BMO–QUASICONFORMAL MAPPINGS AND Q-HOMEOMORPHISMS IN $\mathbb{R}^n$

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Abstract. Space BMO-quasiconformal mappings satisfy a special modulus inequality that is used to define the class of Q-homeomorphisms. In this class we study distortion theorems, the reflection principle, boundary behavior, removability and mapping problems. Our proofs are based on extremal length methods and properties of BMO functions.

1. Introduction

Throughout this paper, $f : D \to \mathbb{R}^n$ include the assumption that D is a domain (an open connected set) in $\mathbb{R}^n, n \geq 2$, and that f is continuous. For $f : D \to \mathbb{R}^n$ with partial derivatives a.e. and $x \in D$, we let $f'(x)$ denote the Jacobian matrix of f at x or the differential operator of f at x, if it exists, by $J(x) = J(x, f) = \det f'(x)$ the Jacobian of f at x, and by $|f'(x)|$ the operator norm of $f'(x)$, i.e. $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. We also let $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$.

The *outer dilatation* of f at x is defined by

$$K_O(x) = K_O(x, f) = \begin{cases} \frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x, f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{1.1}$$

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the *inner dilatation* of f at x by

$$K_I(x) = K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f) \neq 0 \\ 1, & \text{if } f'(x) = 0 \\ \infty, & \text{otherwise,} \end{cases} \tag{1.2}$$

and the *maximal dilatation*, or in short the *dilatation*, of f at x by

$$K(x) = K(x,f) = \max(K_O(x), K_I(x)), \tag{1.3}$$

cf. [MRV], [Ri] and [Vä]. Note, that $K_I(x) \leq K_O(x)^{n-1}$ and $K_O(x) \leq K_I(x)^{n-1}$, see e.g. 1.2.1 in [Re_1], and, in particular, $K_O(x)$, $K_I(x)$ and $K(x)$ are simultaneously finite or infinite.

Suppose that $f : D \to \mathbb{R}^n$ has partial derivatives a.e. and that $J(x,f) \geq 0$ a.e. We say that f is of *finite dilatation* if one, and hence, all the dilatations K_O, K_I and K are finite a.e. in D. Note, that, if f has partial derivatives a.e., then the condition $K(x,f) < \infty$ a.e. is equivalent to the condition that a.e. either $\det f'(x) > 0$ or $f'(x) = 0$, cf. [GI] and [IŠ].

Definition 1.4. Given a function $Q : D \to [1,\infty)$ we say that a sense-preserving homeomorphism $f : D \to \mathbb{R}^n$ is $Q(x)$-*quasiconformal*, abbr. $Q(x)$-*qc*, if

$$K(x,f) \leq Q(x) \qquad \text{a.e.} \tag{1.5}$$

and $f \in \mathrm{W}^{1,n}_{loc}(D \setminus C)$ where C is a closed set in D with

$$|f(C)| = |C| = 0. \tag{1.6}$$

Obviously, every $Q(x)$–qc mapping f is of finite dilatation and if $Q \in \mathrm{L}^{\infty}(D)$ and $C = \emptyset$, then f is quasiconformal in D, cf. [Vä, 34.6]. We will show in Section 2 that the latter remains true for $C \neq \emptyset$, provided that $|f(C)| = |C| = 0$. This result is related to removability theorem which was proved by Strebel [St] for $n = 2$ and generalized by Väisälä [$\mathrm{Vä}_1$, 35.1] for $n \geq 2$. They used the assumption that C is closed set of σ-finite $(n-1)$-Hausdorff measure.

The removability and many other results in this paper are based on the fact that every $Q(x)$-qc mapping with $Q \in L^{n-1}_{loc}$ satisfies the modulus inequality (1.8) which appears in the following definition.

Definition 1.7. Let D be a domain in $\mathbb{R}^n$ and $Q : D \to [1, \infty]$. We say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int\limits_D Q(x)\rho^n(x)dm(x) \tag{1.8}$$

for every family Γ of paths in D possibly with ends in ∂D and every admissible function ρ for Γ.

Recall that, given a family of paths Γ in a domain D, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in adm\Gamma$, if

$$\int\limits_\gamma \rho(x)|dx| \geq 1$$

for each $\gamma \in \Gamma$. The *modulus* $M(\Gamma)$ of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in adm\Gamma} \int\limits_D \rho^n(x)dm(x).$$

Definition 1.9. We say that $f : D \to \mathbb{R}^n$ is *BMO-quasiconformal*, abbr. *BMO-qc*, if f is $Q(x)$-qc for some BMO function $Q : D \to [1, \infty)$.

BMO stands for bounded mean oscillation in the sense of John and Nirenberg [JN], see also [RR]. Q-homeomorphisms and BMO-qc mappings are Möbius invariants and, hence, the notations are extended to mappings $f : D \to \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ as in the standard qc theory, see [Vä].

In this paper we study various properties as distortion, removability, boundary behavior and mapping properties of Q-homeomorphisms under various conditions on Q. One of the main results in this paper, Theorem 2.19, asserts that every $Q(x)$-qc mapping $f : D \to \mathbb{R}^n, n \geq 2$, with $Q \in \mathrm{L}^{n-1}_{loc}$ is a Q-homeomorphism. A special attention is paid to BMO-qc mappings in $\mathbb{R}^n, n \geq 3$.

The study of related maps for $n = 2$ started by David [Da] and Tukia [Tu]. More recently Astala, Iwaniec, Koskela and Martin [AIKM], [IKM] and [IM] considered mappings with dilatation controlled by BMO functions for $n \geq 2$. The present paper is a continuation of our study of BMO-qc mappings in the plane [RSY$_{1-4}$], see also [Sa], and a similar geometric approach is used throughout.

For $a, b \in \overline{\mathbb{R}^n}$ and $E, F \subset \overline{\mathbb{R}^n}$ we let $q(a,b), q(E)$ and $q(E,F)$ denote the spherical (chordal) distance between the points a and b, the spherical diameter of E and the spherical distance between E and F, respectively. We denote by $B^n(a,r)$ the Euclidean ball $|x-a|<r$ in $\mathbb{R}^n$ with center a and radius r, $S^n(a,r) = \partial B^n(a,r)$, and by $\mathbb{H}^n$ the upper half space $x_n > 0$ in $\mathbb{R}^n$. We also let $B^n(r) = B^n(0,r)$ and $\mathbb{B}^n = B^n(1)$, $S^n = \partial\mathbb{B}^n$.

2. Preliminaries

2.1. Proposition. *Let $f : D \to \mathbb{R}^n$ be a $Q(x)$-qc mapping. Then*

(i) f is differentiable a.e.,

(ii) f satisfies Lusin's property (N),

(iii) $J_f(x) \geq 0$ a.e.

If, in addition, $Q \in BMO(D)$, or more generally $Q \in L^{n-1}_{loc}$, then $f^{-1} \in W^{1,n}_{loc}(f(D)\setminus f(C))$ and

(iv) f^{-1} is differentiable a.e.,

(v) f^{-1} has the property (N),

(vi) $J_f(x) > 0$ a.e.

Proof. (i) and (ii) follow from the corresponding results for $W^{1,n}_{loc}$ homeomorphisms, see [Re$_2$] and [Re$_3$]. In view of (i) and the fact that f is sense-preserving, (iii) follows by Rado-Reichelderfer [RR*, p. 333].

Now, if $Q \in$BMO, then Q and hence $K(x,f)$ belongs to L^p_{loc} for all $p < \infty$ and, in particular, to L^{n-1}_{loc}. Hence, by Theorem 6.1 in [HK], $f^{-1} \in \mathrm{W}^{1,n}_{loc}(f(D)\setminus f(C))$ and thus (iv)-(vi) follow.

2.2. L^1–BMO domains and BMO extension domains. Let D be a domain in $\mathbb{R}^n$. We say that D is an *L^1–BMO domain*, if $u \in L^1(D)$ whenever $u \in BMO(D)$. We say that D is a BMO *extension domain*, if every $u \in BMO(D)$ has an extension to $\mathbb{R}^n$ which belongs to $BMO(\mathbb{R}^n)$. Note that, by [Jo], every domain which is bounded by a quasisphere, i.e., the image of $\partial\mathbb{B}^n$ under a qc automorphism of $\overline{\mathbb{R}^n}$, is by the reflection a BMO extension domain. The converse is not true. The domain $D = \mathbb{H}^n \setminus \overline{B}(e_n, 1)$ in $\mathbb{R}^n$, $n \geq 3$, for instance, is not a uniform domain but it is a BMO extension domain. Indeed, given $u \in BMO(D)$, we can apply the reflection principle for BMO functions, see [RR], in $\partial\mathbb{H}^n$ and $\partial\overline{B}(e_n, 1)$ and their images under the group which is generated by these reflections ending with a BMO function in $\mathbb{R}^n \setminus \{0\}$ which is in fact BMO in $\mathbb{R}^n$. Note also that, if a bounded domain D in $\mathbb{R}^n$ is a BMO extension domain, then D is an L^1–BMO domain.

For $u : D \to \overline{\mathbb{R}^+} = [0, \infty]$, the abbreviation $u \in L^1$–$BMO(D)$ will mean that $u \in L^1(D)$ and $u \in BMO(D)$. We also write $u \in BMO(\overline{D})$ if u has a BMO extension to an open set $U \subset \mathbb{R}^n$ such that $\overline{D} \subset U$. Clearly both conditions are automatically held for all functions $u \in BMO(D)$ if D is a BMO extension domain.

2.3. Lemma. *Let Q be a positive BMO function in $\mathbb{B}^n, n \geq 3$, and let $A(t) = \{x \in \mathbb{R}^n : t < |x| < e^{-1}\}$. Then for all $t \in (0, e^{-2})$,*

$$\int\limits_{A(t)} \frac{Q(x)dm(x)}{(|x| \log\ 1/|x|)^n} \leq c \tag{2.4}$$

where $c = c_1||Q||_* + c_2 Q_1$, and c_1 and c_2 are positive constants which depend only on n. Here $||Q||_*$ is the BMO norm of Q and Q_1 is the average of Q over $B^n(1/e)$.

Proof. Fix $t \in (0, e^{-2})$, and set

$$\eta(t) = \int\limits_{A(t)} \frac{Q(x)dm(x)}{(|x| \log\ 1/|x|)^n}. \tag{2.5}$$

For $k = 1, 2, ...,$ write $t_k = e^{-k}$, $A_k = \{x \in \mathbb{R}^n : t_{k+1} < |x| < t_k\}, B_k = B^n(t_k)$ and let Q_k be the mean value of $Q(x)$ in B_k. Choose an integer N, such that $t_{N+1} \leq t < t_N$. Then $A(t) \subset A(t_{N+1}) = \cup_{k=1}^{N+1} A_k$, and

$$\eta(t) = \int\limits_{A(t)} \frac{Q(x)}{|x|^n \log^n\ 1/|x|} dx \leq \int\limits_{A(t_{N+1})} \frac{Q(x)}{|x|^n \log^n\ 1/|x|} dx = \tag{2.6}$$

$$\sum_{k=1}^{N+1} \int\limits_{A_k} \frac{Q(x)}{|x|^n \log^n\ 1/|x|} dx = S_1 + S_2$$

where

$$S_1 = \sum_{k=1}^{N+1} \int\limits_{A_k} \frac{Q(x) - Q_k}{|x|^n \log^n \ 1/|x|} dx \tag{2.7}$$

and

$$S_2 = \sum_{k=1}^{N+1} Q_k \int\limits_{A_k} \frac{dx}{|x|^n \log^n \ 1/|x|} \tag{2.8}$$

Since $A_k \subset B_k$, and for $x \in A_k$, $|x|^{-n} \leq \Omega_n e^n/|B_k|$, where $\Omega_n = |\mathbb{B}^n|$ and since $log\ 1/|x| > k$, it follows that

$$|S_1| \leq \Omega_n \sum_{k=1}^{N+1} \frac{1}{k^n} \frac{e^n}{|B_k|} \int\limits_{B_k} |Q(x) - Q_k| dx \leq \Omega_n e^n ||Q||_* \sum_{k=1}^{N+1} \frac{1}{k^n}.$$

Since $p \geq 2$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} < 2. \tag{2.9}$$

it follows that

$$|S_1| \leq 2\Omega_n e^n ||Q||_* \tag{2.10}$$

To estimate S_2, we first obtain from the triangle inequality that

$$Q_k = |Q_k| \leq \sum_{l=2}^{k} |Q_l - Q_{l-1}| + Q_1. \tag{2.11}$$

Next we show that, for $l \geq 2$,

$$|Q_l - Q_{l-1}| \leq e^n ||Q||_*. \tag{2.12}$$

Indeed,

$$|Q_l - Q_{l-1}| = \frac{1}{|B_l|} |\int\limits_{B_l} (Q(x) - Q_{l-1}) dx| \leq$$

$$\frac{1}{|B_l|}\int\limits_{B_l}|Q(x)-Q_{l-1}|dx=\frac{e^n}{|B_{l-1}|}\int\limits_{B_l}|Q(x)-Q_{l-1}|dx\le$$

$$\frac{e^n}{|B_{l-1}|}\int\limits_{B_{l-1}}|Q(x)-Q_{l-1}|dx\le e^n||Q||_*.$$

Thus, by (2.11) and (2.12),

$$Q_k\le Q_1+\sum_{l=2}^{k}e^n||Q||_*\le Q_1+ke^n||Q||_*,$$

and, since

$$\int\limits_{A_k}\frac{dx}{|x|^n\ \log^n 1/|x|}\le\frac{1}{k^n}\int\limits_{A_k}\frac{dx}{|x|^n}=\omega_{n-1}\frac{1}{k^n},$$

where ω_{n-1} is the $(n-1)$-measure of S^{n-1}, it follows that

$$S_2\le\omega_{n-1}\sum_{k=1}^{N+1}\frac{Q_k}{k^n}\le\omega_{n-1}\sum_{k=1}^{N+1}\frac{Q_1+ke^n||Q||_*}{k^n}$$

or

$$S_2\le\omega_{n-1}Q_1\sum_{k=1}^{N+1}\frac{1}{k^n}+\omega_{n-1}e^n||Q||_*\sum_{k=1}^{N+1}\frac{1}{k^{(n-1)}}.$$

Since $n\ge 3$, it follows by (2.9), that

$$S_2\le 2\omega_{n-1}Q_1+2\omega_{n-1}e^n||Q||_*. \tag{2.13}$$

Finally, from (2.10) and (2.13) we obtain

$$\eta(t)\le c \tag{2.14}$$

where $c=c_1Q_1+c_2||Q||_*$, and c_1 and c_2 are constants which depend only on n. The proof is complete.

2.15. Remark. For $n\ge 2$, $0<t<e^{-2}$, and $A(t)$ as in Lemma 2.3, let Γ_t denote the family of all paths joining the spheres $|x|=t$ and $|x|=e^{-1}$ in $A(t)$. Then the function ρ which is given by

$$\rho(x)=\frac{1}{(\log\log 1/t)|x|\log 1/|x|} \tag{2.16}$$

for $x \in A(t)$ and $\rho(x) = 0$ otherwise, belongs to $adm\Gamma_t$.

The following lemma provides the standard lower bound for the modulus of all paths joining two continua in $\mathbb{R}^n$, see [Ge$_1$], [Vu, 7.37]. The lemma involves the constant λ_n which depends only on n and which appears in the asymptotic estimates of the modulus of the Teichmüller ring $R_n(t) = \mathbb{R}^n \setminus ([-\infty, 0] \cup [t, \infty])$.

2.17. Lemma. *Let E and F be two continua in $\overline{\mathbb{R}^n}$, $n \geq 2$, with $q(E) \geq \delta_1 > 0$ and $q(F) \geq \delta_2 > 0$, and let Γ be the family of paths joining E and F. Then*

$$M(\Gamma) \geq \frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta_1 \delta_2})^{n-1}} \tag{2.18}$$

where ω_{n-1} is the $(n-1)$-measure of S^{n-1}.

2.19. Theorem. *Let $f : D \to \mathbb{R}^n$ be a $Q(x)$-qc mapping with $Q \in L^{n-1}_{loc}(D)$. Then, for every family Γ of paths in D and every $\rho \in adm\Gamma$,*

$$M(f\Gamma) \leq \int\limits_D Q(x)\rho^n(x)dm(x), \tag{2.20}$$

i.e., f is a Q-homeomorphism.

Proof. By Proposition 2.1, $f^{-1} \in \mathrm{W}^{1,n}_{loc}(f(D))$ and hence $f^{-1} \in \mathrm{ACL}^n_{loc}(f(D)$, see e.g. [Maz, p. 8]. Therefore, by Fuglede's theorem, see e.g. [Vä$_1$, p. 95], if $\tilde{\Gamma}$ is the family of all paths $\gamma \in f\Gamma$ for which f^{-1} is absolutely continuous on all subarcs of γ, then $M(f\Gamma) = M(\tilde{\Gamma})$.

Given a function $\rho \in adm\Gamma$, we let $\tilde{\rho} = \rho(f^{-1}(y))|(f^{-1})'(y)|$ if $y \in f(D)$, $\tilde{\rho}(y) = \infty$ otherwise. Then we obtain that for $\tilde{\gamma} \in \tilde{\Gamma}$

$$\int\limits_{\tilde{\gamma}} \tilde{\rho} ds \geq \int\limits_{f^{-1}\circ\tilde{\gamma}} \rho ds \geq 1$$

and hence $\tilde{\rho} \in adm\tilde{\Gamma}$.

By Proposition 2.1, it follows that f^{-1} has the property (N) and f^{-1} is differentiable a.e. Since f is a homeomorphism, we can use the integral transformation formula to obtain

$$M(f\Gamma) = M(\tilde{\Gamma}) \leq \int\limits_{f(D)} \tilde{\rho}^n dm(y) = \int\limits_{f(D)} \rho(f^{-1}(y))^n |(f^{-1})'(y))|^n dm(y) \leq$$

$$\int\limits_{f(D)} \rho(f^{-1}(y))^n K_O(y, f^{-1}) J(y, f^{-1}) dm(y) =$$

$$\int\limits_{f(D)} \rho(f^{-1}(y))^n K_I(f^{-1}(y), f) J(y, f^{-1}) dm(y) =$$

$$\int\limits_{D} \rho(x)^n K_I(x, f) dm(x) \leq \int\limits_{D} Q(x) \rho(x)^n dm(x).$$

In the above computation we also used the fact that $K_O(y, f^{-1}) = K_I(f^{-1}(y), f)$ a.e. The proof follows.

The following corollary says that our definition of $Q(x)$-quasiconformality is equivalent to the classical definition of quasiconformality when $Q(x) \equiv K < \infty$, cf. [Vä$_1$, 34.6].

2.21. Corollary. *If $Q(x) \equiv K < \infty$, then f is K-qc according to the geometric definition*

$$\frac{1}{K} M(\Gamma) \leq M(f(\Gamma)) \leq K M(\Gamma) \tag{2.22}$$

for every path family Γ in D.

The corollary may also be viewed as a removability theorem. A related removability theorems appeared in [Vä$_1$, 35.1] with the assumption that C is a closed set of σ-finite $(n-1)$-Hausdorff measure with no assumption on $f(C)$.

3. Distortion theorems

3.1. Theorem. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ a Q-homeomorphism with $Q \in BMO(\mathbb{B}^n)$. If $q(\overline{\mathbb{R}^n} \setminus f(B^n(1/e))) \geq \delta > 0$, then for all $|x| < e^{-2}$*

$$q(f(x), f(0)) \leq \frac{C}{(\log 1/|x|)^\alpha} \tag{3.2}$$

where C and α are positive constants which depend only on n, δ, the BMO norm $||Q||_*$ of Q and the average Q_1 of Q over the ball $|x| < 1/e$.

Proof. Fix $t \in (0, e^{-2})$. Let $A(t), \Gamma_t$ and ρ be as in Remark 2.15 and let $\delta_t = q(f(B^n(t)))$. Then, by Remark 2.15, $\rho \in adm\Gamma_t$, and

$$M(f\Gamma_t) \leq \int\limits_{\mathbb{R}^n} Q\rho^n \, dm. \tag{3.3}$$

In view of (2.5),

$$\int\limits_{\mathbb{R}^n} Q\rho^n dm = \int\limits_{A(t)} Q\rho^n dm \leq \frac{c}{(\log\log 1/t)^{n-1}} \tag{3.4}$$

where c is the constant which appears in Lemma 2.3. On the other hand, Lemma 2.17 applied to $M(f\Gamma_t)$ with $E = f(\overline{\mathbb{B}}^n(t))$ and $F = \overline{\mathbb{R}^n} \setminus f(B^n(1/e))$ yields

$$M(f\Gamma_t) \geq \frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta\delta_t})^{n-1}}, \tag{3.5}$$

and the result follows by (3.3)-(3.5) and the fact that $q(f(x), f(0)) \leq \delta_t$ for $|x| = t$.

3.6. Corollary. *Let $\mathcal{F}$ be a family of Q-homeomorphisms $f : D \to \overline{\mathbb{R}^n}$, with $Q \in BMO(D)$, and let $\delta > 0$. If every $f \in \mathcal{F}$ omits two points a_f and b_f in $\overline{\mathbb{R}^n}$ with $q(a_f, b_f) \geq \delta$, then $\mathcal{F}$ is equicontinuous.*

3.7. Theorem. *Let $f : \mathbb{B}^n \to \overline{\mathbb{R}^n}$ be a Q-homeomorphism with $Q \in L^1(\mathbb{B}^n), f(0) = 0$, $q(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n)) \geq \delta > 0$ and $q(f(x_0), f(0)) \geq \delta$ for some $x_0 \in \mathbb{B}^n$. Then, for all $|x| < r = \min(|x_0|/2, 1 - |x_0|)$,*

$$|f(x)| \geq \psi(|x|) \tag{3.8}$$

where $\psi(t)$ is a strictly increasing function with $\psi(0) = 0$ which depends only on the L^1-norm of Q in $\mathbb{B}^n$, n and δ.

Proof. Given y_0 with $|y_0| < r$ choose a continuum E_1 which contains the points 0 and x_0 and a continuum E_2 which contains the point y_0 and $\partial\mathbb{B}^n$, so that $dist(E_1, E_2 \cup \partial\mathbb{B}^n) = |y_0|$. More precisely, denote by L the straight line generated by the pair of points 0 and x_0 and by P the plane defined by the triple of the points 0, x_0 and y_0 (if $y_0 \in L$, then P is an arbitrary plane passing through L). Let C be the circle under intersection of P and the sphere $S^n(y_0, |y_0|) \subset B^n(|x_0|)$. Let t_0 is the tangency point to C of the ray starting from x_0 which is opposite to y_0 with respect to L (an arbitrary one of the two posible if $y_0 \in L$). Then $E_1 = [x_0, t_0] \cup \gamma(0, t_0)$ where $\gamma(0, t_0)$ is the shortest arc of C joining 0 and t_0, and $E_2 = [y_0, i_0] \cup S^n$ where $S^n = \partial\mathbb{B}^n$ is the unit sphere and i_0 is the point (opposite to t_0 with respect to L) of the intersection of S^n with the straight line in P which is perpendicular to L and passing through y_0.

Let Γ denote the family of paths which join E_1 and E_2. Then

$$\rho(x) = |y_0|^{-1}\chi_{\mathbb{B}^n}(x) \in adm\Gamma$$

and hence,

$$M(f\Gamma) \le \int \rho^n(x)Q(x)dm(x) \le |y_0|^{-n} \int\limits_{\mathbb{B}^n} Q(x)dm(x) = \frac{||Q||_1}{|y_0|^n}. \tag{3.9}$$

The ring domain $A' = f(\mathbb{B}^n \setminus (E_1 \cup E_2))$ separates the continua $E_1' = f(E_1)$ and $E_2' = \overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n \setminus E_2)$, and since

$$q(E_1') \ge q(f(x_0), f(0)) \ge \delta, \quad q(E_2') \ge q(\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n)) \ge \delta$$

and

$$q(E_1', E_2') \le q(f(y_0), f(0)$$

it follows that

$$M(f(\Gamma)) \ge \lambda(q(f(y_0), f(0))) \tag{3.10}$$

where $\lambda(t) = \lambda_n(\delta, t)$ is a strictly decreasing positive function with $\lambda(t) \to \infty$ as $t \to 0$, see [Vä$_1$, 12.7]. Hence, by (3.9) and (3.10),

$$|f(y_0)| > q(f(y_0), f(0)) \ge \psi(|y_0|)$$

where

$$\psi(t) = \lambda^{-1}\left(\frac{||Q||_1}{t^n}\right)$$

has the required properties.

3.11. Remark. In view of Theorem 2.19, Theorem 3.7 is valid for $Q(x)$-qc mappings with $Q \in L^{n-1}(\mathbb{B}^n)$, and Theorem 3.1 and Corollary 3.6 are valid for $Q(x)$-qc mappings with $Q \in BMO(\mathbb{B}^n)$.

4. Removability of isolated singularities

In this section we describe, Theorems 4.1 and 4.5, two unrelated cases where isolated singularities are removable.

4.1. Theorem. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a Q-homeomorphism. If*

$$\limsup_{r \to 0} \frac{1}{|B^n(r)|} \int\limits_{B^n(r)} Q(x)dm(x) < \infty, \tag{4.2}$$

then f has an extension to $\mathbb{B}^n$ which is a Q-homeomorphism.

Proof. As the modulus of a family of paths which pass through the origin vanishes, it suffices to show that f has a continuous extension on $\mathbb{B}^n$. Suppose that this is not the case. Since f is a homeomorphism, $\overline{\mathbb{R}^n} \setminus f(\mathbb{B}^n \setminus \{0\})$ consists of two connected compact sets F_1 and F_2 in $\overline{\mathbb{R}^n}$ where F_1 contains the cluster set $E = C(0, f)$ of f at 0. Here F_1 is a nondegenerate continuum and using an arbitrary Möbius transformation we may assume that $F_1 \subset \mathbb{R}^n$. Now $U = F_1 \cup f(\mathbb{B}^n \setminus \{0\})$ is a neighborhood of E. Thus there exists $\delta > 0$ such that all balls $B_z = B^n(z, \delta), z \in F_1$, are contained in U. Let $V = \cup B_z$.

Now, choose a point $y \in F_1$ such that $dist(y, \partial V) = \delta$, and a point $z \in B_y \setminus F_1$. Next, choose a path $\beta : [0, 1] \to B_y$ with $\beta(0) = y, \beta(1) = z$ and $\beta(t) \in B_y \setminus F_1$ for $t \in (0, 1]$. Let $\alpha = f^{-1} \circ \beta$. For $r \in (0, |f^{-1}(z)|)$, let α_r denote the connected component of the curve $\alpha(I) \setminus B^n(r), I = [0, 1]$, which contains the point $f^{-1}(z) = \alpha(1)$, and let Γ_r denote the family of all paths joining α_r and the point 0 in $\mathbb{B}^n \setminus \{0\}$.

Then the function $\rho(x) = 1/r$ if $x \in B^n(r) \setminus \{0\}$ and $\rho = 0$ otherwise is in $adm\Gamma_r$, and by (4.2),

$$\limsup_{r\to 0} \int\limits_{B^n(r)\setminus\{0\}} Q(x)\rho^n(x)dm(x) = \tag{4.3}$$

$$\Omega_n \limsup_{r\to 0} \frac{1}{|B^n(r)|} \int\limits_{B^n(r)\setminus\{0\}} Q(x)dm(x) < \infty.$$

On the other hand, if we let Γ_r' denote the family of all paths joining two continua $f(\alpha_r)$ and E in $B_y \setminus E$, then $\Gamma_r' \subset f(\Gamma_r)$, and thus

$$M(\Gamma_r') \le M(f\Gamma_r). \tag{4.4}$$

Evidently, $dist(f(\alpha_r), E) \to 0$, and the diameter of $f(\alpha_r)$ increases as $r \to 0$, and as both $f(\alpha_r)$ and E are subsets of a ball, $M(f\Gamma_r) \to \infty$ as $r \to 0$. This together with (4.3) and (4.4) contradicts the modulus inequality (2.20).

4.5. Theorem. *Let $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ be a Q-homeomorphism with $Q \in BMO(\mathbb{B}^n \setminus \{0\})$. Then f has a $Q(x)$-homeomorphic extension to $\mathbb{B}^n$.*

Proof. The case $n = 2$ was considered in [RSY$_3$] for a wider class of mappings. Suppose that $n > 2$. Fix $t \in (0, e^{-2})$ and let $A(t), \Gamma_t$ and ρ be as in Remark 2.15. Then, by Lemma 2.17,

$$\frac{\omega_{n-1}}{(\log \frac{2\lambda_n}{\delta\delta_t})^{n-1}} \le M(f\Gamma_t) \le \int\limits_{A(t)} Q\rho^n dm, \tag{4.6}$$

where $\delta = q(f(\partial B^n(e^{-1})))$ and $\delta_t = q(f(\partial B^n(t)))$. Since isolated singularities are removable for BMO functions, see [RR], we may assume that Q is defined in $\mathbb{B}^n$ and that $Q \in$BMO$(\mathbb{B}^n)$. Thus, by 2.2 and Lemma 2.3,

$$\int\limits_{A(t)} Q(x)\rho^n dm \le \frac{c}{(\log\log 1/t)^{n-1}} \tag{4.7}$$

where c is a constant as in Lemma 2.4. Since here c depends only on $n, ||Q||_*$ and $Q_1 = Q_{B^n(1/e)}$, we obtain by (4.6)-(4.7) that $\delta_t \to 0$ as $t \to 0$, and hence that $\lim_{x\to 0} f(x)$ exists.

In view of the last theorem, Theorem 2.19 and the Möbius invariance of BMO-qc mappings, we obtain the following corollary.

4.8. Corollary. *Isolated points are removable for BMO-qc mappings, and, in particular, if* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a BMO-qc, then* f *has a BMO-qc extension to* $\overline{\mathbb{R}^n}$ *and* $f(\mathbb{R}^n) = \mathbb{R}^n$.

5. The reflection principle

5.1. Theorem. *If* f *is a BMO-qc mapping of* $\mathbb{H}^n$ *onto itself, then*
(i) f *has a homeomorphic extension to* $\overline{\mathbb{H}^n}$.
(ii) The symmetric extension of f *through* $\partial\mathbb{H}^n$ *to* $\mathbb{R}^n$ *is BMO-qc.*

Proof. The case $n = 2$ was considered in [RSY$_3$] for a wider class of mappings. Suppose that $n > 2$. Since f is BMO–qc in $\mathbb{H}^n$, $f \in W^{1,n}_{loc}(\mathbb{H}^n \setminus C)$ for some set C which is relatively close in $\mathbb{H}^n$ with $|f(C)| = |C| = 0$, and f is $Q(x)$–qc for some positive function Q in $BMO(\mathbb{H}^n)$. Since BMO functions and BMO-qc mappings are Möbius invariant, it suffices to treat the case where $x_0 = 0$. Fix $t \in (0, e^{-2})$, and let $A(t)$ be as in Lemma 2.3, $\rho(x)$ as in (2.16), i.e.

$$\rho(x) = \frac{1}{(\log\log 1/t)|x|\log 1/|x|}$$

and $\delta_t = q(f(B^n(t) \cap \mathbb{H}^n))$, $\delta = q(f(\partial B^n(1/e) \cap \mathbb{H}^n))$ and let Γ_t denote the family of curves which join the spheres $|x| = t$ and $|x| = 1/e$ in $\mathbb{H}^n$. Then $\rho \in adm\Gamma_t$, and by Lemma 2.17, a symmetry principle for the modulus of path families and Theorem 2.19,

$$\frac{\omega_{n-1}}{2(\log\frac{2\lambda_n}{\delta\delta_t})^{n-1}} \le M(f\Gamma_t) \le \int\limits_{\mathbb{H}^n} Q\rho^n\,dm. \tag{5.2}$$

Choose any symmetric extension of Q to $\mathbb{R}^n$ and denote it again by Q. Then by the reflection principle for BMO functions, see [RR], Q is BMO in $\mathbb{R}^n$ and, in view of Lemma 2.3 and (2.16),

$$\int\limits_{\mathbb{H}^n} Q\rho^n\,dm = \frac{1}{2}\int\limits_{\mathbb{R}^n} Q\rho^n\,dm \le \frac{c}{(\log\log 1/t)^n} \tag{5.3}$$

where c is the constant in Lemma 2.3 which depends only on $||Q||_*$ and the average Q_1 of Q over $B^n(1/e)$. From (5.2) and (5.3) we obtain $\delta_t \to 0$ as $t \to 0$ which proves that $\lim_{x\to 0} f(x)$ exists. Consequently, f has a continuous extension on $\overline{\mathbb{H}^n}$. The extended map will be denoted again by f.

We now show that f is injective in $\partial\mathbb{H}^n$. Suppose that $f(x_1) = f(x_2)$ for some distinct points x_1 and x_2 in $\partial\mathbb{H}^n$. In view of the Möbius invariance, we may assume that $x_1 = 2a$ and $x_2 = -2a$ for some $a \in \partial\mathbb{H}^n$ with $|a| = 1$. Let Γ denote the family of curves in $\mathbb{H}^n$ which join the balls B and $-B$ of radius 1 which are centered at $2a$ and $-2a$, respectively, and let $\rho(x) = 1/2$ in $B^n(4)$ and $\rho(x) = 0$ elsewhere. Then $\rho \in adm\Gamma$, and by Theorem 2.19,

$$M(f\Gamma) \le \int_{\mathbb{H}^n} Q\rho^n dm = \frac{1}{2} \int_{\mathbb{H}^n \cap B^n(4)} Q dm.$$

Since $f(a) = f(-a)$, $M(f\Gamma) = \infty$. On the other hand, by the reflection principle for BMO functions the right hand side is finite. This leads to a contradiction which shows that f is injective in $\overline{\mathbb{H}^n}$. The proof of (ii) is trivial and omitted.

5.4. Remark. If D and D' are subdomains of $\mathbb{H}^n$ whose boundaries include $\partial\mathbb{H}^n$ and for every $x \in \partial\mathbb{H}^n$ the cluster set $C(x, f)$ lie in $\partial\mathbb{H}^n$, then the conclusions of Theorem 5.1 are valid. The same is also true if the correspondence holds only for a proper part of $\partial\mathbb{H}^n$. Moreover, we can extend the reflection principle to BMO-qc mappings between balls in view of their invariance with respect to Möbius transformations. For a generalization of the reflection principle, see Theorem 7.1.

6. Boundary behavior

6.1. For a domain D in $\overline{\mathbb{R}^n}$ and two nondegenerate continua E and F in $\overline{D}$, $\Gamma(E, F; D)$ denotes the family of paths joining E and F in D. We say that ∂D is *weakly flat* if, for E and F in $\overline{D}$,

$$M(\Gamma(E, F; D)) > 0 \tag{6.2}$$

and that ∂D is *weakly accessible* if, for E and F in $\overline{D}$ with $E \cap F \ne \emptyset$,

$$M(\Gamma(E, F; D)) = \infty \tag{6.3}$$

and that ∂D is *weakly regular* if ∂D is weakly flat and weakly accessible These properties are closely related to the notations of quasiconformal flatness and quasiconformal accessibility by Näkki in [Nä], to properties P_1 and P_2 by Väisälä, see 17.5 in [Vä$_1$], and to the QED condition by Gehring and Martio in [GM], and are preserved under qc mappings of neighborhoods of ∂D. Note that the conditions (6.2) and (6.3) automatically hold if E and F are in D, but, generally speaking, not if E and F are in ∂D. Recall that D is said to be *locally connected* at ∂D if every point $x \in \partial D$ has an arbitrary small neighborhood U such that $U \cap D$ is connected. Note that every Jordan domain D in $\mathbb{R}^n$ is locally connected at ∂D, see [Wi], p. 66.

6.4. Theorem. *Let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in BMO(\overline{D})$ and suppose that D is locally connected at ∂D. If $\partial D'$ is weakly flat, then f has a continuous extension $\tilde{f} : \overline{D} \to \overline{D}'$.*

Proof. Let $x_0 \in \partial D$. As BMO functions and Q-homeomorphisms are Möbius invariant, we may assume that $x_0 = 0$ and that $\partial \mathbb{B}^n \cap D \neq \emptyset$. We will show that the cluster set $E = C(0, f)$ of f at 0 is a point, which will prove that $f(x)$ has a limit at 0.

Since D is locally connected at $0, E$ is a continuum. Suppose that E is nondegenerate. For $t \in (0, 1/e)$, let D_t denote the component of $B^n(t) \cap D$ for which $0 \in \overline{D}_t$. Note, that D_t is well defined, since D is locally connected at 0, and that $D_t \subset D_{t'}$ for $t < t'$. For $t \in (0, e^{-2}]$, let Γ_t denote the family of path joining D_t and the set $S = D \cap \partial B^n(1/e)$ in $D_{1/e} \setminus \overline{D}_t$. As in Lemma 2.3, we let $A(t)$ denote the spherical ring $t < |x| < 1/e$. Then the function $\rho(x)$ which is defined in 2.15 is admissible for Γ_t, and hence

$$M(f\Gamma_t) \leq \int_{D_t} Q(x)\rho^n(x)dm(x) \tag{6.5}$$

As $Q \in$BMO$(\overline{D})$ we may apply (2.4), and get

$$\int_{D_t} Q(x)\rho^n(x)dm(x) \leq \int_{A(t)} Q(x)\rho^n(x)dm(x) \to 0 \tag{6.6}$$

as $t \to 0$. On the other hand

$$M(f\Gamma_t) \geq M(\Gamma(\overline{f(S)}), E; D'). \tag{6.7}$$

Now, $\partial D'$ is weakly flat, $f(S)$ contains a nondegenerate continuum and E is presumably nondegenerate. Therefore, the right hand side in (6.7) is positive contradicting (6.5) and (6.6). This shows that the cluster set of f at every point of ∂D is degenerate and thus f has a continuous extension on $\overline{D}$.

6.8. Theorem. *Let D be a domain in $\mathbb{R}^n$ and f a Q-homeomorphism of D onto a domain D' in $\mathbb{R}^n$ with $Q \in L^1(D)$. Suppose that D is locally connected at ∂D. If $\partial D'$ is weakly accessible, then $C(x_1, f) \cap C(x_2, f) = \emptyset$ for every two distinct points x_1 and x_2 in ∂D.*

Proof. For $i = 1, 2$, let $E_i = C(x_i, f)$ and suppose that $E_1 \cap E_2 \neq \emptyset$. Let $d = |x_1 - x_2|$. Since D is locally connected in ∂D, there are neighborhoods U_i of x_i, such that $W_i = D \cap U_i$ is connected and $U_i \subset B^n(x_i, d/3), i = 1, 2$. Then the function $\rho(x) = 3/d$ if $x \in D \cap B^n((x_1 + x_2)/2, d)$ and $\rho(x) = 0$ elsewhere is admissible for the family $\Gamma = \Gamma(\overline{W}_1, \overline{W}_2; D)$. Thus,

$$M(f\Gamma) \leq \int\limits_D Q(x)\rho^n(x)dm(x) \leq \frac{3^n}{d^n} \int\limits_D Q(x)dm(x) < \infty. \tag{6.9}$$

On the other hand

$$M(\Gamma(E_1, E_2; D)) \leq M(f\Gamma). \tag{6.10}$$

But as $\partial D'$ is weakly flat, and $E_i, i = 1, 2$, are nondegenerate continua in $\overline{D'}$ with non-empty intersection, it follows that the left, and hence, the right hand side in (6.10) is not finite, contradicting (6.9). The assertion follows.

6.11. Corollary. *Let D, D', f and Q be as in Theorem 6.8. Then f^{-1} has a continuous extension to $\overline{D'}$.*

6.12. Corollary. *Let E be a nondegenerate continuum in $\mathbb{B}^n$ and $Q \in L^1(\mathbb{B}^n \setminus E)$. Then there exist no Q-homeomorphisms of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$. If $Q \in L^{n-1}(\mathbb{B}^n \setminus E)$, then also there exist no $Q(x)$-qc mappings of $\mathbb{B}^n \setminus E$ onto $\mathbb{B}^n \setminus \{0\}$.*

6.13. Corollary. *Let $f : D \to D' \subset \mathbb{R}^n$ be a Q-homeomorphism onto D' with $Q \in BMO(\overline{D})$. If D locally connected at ∂D and $\partial D'$ is weakly regular, then f has a homeomorphic extension $\tilde{f} : \overline{D} \to \overline{D'}$.*

6.14. Corollary. *If D is a domain in $\mathbb{R}^n$ which is locally connected at ∂D but not a Jordan domain, then D cannot be mapped by a Q-homeomorphism with $Q \in BMO(\overline{D})$ onto $\mathbb{B}^n$.*

6.15. Corollary. *If a domain D in $\mathbb{R}^n$ is uniform but not Jordan, then there is no BMO–qc mapping of D onto $\mathbb{B}^n$.*

7. BMO-quasiballs and BMO-quasispheres

We say that a domain D in $\mathbb{R}^n, n \geq 2$, is a *quasiball*, respectively, *BMO-quasiball* if there exists a homeomorphism of D onto $\mathbb{B}^n$ which is qc, respectively, BMO-qc. We say that a set S in $\overline{\mathbb{R}^n}$ is a *quasisphere*, respectively, *BMO-quasisphere* if there exists a qc mapping, respectively, BMO-qc mapping f of $\overline{\mathbb{R}^n}$ onto itself such that $f(S) = \partial\mathbb{B}^n$.

7.1. Theorem. *Let $f : D \to D' \subset \mathbb{R}^n, n \geq 2, n \neq 4$, be a BMO-qc mapping onto D'. If ∂D and $\partial D'$ are quasispheres, then f has a BMO-qc extension to $\overline{\mathbb{R}^n}$.*

Proof. Since ∂D and D' are quasispheres, it follows by the qc invariance of BMO functions and the reflection principle for BMO functions that D is a BMO extension domain, that D is locally connected on ∂D and D' has property P. Therefore, by Corollary 6.13, f has a homeomorphic extension, denoted again by f.

Since ∂D and $\partial D'$ are quasispheres, each admits quasiconformal reflection, which by Tukia and Väisälä [TV, 7.15] can be changed by a bilipschitz one if $n \neq 4$, see also Ahlfors [Ah, p. 80] for $n = 2$. Denote these reflections by φ and ψ, respectively. Define $F : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ by letting $F(x) = f(x)$ if $x \in \overline{D}$, and $F(x) = \psi \circ f \circ \varphi(x)$ if $x \in \overline{\mathbb{R}^n} \setminus \overline{D}$. Then F is a homeomorphic extension of f to $\overline{\mathbb{R}^n}$. Furthermore, if $f \in W^{1,n}_{loc}(D \setminus C)$ for some set C which is closed in D with $|C| = |f(C)| = 0$, then $F \in W^{1,n}_{loc}(\overline{\mathbb{R}^n} \setminus (\partial D \cup C \cup \varphi(C))$ and $|\partial D \cup C \cup \varphi(C)| = |F(\partial D \cup C \cup \varphi(C))| = 0$ and $K_F(x) \leq \tilde{Q}(x)$ a.e. in $\overline{\mathbb{R}^n}$. Here $\tilde{Q}(x) = KK'Q(x)$ if $x \in D$ and $\tilde{Q}(x) = KK'Q(\varphi(x))$ otherwise. Here K and K' are the coefficients of quasiconformality of φ and ψ, respectively, and it is clear that $\tilde{Q}$ is a BMO function in $\overline{\mathbb{R}^n}$, see e.g. [Jo] and [RR]. Thus, F is a BMO-qc extension of f to $\overline{\mathbb{R}^n}$.

7.2. Remark. We do not know if the assumption in the theorem that ∂D and $\partial D'$ are quasispheres can be replaced by the weaker condition that ∂D and $\partial D'$ are BMO-quasispheres. Nor do we know if $n \neq 4$ is dispensible here or in the theorem of Tukia and Väisälä [TV, 7.15] about Lipschitz reflections. Clearly the cases are related. Nor we know if BMO-quasispheres admit a Lipschitz reflection for $n \geq 3$.

7.3. Theorem. *There are a BMO-quasicircles which are not quasicircles.*

Proof. Consider the curves $\gamma_1 = [0, \infty]$ and $\gamma_2 = [-\infty, -1/e]$, $\gamma_3 = \{te^{i\pi/\log 1/t} : 0 < t < 1/e\}$ and $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$. Then, clearly, γ does not satisfy Ahlfors's three points condition, and hence it is not a quasicircle.

We will show that γ is a BMO-quasicircle. Indeed, the map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is identity in $\overline{\mathbb{C}} \setminus \mathbb{B}^2$ and is given for $|z| < 1$ by

$$f(re^{i\theta}) = \begin{cases} r \exp i(\theta \log 1/r), & \text{if } 0 \leq \theta \leq \frac{\pi}{\log 1/r} \\ r \exp i\pi(1 + \frac{1-\theta/\pi \log 1/r}{1-2\log 1/r}), & \text{if } \frac{\pi}{\log 1/r} \leq \theta < 2\pi \end{cases}$$

is $Q(z)$-qc with $Q(re^{i\theta}) = \max(1, \log 1/r)$ which is BMO-qc in $\overline{\mathbb{C}}$ and maps γ onto $\overline{\mathbb{R}}$.

7.4. Theorem. *For every $n \geq 3$, there are topological balls in $\mathbb{R}^n$ which are not BMO-quasiballs.*

Proof. The domain $D_n = \{x \in \mathbb{R}^n : |x_n| < 1\}$ between the hyperplanes is not a Jordan domain. However, every BMO function Q in D can be extended by repeated reflections in hyperplanes to a BMO function in $\mathbb{R}^n$, and, consequently, to $\overline{\mathbb{R}^n}$, see [RR]. Thus, the statement follows by Corollary 6.14.

8. Mapping problems

In Section 4, we showed that there are no BMO-qc mappings of $\mathbb{R}^n$ onto a proper subset of $\mathbb{R}^n$, nor BMO-qc mappings of a punctured ball onto a domain that has two nondegenerate boundary components. We may consider the following two questions.

(a) Are there any proper subsets of $\mathbb{R}^n$ that can be mapped BMO-quasiconformally onto $\mathbb{R}^n$?

(b) Are there any nondegenerate continua E in $\mathbb{B}^n$ such that $\mathbb{B}^n \setminus E$ can be mapped BMO-quasiconformally onto $\mathbb{B}^n \setminus \{0\}$?

In [RSY$_3$] we showed that the answer to these questions is negative if $n = 2$. The proofs were based on the Riemann mapping theorem and on the existence of a homeomorphic solution to the Beltrami equation

$$w_{\overline{z}} = \mu(z) w_z$$

for measurable functions μ with $||\mu||_\infty \le 1$ which satisfy $(1 + |\mu(z)|)/(1 - |\mu(z)|) \le Q(z)$ a.e. for some BMO function Q. One may modify Questions (a) and (b) by replacing the words "BMO-quasiconformally" by "by a Q-homeomorpism". Below, we provide a negative answer to the modified and hence to Questions (a) and (b) in some special cases when $n > 2$.

8.1. Theorem. *Let D be a domain in $\mathbb{R}^n, D \neq \mathbb{R}^n$, and $f : D \to \mathbb{R}^n$ a Q-homeomorphism. If there exist a point $b \in \partial D$ and a neighborhood U of b such that $Q|_{D \cap U} \in L^1$, then $f(D) \neq \mathbb{R}^n$.*

Proof. The statement is trivial if D is not a topological ball. Suppose that D is a topological ball. By the Möbius invariance, we may assume that $b = 0$ and that $\infty \in \partial D$. Let $r > 0$ be such that $B^n(r) \subset U$. Then Q is integrable in $B^n(r) \cap D$. Choose two arcs E and F in $B^n(r/2) \cap D$ each having exactly one end point in ∂D such that $0 < dist(E, F) < r/2$. Such arcs exist. Indeed, since ∂D is connected and 0 and ∞ belong to ∂D, the sphere $\partial B^n(r/2)$ meets ∂D and contains a point x_0 which belongs to D. Then one can take E as a maximal line segment in $(0, x_0] \cap D$ with one end point at x_0 and the other one in ∂D, and F as a circular arc in the maximal spherical cap in $\partial B^n(r/2) \cap D$ which is centered at x_0, so that F has one end point in ∂D and the other one in D.

Now, let Γ denote the family of all paths which join E and F in D. Then $\rho(x) = dist(E, F)^{-1}$ if $x \in B^n(r) \cap D$ and $\rho(x) = 0$ otherwise is admissible for Γ. Then by (1.8)

$$M(f\Gamma) \le \int_D Q\rho^n \, dm \le \frac{1}{dist(E, F)^n} \int_{B^n(r) \cap D} Q \, dm < \infty. \tag{8.2}$$

On the other hand, if $f(D) = \mathbb{R}^n$, then $f(E)$ and $f(F)$ meet at ∞ and $f\Gamma$ is the family of paths joining $f(E)$ and $f(F)$ in $\mathbb{R}^n$. Thus $M(f\Gamma) = \infty$. The contradiction shows that $f(D) \neq \mathbb{R}^n$.

As a consequence of Theorem 8.1, we have the following corollaries which say that a proper subdomain D of $\mathbb{R}^n$ having locally a nice boundary at least at one point of ∂D cannot be mapped BMO-quasiconformally onto $\mathbb{R}^n$.

8.3. Corollary. *Let D be a domain in $\mathbb{R}^n, D \neq \mathbb{R}^n$ and let $f : D \to \mathbb{R}^n$ be a Q-homeomorphism with $Q \in BMO(D)$. If there exists a point $b \in \partial D$ and a neighborhood U of b such that $D \cap U$ is an L^1-BMO domain or, in particular, $\partial(D \cap U)$ is a quasisphere, then $f(D) \neq \mathbb{R}^n$.*

By the techniques which was used in the proof of Theorem 8.1 and Corollary 8.3, one can establish the following theorem which gives partial answers to (b).

8.5. Theorem. *Let E be a nondegenerate continuum in $\mathbb{B}^n$, $D = \mathbb{B}^n \setminus E$, and let $f : D \to \mathbb{R}^n$ be a Q-homeomorphism. If there exist a point $x_0 \in \partial D \cap \mathbb{B}^n$ and a neighborhood U of x_0 such that $Q|_{D\cap U} \in L^1$, then $f(D)$ is not a punctured topological ball.*

8.6. Corollary. *Let E be a nondegenerate continuum in $\mathbb{B}^n$ and $D = \mathbb{B}^n \setminus E$. If there exist a point $x_0 \in \partial D \cap \mathbb{B}^n$ and a neighborhood U of x_0 such that $U \setminus E$ is an L^1-BMO domain or, in particular, $\partial(U \setminus E)$ is a quasisphere, then D cannot be mapped BMO-quasiconformally onto $\mathbb{B}^n \setminus \{0\}$.*

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