Tangent cones of certain spaces admitting Poincaré inequality

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0. Introduction

We constructed in [L] Ahlfors-regular spaces having given Hausdorff-dimension greater than 1 and admitting weak $(1,1)$ –Poincaré inequality. We also considered tangent cones of these spaces . By a tangent cone we a mean pointed Gromov–Hausdorff limit $(\bar{X}, \bar{m}, \bar{x})$ of $(X, r_k m, x)$, where $x \in X$, the scaling sequence r_k goes to infinity and m is a metric in X. See [GLP] or [G] for the definition of pointed Gromov-Hausdorff limit.

We have to make a modification to the construction of tangent cones as presented in 5.2 of [L]. In certain cases the real tangent cone consists of two copies of the claimed tangent cone that are glued together at a certain height. This happens when in the construction 2.3 below the so called $-\infty$ *-level* occurs. In the earlier version we made the $-\infty$ -level to the tangent cone only when the center point was in a so called k -level in the original space for some k, but this is not sufficient. The $-\infty$ -level can occur much more generally, as explained below, but only once in a fixed tangent cone. In the earlier version we used the term *wormhole level of order* k of what is below called k -level. This term will be explained in the construction. (In the earlier version we did not explicitly talk about $-\infty$ -levels, but now we have given name to this thing. Anyhow, such only appear in the tangent cones, not in the original construction.) For self-containedness, we recall part of Section 1 and Remark 5.1.2 from [L]. In Section 1 we had the construction of the original spaces, whose tangent cones we are to consider here. Section 5.2 is represented here with much more details. The corrected version of 5.2 of [L] is called Section 2 here. We include also 5.1.2 from [L] into it, with number 2.1.

Recall that a metric measure space (X, d, m) is doubling if for all x and r and for some C independent of x and r we have $m(B(x, 2r)) < Cm(B(x, r))$. Theorem [C, 9.6, page 486] says that if a sequence of complete metric measure spaces satisfies $(1,p)$ –Poincaré inequality and doubling condition with a uniform constant, then its pointed measured Gromov–Hausdorff limit satisfies $(1,q)$ –Poincaré inequality for any $q > p$. See [C, 4.1-4.3, page 449] for the definition of Poincaré inequalities and [C, page 484] for the pointed measured Gromov–Hausdorff convergence. Cheeger and Koskela have separately realized that actually one can get $q = p$ here, using Cheegers results. The same result has been proved by Steven Keith by different method. It is well known that Ahlfors–regularity passes to Gromov-Hausdorff -limits when satisfied with common constant by the converging sequence of spaces. (See [DS, Lemma 8.29, p. 63].) Recall also that Poincaré inequalities and Ahlfors regularity are scale invariant. So we know without any special arguments that tangent cones of spaces constructed in [L] have essentially the same analytic properties as the original spaces themselves. It is also well know from Gromov's compactness theorem that a sequence of uniformly doubling spaces has Gromov-Hausdorff converging subsequences. We mean here that the metric is doubling, i.e. any ball can be covered by bounded number of balls with half radius. See [G] or [GLP]. All spaces considered here are doubling. Thus tangent cones exist for some subsequences.

So why do we need to know the exact structure of the tangent cone if we already know it has the properties one can hope for? There seems to be at least one good reason. The precise structure of the tangent cones is needed in a theorem that will be published later. It says: For almost every dimension $Q > 1$ the following holds. Let F be any of the spaces as constructed below with Hausdorff dimension Q. Let f be a bilipschitz mapping into F from a measurable subset of F with positive measure. Then at almost every point tangent mappings of f are isometries. Moreover, open bilipschitz mappings defined on open sets are locally isometries. The same is true for open quasisymmetric mappings defined on open sets. This is a recently invented result not yet written and not checked by anybody else. We mark with $(*)$ those points that are not needed in fixing the tangent cone consideration of [L]. For example in 2.7 we prove that the error of [L] is not insignificant; at almost every point there are tangent cones with a $-\infty$-level. To prove the abomentioned theorem about bilipschitz and quasiconformal mappings we need none of these points marked with $(*)$, only the fixed version of the tangent cone study from [L].

1. Construction

We do first a special case of the contruction and then generalize in a straightforward way. In the next section things will be mainly explained as for the special case, denoting the modifications when needed and not self-evident.

Restrict the dimension Q first to the interval $(1,2)$. Let t satisfy $1+\ln(2)/\ln(1/t) = Q$. We have thus $t \in (0,1/2)$. Denote by $K = K(t)$ the closed Cantor set in $I = [0,1]$ satisfying $K = tK \cup (tK+1-t)$. The Hausdorff dimension of K is thus $Q-1$. Set $K_0 = tK$ and $K_1 = tK+1-t$. Denote by K_{00} and K_{01} the similar parts of K_0, i.e. t^2K and $t\,(tK+1-t)$ respectively. Continue this naming inductively. For K_a, $|a|$ denotes the length of the binary string a and $a1$ denotes concatenation of the strings a and 1.

We construct our space F from $K \times I$ as follows. (We write the product in reversed order compared to [L].) In $K \times I$ we take the metric induced from $\mathbb{R}^2$. Let j be an infinite sequence of integers greater than or equal to two such that

$$C^{-1}\prod_{i=1}^{m} j_i^{-1} \le t^m \le C\prod_{i=1}^{m} j_i^{-1} \tag{1.0}$$

for every m and for some C independent of m.

Set $w(i) = i\, j_1^{-1}$, where i ranges from 0 to j_1-1. Similarly, $w(h,i) = j_1^{-1}(h+i\, j_2^{-1})$, with integers h and i in ranges $0 \le h < j_1$ and $0 \le i < j_2$. In general,

$$w(m_1,\ldots,m_k) = \sum_{i=1}^{k} m_i \prod_{h=1}^{i} j_h^{-1},$$

where $0 \leq m_i < j_i$. Call sets $K \times w(m_1, \ldots, m_k)$ in $K \times I$ *levels of order* k or shortly k *-levels* provided that $m_k > 0$. The requirement $m_k > 0$ is needed to exclude the overlapping of several levels of different orders. In a k -level identify $K_{a0} \times w(m_1, \ldots, m_k)$ with $K_{a1} \times w(m_1, \ldots, m_k)$ for each binary string a of length $k-1$. A point (x_1, x_2) in $K_{a0} \times w(m_1, \ldots, m_k)$ is thus identified with $(x_1 + t^{k-1}(1-t), x_2)$. Call the resulting space F. Denote the natural projection from $K \times I \to F$ by s. We call m -levels for $m \leq k$ also $\leq k$ -levels. Similarly one defines $> k$ -levels, etc. By *levels* we mean k -levels for any k. We also call s -images of k -levels by the same name. Similarly we push to F all variations of the terminology like $\leq k$ -levels. So, for example, the $\leq k$ -levels of F appear periodically with respect to the height, but k -levels are not periodical, since there are no k -levels in the heights where $k-1$ -levels or more generally $< k$ -levels appear.

In the space F we define the metric as follows. Set

$$d(x, y) = \inf\{H^1(p) |\ s(p) \text{ is a path joining } x \text{ and } y\},$$

where naturally $p \subset K \times I$. Note that s identifies only points that have the same I -component. The projection to I, denoted by h, is therefore also well defined and continuous in F, too. Actually h is even 1-Lipschitz. We call h the *height*. With respect to height we define concepts upwards, downwards, above and below.

It was proved in [L] that F is Ahlfors Q-regular and admits a weak $(1,1)$-Poincaré inequality. To get a space with similar properties and arbitrary dimension greater than 2 we can for example take Cartesian product of F with I^n. This has the same properties. Of course the tangent cone considerations of the following chapter can be easier generalised to $F \times I^n$. There is also another way to generalise the construction and also that was used in [L].

Let t and j be as before and let η be an integer greater than 0. Recall that t is the scaling factor of K and controls j through (1.0). Consider $F(t, j, \eta)$ that is constructed by identifying points of $K^\eta \times I$ as follows. For $1 \leq m \leq \eta$ let p_m be the projection to mth component of the Cartesian product $K^\eta \times I$. The height h is the projection to the last component. We identify x and x' if for some integer $0 < n < \eta + 1$ we have that when $m \neq n$, points $\{p_m(x)\} \times \{h(x)\}$ and $\{p_m(x')\} \times \{h(x')\}$ are equal and when $m = n$, they are identified in the previous construction. Thus x will be identified with $2^\eta - 1$ points if x is identified at all. The earlier construction corresponds thus $F(t, j, 1)$. We denote $F(t, j, 1)$ also by $F(t, j)$ or just F. We will in the following talk mostly of the $F(t, j, 1)$, but the same facts are true and arguments work also for $F(t, j, \eta)$. Sometimes one has to express some facts sligthly differently for $F(t, j, \eta)$ even though it is essentially the same. The Hausdorff dimension of $F(t, j, \eta)$ is $1 + \eta \ln(2)/ln(1/t)$.

2. Gromov–Hausdorff -limits

Remark 2.1. For any $k \in \mathbb{N}_+$ there is a height–preserving isometry i_k that maps $s(K_{a0b} \times I)$ to $s(K_{a1b} \times I)$ and vice versa simultaneously for all a with $|a| = k-1$ and for all b of any length. (When $k = 1$ the string a is empty. Also b can be the empty string.) We call i_k *side-switching of order* k. Composing suitably chosen side-switchings leads to an isometry that maps a given point x to $s(\{0\} \times I)$ and preserves the height. To this we need a composition of infinitely many side-switchings, but this is not a problem, since $\sup_{z\in F} d(i_k(z),z)$ goes to zero when k goes to infinity. Hence the sequence of compositions of suitable i_k converges uniformly, and the limit is an isometry with the limit of the image of x in $s(\{0\}\times I)$. Similar thing is true also for $F(t,j,\eta)$. There we have 2^n "sides" for each order. We can switch any two of them with height preserving isometry.

2.2 Preliminary considerations We consider a tangent cone at $p\in F$. By 2.1 only the height of p and the scaling sequence matter. To avoid the need of easy exceptions in the argument, we assume that $0<h(p)<1$. We first consider the special scaling sequence r_k as below and then generalize.

Definition 2.2.1; The representation b. Let j be as in the construction of F. We use the following representation of $h(p)$. Let $b(p)$ be an infinite sequence of integers with indices in $\mathbb{N}_+$ such that $0\le b_i(p)<j_i$ for each i and

$$h(p)=\sum_{i=1}^{\infty} b_i(p)\prod_{m=1}^{i} j_m^{-1}.$$

So if $j_k=c$ for all k, then $b(p)$ is the ordinary c –ary representation of $h(p)$. Just as in the ordinary representation we make the sequence $b(p)$ unique by picking, if possible, the alternative with $b_k(p)=0$ when $k>M_{b(p)}$. So, when the 1-levels divide I to j_1 pieces, $b_1(p)$ expresses, in which one of the pieces the point p is (counted from the bottom so that the lowest piece is number zero). The 1-levels themselves are counted to the upper piece they touch by the uniqueness requirement. Similarly, when each of these pieces are divided into j_2 pieces, $b_2(p)$ says in which piece the point is. We omit the p from $b(p)$ when no confusion arises.

Let a and b be heights of two successive $\le k$ -levels and let c be a binary string with $|c|=k$. Scaling $s(K_c\times[a,b])$ by $r_k=(b-a)^{-1}$ makes it isometric to a space $F(t,j^{(k)})$ with t being the same as in the original space and $j^{(k)}$ being a left-shifting of the original j, i.e. $j_m^{(k)}=j_{k+m}$. For any $n>k$ the n-levels of $s(K_c\times[a,b])$ become $n-k$-levels of $F(t,j^{(k)})$, at least when using the natural isometry. We consider the entire F scaled by r_k as a new metric space (F_k,d_k) with orders of levels decreased by k also to some negative orders. We denote by p_k the point p understood as a point of the scaled metric space F_k. Now $j_n^{(k)}$ as above determines the ratio of distances of successive $\le n-1$-levels and successive $\le n$-levels in F_k, for all $n>-k$. Because of (1.0), distances of successive n-levels are comparable to t^n in F_k with uniform constants not depending on k. Roughly speaking, for each n the n-levels control the metric at scale Ct^n in any F_k. Moreover, in the end n -levels of F_k will converge to n -levels of the tangent cone, for a suitable subsequence at least.

We further extend the definition of $j^{(k)}$ by setting $j_m^{(k)}=0$ for all $m\in\mathbb{Z}$ such that $m\le -k$. Now we can consider any sequence $j^{(k)}$ as a point in the same space $\mathbb{N}^{\mathbb{Z}}$. This extension is just a formal one and has no counterpart in the construction of F_k that remains a scaled version of F with levels relabeled. Because of inequality (1.0), j has a bounded range. The same holds naturally to any left shifted sequence $j^{(k)}$, even with a uniform bound. So any $j^{(k)}$ is contained in $\{1,2,...,M\}^{\mathbb{Z}}$ that is a compact space in the product topology. Therefore we can take a subsequence $j^{(k_i)}$ converging to a sequence $\bar{j}$ with indices in $\mathbb{Z}$. The extension of $j^{(k)}$ that we used to include it to $\mathbb{N}^{\mathbb{Z}}$ does not affect the limit or the convergence; we could have given any values to the missing indices.

Now b_n expresses, how many $\leq n$-levels there are below p and above the highest $\leq n-1$-level that is also below p. (Strictly speaking we need $n > 1$ here.) When the orders of levels are decreased when moving from F to F_k, we have to left–shift b just like j to get the same information from it. So, when scaled by r_k, we replace $b(p)$ by $b^{(k)}(p)$ that are left–shiftings of the original $b(p)$. More precisely $b_n^{(k)}(p) = b_{n+k}(p)$. Since $0 \leq b_n < j_n$, also b has a bounded range, so we can take another subsequence of $b^{(k_i)}(p)$ such that it converges to a sequence $\bar{b}$ with indices in $\mathbb{Z}$ similarly to j. We construct the tangent cone from these.

2.3. Construction of $\bar{F}(t,\bar{j},\bar{b},\eta)$. We begin with $\eta = 1$. Recall that $t < 1$ denotes the scaling factor of K. Let $\bar{K}$ be $\cup_{n=1}^{\infty} t^{-n} K$. We construct first a preliminary version F^* of the tangent cone $\bar{F}$. In $\bar{K} \times \mathbb{R}$ we mean by the height the natural projection to the second component $\mathbb{R}$. We construct the topological space F^* through a projection mapping $\bar{s}$ that identifies only points that have the same height. We then make the metric $\bar{d}$ in F^* as in F. Let $\bar{s} : \bar{K} \times \mathbb{R} \to F^*$ be as follows. We extend the notion of n -level to negative n and to the unbounded space $\bar{K} \times \mathbb{R}$ in the natural way: Suppose we have an n -level at height y. If $|q| - m = n$ and $x \in t^{-m-1} K_{q0}$, we identify (x,y) to $(x + t^{n-1}(1-t), y)$ by $\bar{s}$.

Next we determine the heights of the n-levels. Set $l_0 = 1$ and $l_n / l_{n-1} = \bar{j}_n$ for $n \in \mathbb{Z}$. We set the difference of heights of successive $\leq n$ -levels to be l_n, analogously to F and F_k. Set $\bar{p} = \bar{s}(\{0\} \times \{0\})$. This will be the limit of p_k in $\bar{F}$. We set an $\leq n$ -level to height

$$u(n) = -\sum_{i=n+1}^{\infty} \bar{b}_i l_i, \tag{2.3.1}$$

analogously to F and F_k. Together with the period l_n we know hence the heights of $\leq n$ -levels. Since we know this also for the order $n-1$, we know exactly, where to put the n -levels. (The $\leq n-1$ -levels are included in the $\leq n$ -levels, since the period l_{n-1} is an integer multiple of l_n and $u(n) - u(n-1)$ is an integer multiple of the period l_n.) If some height is contained in the heights of $\leq n$ -levels for all n, then we put there $-\infty$ *-level* as follows: We glue two copies of F^* at that height. Such can occur only once, since distances of successive $\leq n$ -levels go to infinity when n decreases. The $-\infty$ -level occurs for example if p is at a k -level in F for some k or $B(p_{k_i}, r) \subset F_{k_i}$ contains a level of an order decreasing to $-\infty$ with i for some r independent of i. Here p_{k_i} is the correspondent of p in F_{k_i} and k_i is a sequence of indices such that both $j^{(k_i)}$ and $b^{(k_i)}$ converge to the $\bar{j}$ and $\bar{b}$ used in the construction.

We let the tangent cone $\bar{F}$ be the glued version if the $-\infty$ -level occurs. Otherwise it is F^*. We set $\mathbb{R}^*$ to consist of two copies of $\mathbb{R}$ glued at the height of the $-\infty$ -level if such exists. Otherwise we set $\mathbb{R}^* = \mathbb{R}$. We extend, if needed, the projection $\bar{s}$ to be defined on $\bar{K} \times \mathbb{R}^*$ and have as its image the entire $\bar{F}$. We denote by $\bar{h}$ the height in $\bar{F}$, i.e. the natural projection to $\mathbb{R}$. We denote by $\bar{d}$ the similar metric in $\bar{F}$ that we used in F. Similarly we denote by h_k the height in F_k. Naturally we scale also the image of the height mapping when we scale F to F_k. We normalize these such that $\bar{h}(\bar{p}) = h_k(p_k) = 0$.

2.3.2. Construction of $\bar{F}(t, \bar{j}, \bar{b}, \eta)$ for any η. To construct tangent cones $\bar{F}(t, \bar{j}, \bar{b}, \eta)$ of $F(t, j, \eta)$ we do the following modifications to the earlier construction. To construct the preliminary version F^* of the tangent cone $\bar{F}$ we do as follows. For $1 \leq m \leq \eta$ let p_m be the projection to the mth component of the Cartesian product $\bar{K}^\eta \times \mathbb{R}$. Let the height $\bar{h}$ be the projection to the last component. We identify x and x' if for some integer $0 < n < \eta + 1$ we have that when $m \neq n$, points $\{p_m(x)\} \times \{\bar{h}(x)\}$ and $\{p_m(x')\} \times \{\bar{h}(x')\}$ are equal and when $m = n$ they are identified in the construction of $\bar{F}(t, \bar{j}, \bar{b}, 1)$. Thus each point in a k-level in F^* will have 2^η preimages in the product space. If a $-\infty$-level is needed, we glue naturally 2^η copies of F^* at the appropriate height to get the tangent cone $\bar{F}$.

$(*)$ **Remark 2.3.3.** Let $\bar{j}^+$ be the $\bar{j}$ restricted to positive indices. Then $\bar{F} = \cup_{n=1}^{\infty} \bar{B}(x_n, 1/2)$ and each closed ball $\bar{B}(x_n, 1/2)$ is isometric to $F(t, \bar{j}^+, \eta)$ and the corresponding open balls are pairwise disjoint. This is a kind of "self–similarity" condition. If the sequence j is periodic, its left–shiftings are essentially the same sequence and $\bar{F}$ consists of copies of F with some scale factor. In such case F is also self similar, and the previous statement is a special case of a more general fact.

$(*)$ **Remark 2.3.4.** The spaces $\bar{F}(t, \bar{j}, \bar{b}, \eta)$ $\bar{F}(t, \bar{j}, \bar{b}', \eta)$ are isometric if there is $M \in \mathbb{Z}$ such that $\bar{b}_n = \bar{b}'_n$, when $n < M$. They are also isometric if $\bar{b}_n = 0$ for any n and $\bar{b}'_n = \bar{j}'_n - 1$ for any n. The latter is analogous to non-uniqueness of c -ary representation; in both spaces there is a $-\infty$-level at height 0.

2.4. Gromov–Hausdorff -convergence Denote also by (F_k, d_k, p_k) the suitable subsequence of the original (F_k, d_k, p_k) such that $j_i^{(k)} = \bar{j}_i$ and $b_i^{(k)} = \bar{b}_i$, when $|i| \leq k$, and distance from p_k to the bottom and to the top of the space is more than Ct^{-k} for some $C > 0$. We show that $(\bar{F}, \bar{d}, \bar{p})$ is a pointed Gromov–Hausdorff limit of (F_k, d_k, p_k). Roughly this is because the convergence conditions of the sequences say that $\bar{F}$ and F_k are the same at scales from t^{-k} to t^k, since levels of orders from $-k$ to k control these scales. We will extract from F_k a set A_k whose Ct^k-neighborhood contains $B(p_k, t^{-k})$ and that embeds isometrically to $\bar{F}$ such that the Ct^k-neighborhood of the image contains $B(\bar{p}, t^{-k})$ and p_k is mapped close to $\bar{p}$.

Because all $j^{(k)}$ satisfy (1.0) with a uniform constant, the maximal distance of successive n -levels is less than $C_1 t^n$ for every n in F_k and also in $\bar{F}$. Fix such constant $C_1 > 0$. Moreover, for some C independent of k there can be only one $< -k$ -level in $B(x, Ct^{-k})$. This holds in both $\bar{F}$ and F_k. Fix $C_0 > 0$ such that any $< -k$ -levels have distance more than $10C_0 t^{-k}$ from each other in $\bar{F}$ and in F_k. This is to guarantee that $< -k$ -levels outside the ball do not affect the metric in $B(\bar{p}, C_0 t^{-k})$ or in a somewhat larger ball. We further require that $B(\bar{p}, 3C_0 t^{-k})$ does not meet the bottom or the top of

F_k. We assume that $k > k_0$ to make the previous requirement possible and to make t^k small compared to t^{-k}. So, C_0 is usually a small constant and C_1 is a big constant.

For any $c \in \mathbb{R}$, $d \in \mathbb{R}$ and $x \in \bar{K}$ we call $\bar{s}(\{x\} \times [c, d])$ *lines* of $\bar{F}$, if $\bar{F}$ has no $-\infty$ -levels. If such exists, we let $i \in \{1, 2\}$ to express in which copy of $\mathbb{R}$ the interval is and call $\bar{s}(\{x\} \times [c, d]_i)$ lines. Similarly in F_k. Let z be the height of the highest k -level not above p_k in F_k. Let $A_k \subset F_k$ be the union of lines with height interval $[z - 2C_0t^{-k}, z + 2C_0t^{-k}]$ that can be connected to p_k by using only n -levels with $-k \leq n \leq k$ and the lines themselves. By this we mean that A_k is the union of the maximal set of lines with above height interval such that any line can be connected to p_k by a path in A_k that moves from line to another only at n-levels with $-k \leq n \leq k$. The metric restricted to A_k depends only on n -levels with $-k \leq n \leq k$. Recall the meaning of $u(k)$ from (2.3.1) and from the sentence above it. Because of the convergence conditions for $j^{(k)}$ and $b^{(k)}$, the A_k embeds with a natural isometry e to F^*, becoming the union of lines with height interval $[u(k) - 2C_0t^{-k}, u(k) + 2C_0t^{-k}]$ that can be connected to $\bar{p}$ by n -levels with $-k \leq n \leq k$ and the lines themselves. Namely, l_n is the same as the distance of successive $\leq n$ -levels in both F_k and $\bar{F}$ for $-k \leq n \leq k$, because $\bar{j}$ and $j^{(k)}$ coincide for these indices. Remembering the role of $b^{(k)}$ we also see that for $-1 - k \leq n \leq k$ the quantity

$$u(k) - u(n) = \sum_{m=n+1}^{k} l_m \bar{b}_m = \sum_{m=n+1}^{k} l_m b_m^{(k)}$$

is the same as the distance from z to the highest height of a $\leq n$ -level not above p_k in F_k. Recall that it gives, by definition, the same information in $\bar{F}$. So the n-levels for $-k \leq n \leq k$ will occur at the same distance from the center of the height interval of A_k and from the center of the height interval of its image.

(*) **Remark.** We need here the $u(k)$ instead of taking from $\bar{F}$ the height of the highest $\leq n$ -level not above $\bar{p}$, because $\bar{b}$ can take maximal values near ∞, which possibly induces an $\leq i$ -level at height 0 for some $i \leq k$. This would mean in the end that p_k converges with k from below to an i -level and $\bar{p}$ is at such level. Thus "the height of nearest $\leq n$ -level *not above* $\bar{p}$ " is by no means the limit of "the height of nearest $\leq n$ -level *not above* p_k ".

Since e maps height z to height $u(k)$, also p_k is mapped to almost the same height as $\bar{p}$; difference of their heights is at most C_1t^k. Thus we can choose the isometric embedding e of A_k such that the distance between $\bar{p}$ and the image of p_k is less than C_1t^k in the image of A_k. Namely, the location in the Cantor set direction is insignificant because of sides-switching isometries as in 2.1. Side-switching for any order $-k \leq n \leq k$ maps the A_k to itself. Suitable composition of the these side switchings maps a given line of A_k to another given line of A_k. Hence we can choose e such that the image of the line that goes through p_k goes through $\bar{p}$.

Because of $-\infty$ -levels, there are no gaps in the level system. Distance of succesive $\leq -k - 1$ -levels is l_{-k-1} with no exceptions. So if we look at all $\leq -k - 1$ -levels, then they do appear periodically both in $\bar{F}$ and F_k. Because of convergence conditions for $j^{(k)}$ and $b^{(k)}$, the possible $\leq -k-1$ -level meeting A_k appears at the same distance from z in F_k as from $u(k)$ in $\bar{F}$. Their orders might be different in these two spaces, but since A_k meets at most one such level and the other $\leq -k-1$ levels are far, the order of that level does not play any role metrically in balls $B(\bar{p}, C_0 t^{-k})$ and $B(p_k, C_0 t^{-k})$ that we are studying. We can therefore extend A_k to the other side of the $\leq -k-1$ -level similarly in F_k and $\bar{F}$ without knowing what order the $\leq -k-1$ -levels have. We extend it naturally to the same height interval as before. If A_k meets no $\leq -k-1$-level, then neither $B(\bar{p}, C_0 t^{-k})$ nor $B(p_k, C_0 t^{-k})$ can contain parts that are joined by $\leq -k-1$-levels. Therefore a possible $\leq -k-1$-level near A_k cannot affect the metric in the balls or in A_k or its image. If we had not created the $-\infty$ -level, the level we needed in the extension might be missing from $\bar{F}$; there might be a gap in the level pattern. If $\eta > 1$, we naturally extend A_k to all other $2^\eta - 1$ "sides" that meet at the $\leq -k-1$ -level, if it exists.

Now we show that the $C_1 t^k$-neighborhoods of the extended A_k and its image contain the balls $B(p_k, C_0 t^{-k})$ and $B(\bar{p}, C_0 t^{-k})$. We write this for $\bar{F}$, but similar consideration works in F_k, too. In the following $i = 1, 2$ denotes which copy of $\mathbb{R}$ we use if there are two in $\mathbb{R}^*$. Otherwise it expresses nothing and can be omitted. There is a line that is contained in A_k in every $\bar{s}(t^{-m} K_q \times \mathbb{R}_i)$ that intersects $B(\bar{p}, C_0 t^{-k})$ with $|q| - m = k$ and $i \in \{1, 2\}$. This is because every $\bar{s}(t^{-m} K_q \times \mathbb{R}_i)$ can be connected to each other by $\leq k$ -levels and only by them. (And in each $\leq k$ -level every line of $\bar{s}(t^{-m} K_q \times \mathbb{R}_i)$ meets the other part. In addition, the metric is geodesic, so any point in the ball $B(\bar{p}, C_0 t^{-k})$ has to be connected to the center by a geodesic.)

Moreover, points inside any such $\bar{s}(t^{-m} K_q \times \mathbb{R}_i)$ can be connected to each other by $> k$ -levels. Recall that the maximal distance of successive k -levels is $C_1 t^k$. Within that distance one can also find n -levels for any $n > k$. We can hence make for any point of $\bar{s}(\mathbb{R} \times t^{-m} K_q) \cap B(\bar{p}, C_0 t^{-k})$ a path of length at most $C_1 t^k$ that connects it to a line of A_k. Hence A_k "fills" the balls in Gromov–Hausdorff sense in both spaces.

We glue F_k to one end of $[0, C_1 t^k] \times A_k$ and $\bar{F}$ to the other (with using isometry e in the latter case). In the glued space we can identify $\bar{p}$ and p_k with changing the metric neither in F_k nor in $\bar{F}$. Call this resulting metric space Z_k. When restricted to $B(\bar{p}, C_0 t^{-k}) \subset Z_k$, all points of F_k and $\bar{F}$ have distance at most $C_1 t^k$ to the extended A_k or to its isometric image. The Hausdorff–distance of (isometric embeddings of) $B(\bar{p}, C_0 t^{-k}) \cap F_k$ and $B(\bar{p}, C_0 t^{-k}) \cap \bar{F}$ is thus less than $3C_1 t^k$ in Z_k, as required.

2.5. Generalizing remarks We considered so far only tangent cones coming from subsequences of special scaling sequences r_k that were the inverses of distances of successive $\leq k$ -levels as above. It is easy to see that general sequences make no essential difference. Namely, since j has an upper bound, r_k / r_{k+1} has upper bound, too. Thus any scale factors w_k can be represented as $v_k r_{n(k)}$, where $1/M \leq v_k < M$ for some M. Now taking a suitable subsequence makes v_k to converge. Scaling by $r_{n(k)}$ gives spaces $F_{n(k)}$ with left–shifted $j^{(n(k))}$. Thus we can continue as before. So, in particular, any sequence $(F, w_k d, p)$ converging to some (X, d, x) has a subsequence converging to $(\bar{F}, v\bar{d}, \bar{p})$ constructed as above. Here the scaling factor v is the limit of the converging subsequence of v_k. Hence X is isometric to $\bar{F}$ after a scaling.

We get a similar limit space even if we allow the centers to vary when scaling, because the representations $b^{(k)}$ of the centers still vary in a compact space and converge for a suitable subsequence. Strictly speaking we need that the heights of the centers have a uniform lower bound $\epsilon > 0$ and a uniform upper bound $1-\epsilon$. Otherwise we need to make easy modifications to the space and to the proof. All this works also for $F(t,j,\eta)$ by the same arguments. That will say that the Gromov-Hausdorff convergence can be straightforwardly generalised to $\bar{F}(t,\bar{j},\bar{b},\eta)$ and the process to get $\bar{j}$ and $\bar{b}$ is independent of η. We collect our results to a lemma.

Lemma 2.6. Let $F = F(t,j,\eta)$ be any of the spaces constructed in Chapter 1. Suppose that for all k and some $\epsilon > 0$ we have $h(x_k) \in [\epsilon, 1-\epsilon]$ and w_k goes to infinity with k.

∘ If the pointed Gromov–Hausdorff limits of $(F, w_k d, x_k)$ exists, it is isometric to some $(\bar{F}(t,\bar{j},\bar{b},\eta), v\bar{d}, \bar{x})$ as constructed in 2.3.

∘ The sequence $\bar{j}$ in the construction of the $\bar{F}$ above is the limit of some left-shiftings of the original sequence j. It depends only on the scaling sequence w_k and the original j, not on x_k. The sequence $\bar{b}$ is the limit of certain left-shiftings of some of the sequences $b(x_k)$.

∘ (∗) Let $\bar{j}^+$ be the $\bar{j}$ restricted to positive indices. Then $\bar{F} = \cup_{n=1}^{\infty} \bar{B}(x_n, 1/2)$ and each closed ball $\bar{B}(x_n, 1/2)$ is isometric to $F(t, \bar{j}^+, \eta)$ and the corresponding open balls are pairwise disjoint.

2.7. (∗) **How often $-\infty$ -levels?** We consider here the tangent cone case, i.e. fixed center point p. Recall that for the tangent cone only $h(p)$ and the scaling sequence matter. We prove that for almost every height $h(p)$ there exists a scaling sequence that leads to a $-\infty$-level. Moreover, at almost every height there also exists a scaling sequence that does not lead to a $-\infty$-level. We take the "almost every" naturally with respect to the Hausdorff 1-measure in I. We first characterize the existence of the $-\infty$ -level in terms of the sequence $\bar{b}$. Then we consider the heights at which such $\bar{b}$ can arise.

See (2.3.1) and the sentence above it for the meaning of u. Since $0 \le b_n < j_n$, the same holds for the limits of left–shiftings $\bar{b}$ and $\bar{j}$. Therefore $u(n)$ is the greatest negative height of a $\le n$ -level, unless $u(n) = 0$. Since l_n goes to infinity when n goes to minus infinity, the heights of $\le n$ -levels that occur below $u(n)$ go to minus infinity with n. Thus the $-\infty$ -level occurs at a negative height if and only if there is an integer M such that $u(n) = u(M)$, when $n < M$, but not for any n. This means $\bar{b}_n = 0$ for all $n < M$, but $\bar{b}_n > 0$ for some $n \in \mathbb{Z}$. If $\bar{b}_n = 0$ for all $n \in \mathbb{Z}$, then the height of the $-\infty$ -level is zero. Similarly, since $u(n) + 2l_n$ goes to infinity with n, the only possible positive height for $-\infty$ -level is $u(n) + l_n$ that should be independent of n for $n < M$ for some M. Thus the $-\infty$ -level occurs at positive heights if and only if $\bar{b}_n$ takes the greatest possible value $\bar{j}_n - 1$ when n is less than some M but it does not take the value $\bar{j}_n - 1$ for every $n \in \mathbb{Z}$. If the maximal value occurs for all n, the height of the $-\infty$ -level is zero.

So, how frequent are the heights $h(p)$, where the $-\infty$ -level is possible for some scaling sequence? We consider here the easy case $j_n = c$ for all n. Then b is the c -ary representation of the height $h(p)$. The following argument works as well for more general j with easy modifications. To see the easy generalization one needs the inequality (1.0).

If there are arbitrary long pieces of the form $b_n, b_{n+1}, ..., b_{n+m}$ that are entirely zero, then there exists a sequence of left-shiftings such that in the limit $\bar{b}_n = 0$ for all $n < M$ for some M. Hence there also exists a sequence of scalings that gives raise to the $-\infty$ -level. It is easy to show and surely well known that almost every real number has arbitrarily long zero-pieces as above in its c –ary representation. Namely, consider the numbers that have not. For any such number there is an upper bound to the length of zero pieces. We can take a countable set of upper bounds. The set of numbers with a fixed upper limit is contained in a Cantor type set. Namely, if M is the upper limit of lengths of zero pieces in a subset B of the unit interval, then we divide the unit interval to c^{M+1} equal subintervals. The first interval contains no points of B, since the first $M+1$ digits are zero there. We divide each remaining intervals to c^{M+1} equal subintervals. In each interval the first subinterval has no points of B, since that would mean too long zero piece in the c -ary representation. We continue this way and see that B is contained in a Cantor type set with Hausdorff-dimension less than one. In particular, B has measure zero. Similarly at almost every height there are arbitrary long pieces of the form $0, 1, 0, 1, 0, ...$ in b. Then one can take a scaling sequence that leads to a space without $-\infty$-levels.

Remark 2.7.1. It is proved in [L] that the projection $s : (K \times I) \to F$ is David-Semmes -regular. See [L] or [DS] for the definition. For David-Semmes -regular mappings the Hausdorff measure of a set is comparable to the Hausdorff-measure of its image. The comparability constants depend only on the regularity data associated to the mapping. In $K \times I$ the Fubini theorem holds for the Hausdorff-measure. Thence we know that the $-\infty$-levels can occur at almost every point of F with respect to the Hausdorff-measure.

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