ONE-POINT COMPACTIFICATION OF GENERALIZED SPACES

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Abstract. We prove that the one-point compactification of a generalized proper homotopy n-space is a generalized homotopy n-sphere.

1. Introduction

It is well–known that the one–point compactification of the n–dimensional Euclidean space $\mathbb{R}^n$ is homeomorphic to the n–dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. On the other hand, it is also known that the one–point compactification of n–dimensional homotopy (contractible) spaces can be even not simply connected, see e.g. [MS], p. 121. Here we prove that the one-point compactification of a generalized proper homotopy n-space is a generalized homotopy n-sphere.

Recall that a separable metric space M is said to be an *n-manifold* if every point $x \in M$ has a neighborhood $U \subset M$ which is homeomorphic to $\mathbb{R}^n$. All manifolds are assumed to be connected. A locally compact separable metric space X is said to be a *generalized n-manifold* if X is a finite-dimensional ANR, absolute neighborhood retract, which is a *singular homology n-manifold*, i.e., for each $x \in X$, $H_k(X, X \setminus \{x\})$ is isomorphic to $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ for all k (local singular homology over the integers).

Note that among finite-dimensional metric spaces, X is ANR if and only if X is *locally contractible*, i.e., for every $x_0 \in X$ and neighborhood U of x_0 there is a neighborhood V of x_0 such that the inclusion $V \to U$ is homotopic in U to the constant map x_0, see e.g. [MS*], p. 40. Furthermore, a metric space X of finite (covering) dimension $\dim X = n$ is an ANR if and only if X is LC^n, see [Du], p. 244, cf. [Li], p. 142, for metric separable spaces. The last means that X is k–LC, $k \leq n$, at every point $x \in X$, i.e., for each neighborhood U of x there is a neighborhood V of x, $V \subset U$, such that every map $f : S^k \to V$ extends to a map $\overline{f} : \mathbb{B}^{k+1} \to U$.

A (generalized) compact n-manifold G^n of the same homotopy type as S^n is said to be a *(generalized) homotopy n-sphere.* Similarly, a (generalized) open n-manifold G^n of the same homotopy type as $\mathbb{R}^n$ (in the other words, contractible) is called a *(generalized) homotopy n-space.* Finally, a (generalized) open n-manifold G^n which

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is proper homotopy equivalent to $\mathbb{R}^n$ is called a *(generalized) proper homotopy n-space.* Here open means noncompact. For compact spaces, the concept of proper homotopy coincides with the ordinary concept of homotopy, see [Mas], p. 38.

By the well-known theorem of Alexandroff, see e.g. [Bou_1], p. 92, every locally compact topological space X has the one-point compactification $\overline{X} = X \cup \{\infty\}$ where ∞ is an ideal element with open neighborhoods of the form $(X \setminus K) \cup \{\infty\}$ where K is a compact subset of X. Since a natural extension $\overline{f} : \overline{X} \to \overline{Y}$ with $\overline{f}(\infty) = \infty$ for every proper continuous mapping $f : X \to Y$ is continuous, see [Bou_1], p. 104, we have the following.

1.1. Proposition. *Let X and Y be proper homotopy equivalent locally compact topological spaces. Then their one-point compactifications $\overline{X}$ and $\overline{Y}$ are (proper) homotopy equivalent.*

2. The main lemma

A locally compact space X is said to be *σ–compact* or *countable at infinity* if it is a countable union of compact subsets. Note that X is σ-compact if and only if ∞ has a countable fundamental system of neighborhoods in $\overline{X}$, see [Bou_1], p. 94. It is also known that, if a locally compact space X is not σ–compact, then the Alexandroff compactification $\overline{X}$ is not locally connected, see [Ba], p. 397. A space X is said to be 1-LC at ∞, 1*-locally connected at infinity*, if X is not compact and, for every compact set $C \subset X$, there is a compact set K such that $C \subset K \subset X$ and every loop in $X \setminus K$ is contractible in $X \setminus C$. A topological space X is said to *have one end* if X is not compact and if, for every compact set $C \subset X$, there is a compact set K such that $C \subset K \subset X$ and $X \setminus K$ is 0-connected, i.e., path connected.

2.1. Lemma. *Let a space X be 1-LC at ∞, σ-compact and have one end. Then its one-point compactification $\overline{X}$ is 1-LC at the point ∞.*

Proof. Since X is σ-compact and 1-LC at ∞ there is an expanding sequence of compacta $K_l \subset X$, $l = 1, 2, \dots$, exhausting X such that every loop in $X \setminus K_{l+1}$ is contractible in $X \setminus K_l$. We may also assume that $X \setminus K_l$ is path connected for all $l = 1, 2, \dots$ because X has one end.

Let D be the unit closed disk in $\mathbb{R}^2$ and let $\varphi : \partial D \to \overline{X} \setminus K_{l+1}$ be a loop L with $\infty \in \varphi(\partial D)$. We show that the loop is contractible along $\overline{X} \setminus K_l$ with a homotopy keeping ∞. Note that the set $\varphi^{-1}(\infty)$ is closed in ∂D and hence $\partial D \setminus \varphi^{-1}(\infty)$ consists of a countable collection of open subarcs of the unit circle ∂D with their end points at $\varphi^{-1}(\infty)$. The image of every such closed arc under φ is a loop in $\overline{X}$ passing once through ∞. Thus, without loss of generality we may assume that $\varphi^{-1}(\infty)$ consists of one point $c_0 \in \partial D$ only.

Choose a couple of points $c_1, c_2 \in \partial D$ in the component of $\varphi^{-1}(\varphi(\partial D) \setminus K_{l+2})$ including c_0 such that c_0 separates c_1 and c_2. Since $X \setminus K_{l+2}$ is path connected we can join $\varphi(c_1)$ and $\varphi(c_2)$ by a path α in $X \setminus K_{l+2}$. Let β be the path $\varphi(\partial D \setminus \gamma)$ in $X \setminus K_{l+1}$ where γ is an open subarc of ∂D with c_1 and c_2 as ends including c_0. Then α and β form a loop ω in $X \setminus K_{l+1}$. Let $\psi : \partial D \to X \setminus K_{l+1}$ be some parametrization of ω such that $\psi|_\gamma = \varphi|_\gamma$. Then ψ can be extended to a continuous mapping $\psi^* : D \to X \setminus K_l$. Take an arbitrary homotopy $\varphi(\tau, t)$, $t, \tau \in [0, 1]$, of $\overline{\gamma}$ to

$\partial D \setminus \gamma$ inside of D keeping c_1 and c_2. Then $\kappa(\tau, t) = \psi^*(\varphi(\tau, t))$ gives a homotopy of the path β to the path α inside of $X \setminus K_l$ keeping the points $\varphi(c_1)$ and $\varphi(c_2)$ fixed. Thus, the loop L in $\overline{X} \setminus K_{l+1}$ can be homotopied along $\overline{X} \setminus K_l$ to the loop L^* formed by α and $\varphi(\gamma)$ in $\overline{X} \setminus K_{l+2}$ keeping $\varphi(\gamma)$ fixed. Arguing by induction we complete the proof.

3. The main result

3.1. Theorem. *Let X be a generalized proper homotopy n-space. Then its one-point compactification $\overline{X}$ is a generalized homotopy n-sphere.*

Proof. By Proposition 1.1 the space $\overline{X}$ is homotopy equivalent to $\overline{\mathbb{R}^n} \simeq S^n$ and, say by the Urysohn criterion, see e.g. [Ku], p. 241–242, $\overline{X}$ is metrizable, cf. [Bou$_2$], p. 156, and Theorem 2 in [Man], p. 1113. It is also evidently separable and compact.

Now, the one–point compactification $\overline{X}$ is ANR. Indeed, as ANR the space X is locally contractible, consequently, LC^∞, and hence lc_s^∞, singular homology locally connected in all dimension, see Theorem 4.22 in [Gr], p. 228. Thus, X is a cohomology n–manifold by Wilder, see Theorem 2 in [Ra$_1$], p. 1372. Note that the cohomology dimension of X coincides with its topological dimension n, see Theorem 3.2(b) in [Wa], p. 109.

Singular cohomology with compact support is invariant under proper homotopy equivalence, see e.g. Theorem 2.1 in [Mas], p. 39. Therefore, being lc_s^∞, X has the same Cech cohomology with compact support as $\mathbb{R}^n$, see Theorem 1 in [Mar], p. 153. In particular, the latter includes its orientability, see Theorem 4.3(2) in [Bo], p. 12. Then by Corollary 4.7 in [Ra$_2$], p. 958, we have that $\overline{X}$ is also an orientable cohomology n–manifold.

X has one end, and it is σ–compact and 1-LC at ∞ by the invariance of these properties under proper homotopy equivalence. Hence by Lemma 2.1 $\overline{X}$ is 1-LC at the point ∞. Thus, $\overline{X}$ is LC^∞, see Theorem 1 in [Hu], p. 468, cf. Theorem 4.1 in [Ne$_2$], p. 50, and hence $\overline{X}$ is a real ANR.

Finally, since $\overline{X}$ is lc_s^∞ and a cohomology n–manifold, we obtain that $\overline{X}$ is also a singular homology n–manifold, see again Theorem 2 in [Ra$_1$], p. 1372.

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