Omega' \subset\subset \Omega$. Since $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, it is finite $\mu$-almost everywhere on $\Omega$ and hence it cannot be identically $\infty$ in any component of $\Omega$.

Let $\Omega'$ be an open set such that $\Omega' \subset\subset \Omega$ and suppose that $v \in C(\overline{\Omega'}) \cap N^{1,p}(\Omega')$ with $v \le u$ on $\partial\Omega'$. Then $\min(u - v, 0) \in N^{1,p}_0(\Omega')$ and hence we have $\min(u - h(v), 0) \in N^{1,p}_0(\Omega')$. Since $h(v)$ is a minimizer in $\Omega'$, Lemma 3.4 implies that $u \ge h(v)$ for $\mu$-almost everywhere in $\Omega'$ and finally (7.5) implies that $u \ge h(v)$ everywhere in $\Omega'$. □

The following corollary is an immediate consequence of Proposition 7.4 and the fact that every superminimizer has a lower semicontinuous representative, see Theorem 5.1.

**7.6. Corollary.** *For every superminimizer $u$ in $\Omega$ there is a superharmonic function $v$ such that $u = v$ $\mu$-almost everywhere in $\Omega$.*

Suppose that $u_i$, $i = 1, 2, \dots$, is an increasing sequence of lower semicontinuous superminimizers in $\Omega$ given by Theorem 5.1. By Proposition 7.4 every $u_i$ is superharmonic and Lemma 7.1 implies that the limit function $u = \lim_{i\to\infty} u_i$ is superharmonic unless it is identically infinity in any component of $\Omega$. We show that, conversely, every superharmonic function can be locally approximated from below by superminimizers. It is important for us that the approximating superminimizers can be chosen to be continuous.

**7.7. Theorem.** *Suppose that $u$ is a superharmonic function in $\Omega$ and let $\Omega'$ be an open set such that $\Omega' \subset\subset \Omega$. Then there is an increasing sequence of continuous superminimizers $u_i$, $i = 1, 2, \dots$, in $\Omega'$ such that $u = \lim_{i\to\infty} u_i$ everywhere in $\Omega'$.*

*Proof.* Let $\Omega'$ be an open set such that $\Omega' \subset\subset \Omega$. Since $u$ is lower semicontinuous in $\Omega$, there is an increasing sequence $\varphi_i$, $i = 1, 2, \dots$, of Lipschitz continuous functions in $\overline{\Omega'}$ such that $u = \lim_{i\to\infty} \varphi_i$ everywhere in $\overline{\Omega'}$. Let $u_i$, $i = 1, 2, \dots$, be the solution to the $\mathcal{K}_{\varphi_i,\varphi_i}(\Omega')$-obstacle problem in $\Omega'$. Each $u_i$ is a superminimizer and we will show that $u_i$, $i = 1, 2, \dots$, is the required sequence. By Theorem 5.5 the functions $u_i$, $i = 1, 2, \dots$, are continuous in $\Omega'$ and the sets $A_i = \{x \in A \colon u_i(x) > \varphi_i(x)\}$, $i = 1, 2, \dots$, are open. Moreover, $u_i - \varphi_i \in N_0^{1,p}(A_i)$ and by Theorem 5.5 the functions $u_i$, $i = 1, 2, \dots$, are minimizers in $A_i$ with boundary values $\varphi_i$ and hence $u_i = h(\varphi_i)$, $i = 1, 2, \dots$, in $A_i$. Since $\varphi_i$ is continuous in $\overline{A_i}$ and $\varphi_i \le u_i$ on $\partial A_i$, the comparison principle yields $u_i = h(\varphi_i) \le u$ in $A_i$. Since $u_i \le \varphi_i \le u$ in $\Omega' \setminus A_i$, it follows that $u_i \le u$ in $\Omega'$. By continuity $u_i \ge \varphi_i$ in $\Omega'$ and we obtain

$$u = \lim_{i\to\infty} \varphi_i \le \lim_{i\to\infty} u_i \le u$$

in $\Omega'$.

Since $\min(u_i, u_{i+1}) \in \mathcal{K}_{\varphi_i,u_i}(\Omega')$ and $\min(u_i, u_{i+1})$ is a superminimizer in $\Omega'$, by Lemma 3.5 we conclude that $u_{i+1} \ge u_i$ $\mu$-almost everywhere in $\Omega'$ and by continuity everywhere in $\Omega'$. This shows that the sequence $u_i$, $i = 1, 2, \dots$, is increasing and the proof is complete. □

**7.8. Corollary.** *If $u$ is superharmonic in $\Omega$ and, in addition, $u$ is locally bounded above, then $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $u$ is a superminimizer.*

*Proof.* By Theorem 6.1 the limit of a locally bounded and increasing sequence of superminimizers is a superminimizer. Thus the claim follows from Theorem 7.7. □

A superminimizer belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$ by definition. Corollary 7.6 shows that every superminimizer has a superharmonic representative. We show that, conversely, if $u$ is superharmonic function which belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$, then it is a superminimizer.

**7.9. Corollary.** *If $u$ is superharmonic and, in addition, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then $u$ is a superminimizer.*

*Proof.* Since $u$ is superharmonic is locally a limit of an increasing sequence of superminimizers by Theorem 7.7. If the limit function $u$ belongs to $N^{1,p}_{\mathrm{loc}}(\Omega)$, then it is a superminimizer by Remark 6.7 (1). $\square$

Proposition 7.4 implies, in particular, that harmonic functions are both super- and subharmonic. The following lemma provides a converse to this result.

**7.10. Corollary.** *A function is harmonic in $\Omega$ if and only if it is both superharmonic and subharmonic $\Omega$.*

*Proof.* Assume that $u$ is both super- and subharmonic in $\Omega$. Then $u$ is continuous in $\Omega$ and hence locally bounded in $\Omega$. Corollary 7.8 implies that $u$ is both super- and subminimizer in $\Omega$ and consequently it is a minimizer in $\Omega$. $\square$

Observe that there is no assumption on the integrability of a superharmonic function. The following lemma shows that a superharmonic function is locally integrable to a small power.

**7.11. Lemma.** *Suppose that $u \geq 0$ is superharmonic in $\Omega$. Then for every ball $B(z,R)$ with $B(z,5R) \subset \Omega$ we have*

$$\left(\fint_{B(z,R)} u^\sigma \, d\mu\right)^{1/\sigma} \leq c \inf_{B(z,3R)} u \tag{7.12}$$

*where $c < \infty$ and $\sigma > 0$ depend on the data in the same way as in Lemma 4.7.*

*7.13. Remark.* Since lower semicontinous functions are locally bounded from below, and superharmonic functions are not identically infinity, by adding a constant to the function it follows from (7.12) that every superharmonic function is locally integrable to a small power $\sigma > 0$ and, in particular, every superharmonic function is finite $\mu$-almost everywhere.

*Proof of Lemma 7.11.* Let $z \in \Omega$ be such that $u(z) < \infty$. Let $\Omega' \subset\subset \Omega$ such that $B(z,5R) \subset\subset \Omega'$. Let $u_i$, $i = 1, 2, \ldots$, be an increasing sequence of continuous superminimizers in $\Omega'$ such that $u = \lim_{i\to\infty} u_i$ in $\Omega'$. Then $\inf_{B(z,R)} u < \infty$ whenever $R > 0$ is such that $B(z,R) \subset \Omega'$. By Lemma 4.7 for every ball $B(z,R)$ with $B(z,5R) \subset \Omega'$ we have

$$\left(\fint_{B(z,R)} u_i^\sigma \, d\mu\right)^{1/\sigma} \leq c \inf_{B(z,3R)} u_i \leq c \inf_{B(z,3R)} u,$$

where $c < \infty$ and $\sigma > 0$ are as in Lemma 4.7. In particular, they are independent of $i$. Letting $i \to \infty$ we conclude that

$$\left(\fint_{B(z,R)} u^\sigma \, d\mu\right)^{1/\sigma} \leq c \inf_{B(z,3R)} u.$$

$\square$

Next we show that if $u$ is a superminimizer in $\Omega$, then $u$ is a superharmonic in $\Omega$ if and only if $u(x) = \operatorname{ess\,lim\,inf}_{y\to x} u(y)$ for every $x \in \Omega$.

**7.14. Theorem.** *If $u$ is superharmonic in $\Omega$, then*

$$u(x) = \operatorname{ess}\liminf_{y\to x} u(y)$$

*for every $x \in \Omega$.*

As a corollary we obtain the following uniqueness result.

**7.15. Corollary.** *If $u$ and $v$ are superharmonic functions in $\Omega$ and $u = v$ $\mu$-almost everywhere in $\Omega$, then $u = v$ everywhere in $\Omega$.*

The proof of Theorem 7.14 is based on the following lemma.

**7.16. Lemma.** *Suppose that $u$ is a superharmonic function in $\Omega$ such that $u = 0$ for $\mu$-almost everywhere in $\Omega$. Then $u = 0$ everywhere in $\Omega$.*

*Proof.* Since $u$ is lower semicontinuous, we conclude that

$$u(x) \le \liminf_{y\to x} u(y) \le 0$$

for every $x \in \Omega$. On the other hand, $v = \min(u, 0)$ is a superharmonic function which is bounded from above. By Corollary 7.8 it is a supersolution and by Theorem 4.6 we have

$$\inf_{B(z,R)} v \ge -c\Big(\fint_{B(z,2R)} v_-^p \, d\mu\Big)^{1/p}$$

whenever $B(z, 2R) \subset\subset \Omega$. Since $u(x) = 0$ for $\mu$-almost every $x \in \Omega$, we conclude that $\inf_{B(z,R)} v \ge 0$ and consequently $u(x) \ge 0$ for every $x \in B(z, R)$. □

*Proof of Theorem 7.14.* Let $x \in \Omega$. Since $u$ is lower semicontinuous, we have

$$u(x) \le \liminf_{y\to x} u(y) \le \operatorname{ess}\liminf_{y\to x} u(y).$$

Hence it is enough to show that

$$\lambda = \operatorname{ess}\liminf_{y\to x} u(y) \le u(x).$$

For every $\varepsilon > 0$, there is $r > 0$ such that $u(y) > \lambda - \varepsilon$ for $\mu$-almost every $y \in B(x, r)$, where $B(x, r) \subset \Omega$. Then $v = \min(u, \lambda-\varepsilon) - (\lambda-\varepsilon)$ is superharmonic in $\Omega$ and $v = 0$ $\mu$-almost everywhere in $B(x, r)$. By Lemma 7.16 we conclude that $u(y) \ge \lambda - \varepsilon$ for every $y \in B(x, r)$. In particularlr $u(x) \ge \lambda - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have established that $\lambda \le u(x)$. □

## References


[B] J. Björn, *Boundary continuity for quasiminimizers on metric spaces*, to appear in Illinois J. Math.

[BBS] A. Björn, J. Björn and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions on metric measure spaces*, Preprint (2001).

[BMS] J. Björn, P. MacManus and N. Shanmugalingam, *Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces*, to appear in J. Anal. Math.

[C] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428-517.

[Gia] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Birkhäuser Verlag, Zürich, 1993.

[Giu] E. Giusti, *Metodi diretti nel calcolo delle variazioni*, Unione Matematica Italiana, 1994.

[Ha] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, J. Potential Anal. **5** (1995), 403–415.

[HaK] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **688** (2000).

[He] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York, 2001.

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonliear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, Oxford, 1993.

[HeK] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[KKM] T. Kilpeläinen, J. Kinnunen and O. Martio, *Sobolev spaces with zero boundary values on metric spaces*, Potential Anal. **12** (2000), 233-247.

[KL] J. Kinnunen and V. Latvala, *Lebesgue points for Sobolev functions on metric spaces*, to appear in Revista Mat. Iberoamericana.

[KM] J. Kinnunen and O. Martio, *Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. **21** (1996), 367–382.

[KS] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizers on metric spaces*, Manuscripta Math. **105** (2001), 401–423.

[Sh1] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Revista Mat. Iberoamericana **16** (2000), 243–279.

[Sh2] N. Shanmugalingam, *Harmonic functions on metric spaces*, to appear in Illinois J. Math.