On the Degenerate Beltrami Equation

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Abstract

We study the well-known Beltrami equation under the assumption that its measurable complex-valued coefficient $\mu(z)$ has the norm $\|\mu\|_\infty = 1$. We employ the pointwise angular dilatation coefficients and this leads to sufficient conditions on μ which imply the existence of a self-homeomorphism of $\widehat{\mathbb{C}}$ satisfying the Beltrami equation with μ. We also consider the uniqueness of solutions.

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1. Introduction

The analytic theory of plane quasiconformal mappings f is based on the Beltrami partial differential equation

$$f_{\bar z} = \mu(z)\, f_z \quad \text{a.e.} \tag{1.1}$$

with the complex-valued measurable coefficient μ satisfying the uniform ellipticity assumption $\|\mu\|_\infty < 1$. In the case $|\mu(z)| < 1$ a.e. in $\mathbb{C}$ and $\|\mu\|_\infty = 1$, equation (1.1) is called a degenerate Beltrami equation and the structure of the solutions heavily depends on the degeneration of μ. In this article, unless otherwise stated, by a Beltrami coefficient in a domain Ω we mean a complex-valued measurable function μ in Ω such that $|\mu| < 1$ a.e. in Ω, and by a solution to the Beltrami equation (1.1) in a domain Ω we mean a function f in the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega)$ whose partial derivatives satisfy (1.1) in Ω. Then, f is sometimes called μ-conformal in Ω.

The measurable Riemann mapping theorem states that given a measurable function μ in the plane $\mathbb{C}$ with $\|\mu\|_\infty < 1$ there is a quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$, $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$, satisfying (1.1). Contrary to this, the degenerate

Beltrami equation need not have a homeomorphic solution and that a solution, if it exists, need not be unique. Therefore, in order to obtain existence or uniqueness results, some extra constraints must be imposed on μ.

We first recall some earlier results on the degenerate Beltrami equation and then discuss the main results in this paper. Bers [5] and Belinskiĭ[4] showed that if the pointwise maximal dilatation function

$$K(z) = K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} \tag{1.2}$$

is locally essentially bounded in Ω, namely, for each compact subset E of Ω, $|\mu| \leq k$ holds a.e. in E for some $k < 1$, then there exists a homeomorphic solution to the Beltrami equation in Ω and it is unique up to a post-composition by a conformal mapping. Later Pesin [17] proved that if K_μ^p is exponentially integrable in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ for some exponent $p > 1$, then there exists a homeomorphic solution $f : \mathbb{D} \to \mathbb{D}$ to the Beltrami equation (1.1) in $\mathbb{D}$. Moreover, $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ for every $q < 2$, and the inverse mapping f^{-1} belongs to the space $W^{1,2}_{\mathrm{loc}}(\mathbb{D})$. Miklyukov and Suvorov [16] showed that if $K(z)$ defined in $\mathbb{D}$ has a majorant $K_0(z) \in W^{1,2}_0(\mathbb{D})$ in the sense that $K(z) \leq C_0 + K_0(z)$ with some constant C_0, then there exists a unique normalized homeomorphic solution $f : \mathbb{D} \to \mathbb{D}$ in $W^{1,2}(\mathbb{D})$ to the Beltrami equation (1.1) and the inverse mapping f^{-1} belongs to the space $W^{1,2}(\mathbb{D})$.

The mappings with exponentially integrable distortion coefficients are close to classical quasiconformal mappings. Brakalova and Jenkins [6] modified Pesin's condition and showed that the subexponential integrability $\exp\{K/(1+\log K)\} \in L^1_{\mathrm{loc}}(\mathbb{C})$ together with a suitable behavior of $K(z)$ near the infinity guarantees the existence of a homeomorphic solution to the Beltrami equation in the class $\bigcap_{1\leq q<2} W^{1,q}_{\mathrm{loc}}(\mathbb{C})$. They also gave a uniqueness theorem. The most recent results concerning the exponential and subexponential integrability of the pointwise maximal dilatation $K(z)$ are due to Iwaniec and Martin [10]. The proofs of their existence and uniqueness theorems are based on a monotonicity principle in the Orlicz-Sobolev classes and powerful results in harmonic analysis.

Closely related results have been obtained by G. David [7]. He showed that if

$$\mathrm{meas}\{z : |\mu| > 1-\varepsilon\} \leq Ce^{-\alpha/\varepsilon} \tag{1.3}$$

for sufficiently small $\varepsilon > 0$ and some positive constants α and C, then a μ-

conformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ exists and is unique in the above sense. Tukia's compactness theorems [21] and the John-Nirenberg BMO class have recently been successfully employed to extend the aforementioned David's result (see, e.g., [19] or [3]).

All the results mentioned above are based on the suitable growth condition of $|\mu(z)|$ only. In this paper we show that for sharp results not only $|\mu(z)|$ but also the argument of $\mu(z)$ should be involved. For example, consider the Beltrami coefficients $\mu_1(z) = (1 - |z|)z/\bar{z}$ and $\mu_2(z) = (|z| - 1)z/\bar{z}$ defined in the unit disk $\mathbb{D}$. It is immediate that $\|\mu_j\|_\infty = 1$, $j = 1, 2$, and $K_1(z) = K_2(z)$ whenever $z \in \mathbb{D} \setminus \{0\}$. The radial stretching $f_1 : \mathbb{D} \to \mathbb{D}$ defined by

$$f_1(z) = \frac{z}{|z|^2} e^{2(1-1/|z|)} \tag{1.4}$$

for $z \in \mathbb{D} \setminus \{0\}$ and by $f_1(0) = 0$ is μ_1-conformal. The second radial stretching

$$f_2(z) = \frac{z}{|z|(2 - |z|)} \tag{1.5}$$

in the punctured disk $\mathbb{D} \setminus \{0\}$ is μ_2-conformal and has the cavitation effect since it maps $\mathbb{D} \setminus \{0\}$ homeomorphically onto the annulus $1/2 < |z| < 1$. Actually, the continuous solution to the Beltrami equation with $\mu = \mu_2$ is unique up to the post-composition of analytic functions (see Proposition 4.1 below), and hence the cavitation is inevitable in this case. Thus the cavitation problem requires more precise information on μ rather than merely on $|\mu|$.

The idea to employ μ instead of $|\mu|$ in the study of some regularity problems for quasiconformal mappings is due to Andreian Cazacu [2] and Reich and Walczak [18]. O. Lehto [12], [13] was the first who considered the degenerate Beltrami equation from this point of view.

In this paper we study general families of normalized self-homeomorphisms of the extended complex plane $\widehat{\mathbb{C}}$ and establish their normality under some additional assumptions. Then, given a Beltrami coefficient μ with $\|\mu\|_\infty = 1$ and using the angular dilatation coefficient D_{μ,z_0} (see (2.13) below), we generate a special pre-compact family of quasiconformal mappings of $\widehat{\mathbb{C}}$. This together with some well-known regularity results allows us to study the existence problem for the degenerate Beltrami equation. We also give a uniqueness theorem under some geometric conditions on the singular set of μ.

2. Sequences of Self-Homeomorphisms

The degenerate Beltrami equation need not have a homeomorphic solution nor even a nonconstant solution. A usual approach for the existence of a solution is to consider a sequence f_n of quasiconformal homeomorphisms satisfying (1.1) with the Beltrami coefficient μ_n, $\|\mu_n\|_\infty < 1$, such that $\mu_n \to \mu$ a.e. and then to use a normal family argument to obtain a limit mapping f. Some conditions must be imposed on μ in order to guarantee the normality.

In the following we introduce a modulus method to study normal families of homeomorphisms. This will be employed to prove the solvability problem of the degenerated Beltrami equation, however, the method is of independent interest.

We introduce some notation. We denote the Euclidean distance and the spherical distance between z and w by $d(z,w) = |z-w|$ and $d^\sharp(z,w) = |z-w|/\sqrt{(1+|z|^2)(1+|w|^2)}$, respectively. Also we denote by $A(z_0,r,R)$ and by $A^\sharp(z_0,r,R)$ the (circular) annuli in the Euclidean and the spherical metric, respectively, i.e.,

$$A(z_0,r,R) = B(z_0,R) \setminus \overline{B}(z_0,r) \text{ and } A^\sharp(z_0,r,R) = B^\sharp(z_0,R) \setminus \overline{B}^\sharp(z_0,r)$$

for $z_0 \in \widehat{\mathbb{C}}$ and $0 \le r < R$, where $B(z_0,r) = \{z \in \mathbb{C} : |z-z_0| < r\}$ and $B^\sharp(z_0,r) = \{z \in \widehat{\mathbb{C}} : d^\sharp(z,z_0) < r\}$. Here, in the case when $z_0 = \infty$, we set $B(\infty,r) = \{z \in \widehat{\mathbb{C}} : |z| > 1/r\}$, and hence, $A(\infty,r,R) = A(0,1/R,1/r)$. In the sequel, for subsets E, E_0, E_1 of $\widehat{\mathbb{C}}$, $\operatorname{diam} E$ and $\operatorname{dist}(E_0,E_1)$ stand for the diameter of E and the distance between E_0 and E_1, respectively, measured in the Euclidean metric d. Similarly, $\operatorname{diam}^\sharp E$ and $\operatorname{dist}^\sharp(E_0,E_1)$ stand for those measured in the spherical metric $d^\sharp$. We also denote by $\mathcal{A}$ and $\mathcal{A}^\sharp$ the two dimensional Lebesgue measure and the spherical measure, respectively, i.e., $\mathcal{A}(E) = \iint\limits_E dxdy$ and $\mathcal{A}^\sharp(E) = \iint\limits_E (1+|z|^2)^{-2} dxdy$.

A doubly connected domain is called a ring domain. The modulus m of a ring domain A is the number such that A is conformally equivalent to $\{1 < |z| < e^m\}$ and will be denoted by $\operatorname{mod} A$. When A is conformally equivalent to $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we define $\operatorname{mod} A = \infty$. A non-negative function $\rho(z,r,R)$ in

$(z, r, R) \in \widehat{\mathbb{C}} \times (0, +\infty) \times (0, +\infty)$, $r < R$, will be called a *modulus constraint* if $\rho(z_0, r, R) \to +\infty$ as $r \to 0$ for any fixed $R \in (0, +\infty)$ and $z_0 \in \widehat{\mathbb{C}}$.

We denote by $\mathcal{H}_\rho$ the family of all normalized homeomorphisms $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the condition

$$\operatorname{mod} f(A(z_0, r, R)) \geq \rho(z_0, r, R)$$

holds for all $z_0 \in \widehat{\mathbb{C}}$ and $r, R \in (0, +\infty)$ with $r < R$. Here and hereafter, a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said to be normalized if f fixes $0, 1$ and ∞. Note that this condition is not Möbius invariant.

Note also that, without changing the family $\mathcal{H}_\rho$, we may assume the monotonicity condition $\rho(z_0, r_1, R_1) \leq \rho(z_0, r_2, R_2)$ for $z_0 \in \widehat{\mathbb{C}}$ and $r_2 \leq r_1 < R_1 \leq R_2$ by replacing ρ by a suitable one.

Similarly, a non-negative function $\rho^\sharp(z, r, R)$ in $(z, r, R) \in \widehat{\mathbb{C}} \times (0, 1) \times (0, 1)$, $r < R$, will be called a *spherical modulus constraint* if the same set of conditions is satisfied for $\rho^\sharp$. We let $\mathcal{H}^\sharp_{\rho^\sharp}$ be the family of all normalized self-homeomorphisms f such that $\operatorname{mod} f(A^\sharp(z_0, r, R)) \geq \rho^\sharp(z_0, r, R)$ holds for all z_0, r, R.

A family of continuous functions is said to be *pre-normal* or *normal* if it is relatively compact or compact, respectively, with respect to the topology of local uniform convergence.

Now we are ready to state our main propositions. A similar statement can be found in Lehto's paper [13] under stronger assumptions (see also Lemma 1 in Section 4 of [6]).

2.1. **Proposition.** *Let $\rho^\sharp$ be a spherical modulus constraint. Then*

1) *$\mathcal{H}^\sharp_{\rho^\sharp}$ is a normal family with respect to the uniform convergence in $\widehat{\mathbb{C}}$, and*
2) *every $f \in \mathcal{H}^\sharp_{\rho^\sharp}$ has modulus of continuity*

$$d^\sharp(f(z_1), f(z_2)) \leq C e^{-\frac{1}{2}\rho^\sharp(z_0, r_1, r_2)}, \quad z_1, z_2 \in B^\sharp(z_0, r_1) \qquad (2.2)$$

for $z_0 \in \widehat{\mathbb{C}}$ and $0 < r_1 < r_2 < 1/2\sqrt{2}$, where C is an absolute constant.

2.3. **Proposition.** *Let ρ be a modulus constraint. Then*

1) *$\mathcal{H}_\rho$ is a normal family with respect to the uniform convergence in $\widehat{\mathbb{C}}$, and*

2) *for each* $R > 0$ *there is a constant* $C = C(R, \rho) > 0$ *depending only on* R *and* ρ *such that every* $f \in \mathcal{H}_\rho$ *satisfies*

$$|f(z_1) - f(z_2)| \le Ce^{-\rho(z_0,r_1,r_2)}, \quad z_1, z_2 \in B(z_0, r_1) \tag{2.4}$$

for $|z_0| \le R$ *and* $0 < r_1 < r_2 < R$.

For the proof of these propositions we need auxiliary lemmas. The first one is shown by applying the famous Teichmüller's lemma on his extremal ring domain. See, for example, [9], where the authors assert that we can take $C_0 = \pi^{-1}\log 2(1+\sqrt{2}) = 0.50118\ldots$ below.

2.5. **Lemma.** *There exists a universal constant* $C_0 > 0$ *with the property that for a ring domain* B *in* $\mathbb{C}$ *with* $\operatorname{mod} B > C_0$ *which separates a point* z_0 *from* ∞ *we can choose an annulus* A *in* B *of the form* $A = A(z_0, r_1, r_2)$, $r_1 < r_2$, *so that* $\operatorname{mod} A \ge \operatorname{mod} B - C_0$ *holds.*

We need information about the size of components of the complement of a ring domain of sufficiently large modulus. There are several kinds of such estimates, among which the following form due to Lehto-Virtanen [14, Lemma I.6.1] is probably best known. Let B be a ring domain whose complement in $\widehat{\mathbb{C}}$ consists of continua E_0 and E_1. Then

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \le \frac{\pi}{\sqrt{2\operatorname{mod} B}}.$$

This inequality is very explicit, however, the order in $\operatorname{mod} B$ is not best possible. So, we prepare the following result where the order in $\operatorname{mod} B$ is the best.

2.6. **Lemma.** *Let* B *be an arbitrary ring domain in* $\widehat{\mathbb{C}}$ *and let* E_0 *and* E_1 *be the components of* $\widehat{\mathbb{C}} \setminus B$. *Then the inequality*

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \le C_1 e^{-\frac{1}{2}\operatorname{mod} B}$$

holds, where C_1 *is an absolute constant.*

Proof. We may assume that $\infty \in E_1$. Then, we get the desired conclusion by combining Lemma 2.5 with the following elementary but sharp result. In particular, we can use the value $C_1 = 2e^{C_0/2} = 2.56957\ldots$.

2.7. **Lemma.** *Let A be an annulus in $\widehat{\mathbb{C}}$ whose complement consists of disjoint closed disks E_0 and E_1. Then*

$$\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\} \le \frac{1}{\cosh(\frac{1}{2}\operatorname{mod} A)}.$$

Equality holds if and only if $\operatorname{diam}^\sharp E_0 = \operatorname{diam}^\sharp E_1$ *and if the spherical centers of* E_0 *and* E_1 *are antipodal.*

Proof. If $\operatorname{diam}^\sharp E_j > \operatorname{diam}^\sharp E_{1-j}$ for some $j = 0, 1$, we can decrease $\operatorname{diam}^\sharp E_j$ a little while leaving $\operatorname{diam}^\sharp E_{1-j}$ and $\min\{\operatorname{diam}^\sharp E_0, \operatorname{diam}^\sharp E_1\}$ invariant, then the resulting annulus will have larger modulus. Hence, using this argument, we may assume $\operatorname{diam}^\sharp E_0 = \operatorname{diam}^\sharp E_1$ without loss of generality. After a suitable isometric Möbius transformation with respect to the spherical metric, we can further assume that E_0 and E_1 are symmetric in the imaginary axis and that the center of E_0 is a positive real number. Let $E_0 \cap \mathbb{R} = [r, R]$ and δ be the hyperbolic diameter of E_0 in the hyperbolic plane $\mathbb{H} = \{z : \operatorname{Re} z > 0\}$. Note that

$$\delta = \int_r^R \frac{dx}{2x} = \frac{1}{2}\log\frac{R}{r} = \log t,$$

where we set $t = \sqrt{R/r} = e^\delta > 1$. Since $\mathbb{H} \setminus E_0$ is Möbius equivalent to the annulus $A(0, \tanh(\delta/2), 1)$, we can compute the modulus of A as follows;

$$\operatorname{mod} A = 2 \operatorname{mod}(\mathbb{H} \setminus E_0) = 2\log\coth\frac{\delta}{2} = 2\log\frac{t+1}{t-1}.$$

In particular, we have $t = \coth(m/2)$, where we set $m = \operatorname{mod} A/2$. On the other hand,

$$\begin{aligned}
\operatorname{diam}^\sharp E_0 &= d^\sharp(r, R) = \frac{R - r}{\sqrt{1+R^2}\sqrt{1+r^2}} = \frac{(t^2-1)r}{\sqrt{1+t^4r^2}\sqrt{1+r^2}} \\
&= \frac{t^2-1}{\sqrt{1+t^4+r^{-2}+t^4r^2}} \le \frac{t^2-1}{\sqrt{1+t^4+2t^2}} = \frac{t^2-1}{t^2+1} \\
&= \frac{\coth^2(m/2)-1}{\coth^2(m/2)+1} = \frac{1}{\cosh m},
\end{aligned}$$

where equality holds if and only if $rt = 1$, equivalently, $rR = 1$. The last relation means that the spherical center of E_0 is 1 and vice versa. In this case, the spherical center of E_1 is -1, which is the antipode of 1. Hence, the last assertion of the lemma follows.

Lemma 2.6 has the best order in $\operatorname{mod} B$, however, if we restrict ourselves to ring domains in the finite plane $\mathbb{C}$, the order is no longer best possible. The following estimate has the best order in $\operatorname{mod} B$ in this case, though the extra factor $\operatorname{dist}(E_0, E_1)$ will come into.

2.8. **Lemma.** *Let B be a ring domain in $\mathbb{C}$ whose complement in $\widehat{\mathbb{C}}$ consists of the bounded component E_0 and the unbounded component E_1. If $\operatorname{mod} B > C_2$, we have the estimate*

$$\operatorname{diam} E_0 \le C_3 \operatorname{dist}(E_0, E_1) e^{-\operatorname{mod} B},$$

where C_2 and C_3 are positive absolute constants.

Proof. We may assume that $\operatorname{dist}(E_0, E_1) = 1,\ 0 \in E_0$ and $1 \in E_1$. Let a be an arbitrary point in E_0 other than 0. Then, by Teichmüller's modulus theorem (see [14]), we have

$$\operatorname{mod} B \le 2\mu\left(\sqrt{\frac{|a|}{1+|a|}}\right),$$

where $\mu(r)$ denotes the modulus of the Grötszch ring $B(0,1) \setminus [0, r]$. Using the well-known estimate $\mu(r) < \log(4/r)$, we obtain

$$|a| \le \frac{16}{e^{\operatorname{mod} B} - 16} \le 32 e^{-\operatorname{mod} B}$$

if $\operatorname{mod} B > 5\log 2$. Hence, we obtain $\operatorname{diam} E_0 \le 64 e^{-\operatorname{mod} B}$ whenever $\operatorname{mod} B > 5\log 2$. In particular, the assertion holds for $C_2 = 5\log 2$ and $C_3 = 64$.

Proof of Proposition 2.1. Observe first that $d^\sharp(0,1) = d^\sharp(1,\infty) = 1/\sqrt{2}$ and that $d^\sharp(0,\infty) = 1$. In particular, when $r_2 < 1/2\sqrt{2}$, the disk $B = B^\sharp(z_0, r_2)$ cannot contain more than one of the three fixed points $0, 1, \infty$. Therefore, the component $\widehat{\mathbb{C}} \setminus f(B)$ of the complement of $f(A^\sharp(z_0, r_1, r_2))$ has diameter at least $1/\sqrt{2}$. Therefore, if the modulus of $f(A^\sharp(z_0, r_1, r_2))$ is sufficiently large, the image of the disk $\overline{B}^\sharp(z_0, r_1)$ must be the smaller component of the complement of $f(A^\sharp(z_0, r_1, r_2))$. Now inequality (2.2) follows from Lemma 2.6. This inequality in turn implies the equicontinuity of $\mathcal{H}^\sharp_{\rho^\sharp}$. Since $\widehat{\mathbb{C}}$ is compact, then by the Ascoli-Arzela theorem, the family $\mathcal{H}^\sharp_{\rho^\sharp}$ is normal.

Let f be the uniform limit of a sequence f_n in $\mathcal{H}^\sharp_{\rho^\sharp}$. We show that f is a member of $\mathcal{H}^\sharp_{\rho^\sharp}$. Since f is the uniform limit of homeomorphisms f_n, we have

$\deg f = \lim \deg f_n = 1$. In particular, $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a surjective continuous map. We now consider the open set

$$V = \{z \in \widehat{\mathbb{C}}; f \text{ is locally constant at } z\}.$$

First we show the following

Claim. If $z_0 \in \widehat{\mathbb{C}} \setminus V$, then $f(z) \neq f(z_0)$ for any $z \in \widehat{\mathbb{C}} \setminus \{z_0\}$.

By permuting the roles of $0, 1, \infty$ if necessary, we may assume that $z_0 \neq \infty$ and that $f(z_0) \neq \infty$. Pick any point w_0 other than z_0. We will show that $f(w_0) \neq f(z_0)$. Choose a small positive number R so that $R < \min\{d^\sharp(z_0, w_0), d^\sharp(z_0, \infty)\}$. Then $\inf_n \operatorname{mod}(f_n(A^\sharp(z_0, \delta, R))) > C_0$ for sufficiently small $\delta > 0$, where C_0 is the constant in Lemma 2.6. By virtue of Lemma 2.6, we can find an annulus A_n of the form $A(f_n(z_0), r_n, r'_n),\ r_n < r'_n$ in $f_n(A^\sharp(z_0, \delta, R))$ for n large enough.

Since f is not locally constant at z_0, there exists a point z_1 in the disk $d^\sharp(z, z_0) < \delta$ with $f(z_0) \neq f(z_1)$. The annulus A_n separates $f_n(z_0), f_n(z_1)$ from $f_n(w_0), \infty$, so we obtain $|f_n(z_1) - f_n(z_0)| \leq r_n$ and $r'_n \leq |f_n(w_0) - f_n(z_0)|$. In particular, $|f_n(z_1) - f_n(z_0)| \leq |f_n(w_0) - f_n(z_0)|$ for n large enough. Letting $n \to \infty$, we obtain $0 < |f(z_1) - f(z_0)| \leq |f(w_0) - f(z_0)|$, and hence $f(w_0) \neq f(z_0)$.

We next show that V is empty. Suppose that V has a non-empty connected component V_0. Then f takes a constant value, say b, in V_0. Because $V_0 \neq \widehat{\mathbb{C}}$, we can take a point z_0 from ∂V_0. By continuity, we have $f(z_0) = b$. On the other hand, from the above Claim, we can deduce that $f(z) \neq f(z_0) = b$ for any point z other than z_0, which contradicts the fact that $f = b$ in V_0. Thus we conclude that V is empty, namely, f is not locally constant at any point.

By using Claim again, we obtain $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. Thus, the injectivity of f follows. Finally, by the continuity of moduli of ring domains with respect to the Hausdorff convergence (see [14]), we conclude that $f \in \mathcal{H}^\sharp_{\rho^\sharp}$.

Proof of Proposition 2.3. It is easy to see that there is a spherical modulus constraint $\rho^\sharp$ such that $\mathcal{H}_\rho \subset \mathcal{H}^\sharp_{\rho^\sharp}$. From Proposition 2.1, the normality of $\mathcal{H}_\rho$ now follows. We next give an estimate of $\operatorname{dist}(E_0, E_1)$, where E_0 and E_1 are the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus f(A),\ A = A(z_0, r_1, r_2)$, respectively. Since $\mathcal{H}_\rho$ is normal, $S = \sup\{|g(z)| : g \in \mathcal{H}_\rho, |z| \leq 2R\} < +\infty$. Now it is clear that $\operatorname{dist}(E_0, E_1) \leq \operatorname{diam} f(\partial B(z_0, r_2)) \leq 2S$. Set $m = \operatorname{mod} f(A)$. If $m > C_2$, we obtain $\operatorname{diam} E_0 \leq 2SC_3 e^{-m}$ by Lemma 2.8. Otherwise, we have $\operatorname{diam} E_0 \leq 2S \leq 2Se^{C_2}e^{-m}$. Hence, the proof is complete.

Let μ be a Beltrami coefficient defined in $\mathbb{C}$ with $\|\mu\|_\infty \leq 1$. In order to apply Proposition 2.3 to the study of the Beltrami equation, we first specify an approximation procedure. This is done in a standard way. For $n = 1, 2, ...$, we set

$$\mu_n(z) = \mu(z), \quad \text{if } |\mu(z)| \leq 1 - 1/n, \tag{2.9}$$

and $\mu_n(z) = 0$ otherwise, and denote by f_n the sequence of quasiconformal automorphisms of the extended complex plane preserving 0, 1 and ∞, and having μ_n as its complex dilatation. The existence of such f_n is guaranteed by the measurable Riemann mapping theorem. We will call such f_n the *canonical approximating sequence* for μ. The topological structure of the set f_n with respect to the uniform convergence under some additional assumptions on the generating coefficient μ will be described in Theorem 2.15.

A function $H : [0, +\infty) \to \mathbb{R}$ is called a *dominating factor* if the following conditions are satisfied;

1. $H(x)$ is continuous and strictly increasing in $[x_0, +\infty)$ and $H(x) = H(x_0)$ for $x \in [0, x_0]$ for some $x_0 \geq 0$, and
2. the function $e^{H(x)}$ is convex in $x \in [0, +\infty)$.

The convexity of e^H implies that $H(x) \to +\infty$ as $x \to +\infty$. In the sequel, the inverse H^{-1} of H will mean the inverse of the homeomorphism $H : [x_0, +\infty) \to [H(0), +\infty)$.

A dominating factor H is said to be of *divergence type* if

$$\int_1^{+\infty} \frac{H(x)dx}{x^2} = +\infty, \tag{2.10}$$

and otherwise H is said to be of *convergence type*. Denote by $\mathcal{D}$ the set of all dominating factors of divergence type. Note that $H(\eta x)$ and $\eta H(x)$ are dominating factors of the same type as $H(x)$ if $H(x)$ is a dominating factor and if η is a positive constant. The following is the most important feature of the divergence condition (2.10).

2.11. **Lemma.** *Let H be a dominating factor. Then H is of divergence type if and only if*

$$\int_{t_1}^{+\infty} \frac{dt}{H^{-1}(t)} = +\infty \tag{2.12}$$

for sufficiently large number t_1.

Proof. By the change of variables $t = H(x)$ and integration by parts, we see

$$\int_{t_1}^{t_2} \frac{dt}{H^{-1}(t)} = \int_{x_1}^{x_2} \frac{dH(x)}{x} = \frac{H(x_2)}{x_2} - \frac{H(x_1)}{x_1} + \int_{x_1}^{x_2} \frac{H(x)dx}{x^2},$$

where $t_j = H(x_j)$ for $j = 1, 2$. Noting that $H(x)/x$ is positive for x large enough, we easily get to the "only if" part. In order to show the "if" part, assume that (2.10) fails. Then, in view of the above identity, (2.12) would imply that $H(x)/x \to \infty$ if $x \to \infty$. In particular, $H(x)/x > C > 0$ holds for sufficiently large x, where C is a positive constant. Hence, letting x_1 be large enough, we would see that $\int_{x_1}^{x_2} H(x)dx/x^2 > \int_{x_1}^{x_2} C\,dx/x = C\log(x_2/x_1) \to \infty$ as $x \to \infty$, which is a contradiction. Thus, (2.10) follows.

Let $\mu \in L^\infty(\Omega)$ be a Beltrami coefficient with $|\mu| < 1$ a.e. in an open subset Ω of $\widehat{\mathbb{C}}$. We define the angular dilatation D_{μ,z_0} of μ at $z_0 \in \widehat{\mathbb{C}}$ by

$$D_{\mu,z_0}(z) = \frac{\left|1 - \mu(z)\frac{\bar z - \bar z_0}{z - z_0}\right|^2}{1 - |\mu(z)|^2} \tag{2.13}$$

in the case when z_0 is finite, and by $D_{\mu,\infty}(z) = D_{\mu,0}(z)$ in the case when $z_0 = \infty$. Then D_{μ,z_0} is a measurable function in Ω and satisfies the inequality $1/K_\mu(z) \le D_{\mu,z_0}(z) \le K_\mu(z)$ a.e. for each $z_0 \in \widehat{\mathbb{C}}$. Note that $D_{\mu,0}(z) = K_\mu(z)$ holds a.e. if and only if $\mu(z)$ has the form $-\rho(z)z/\bar z$ for a non-negative measurable function ρ. The name of D_{μ,z_0} comes from the following important relation: if f is μ-conformal in Ω and if we write $z = z_0 + re^{i\theta}$, then

$$\left|\frac{\partial f}{\partial\theta}(z)\right|^2 = r^2 D_{\mu,z_0}(z) J_f(z) \tag{2.14}$$

holds for almost all $z \in \Omega$, where J_f is the Jacobian of f.

2.15. **Theorem.** *Let μ be a Beltrami coefficient in $\mathbb{C}$. Assume that, for each $z_0 \in \widehat{\mathbb{C}}$, one of the following conditions holds for some positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$:*

1) $D_{\mu,z_0}(z) \le M$ *a.e. in* $B(z_0, r_0)$;
2) *There is a dominating factor $H = H_{z_0}$ of divergence type such that*

$$\int_{B(z_0,r_0)} e^{H(D_{\mu,z_0}(z))} d\mathcal{A}(z) \le M$$

holds for $z_0 \in \mathbb{C}$*, while the above condition is replaced by*

$$\int_{B(\infty,r_0)} e^{H(D_{\mu,0}(z))}\frac{d\mathcal{A}(z)}{|z|^4} \le M$$

if $z_0 = \infty$*.*

Then the canonical approximating sequence f_n *for* μ *forms a pre-normal family with respect to the uniform convergence in* $\widehat{\mathbb{C}}$ *and every limit function* f *of this sequence is a self-homeomorphism of* $\widehat{\mathbb{C}}$*, and admits the following modulus of continuity estimates according to cases* 1) *or* 2) *at each point* z_0 *with* $|z_0| \le R_0$*, where* R_0 *is an arbitrary positive number:*

$$|f(z) - f(z_0)| \le C|z - z_0|^{1/M} \tag{2.16}$$

or

$$|f(z) - f(z_0)| \le C\exp\left\{-\int_{1+c}^{2m+c}\frac{dt}{2H^{-1}(t)}\right\}, \tag{2.17}$$

respectively, for $|z - z_0| < r_1$ *and* $0 < r_1 \le \min\{r_0, R_0\}$*, where* $m = \log(r_1/|z - z_0|)$*,* $c = \log(M/\pi r_1^2)$ *and* C *is a constant depending only on* μ *and* R_0*.*

2.18. **Remark.** This theorem does not tell anything about the regularity of the limit mapping f nor the existence of μ-conformal homeomorphisms. At the moment, the following can be said at least. Let $\mathrm{Sing}(\mu)$ be the singular set of a Beltrami coefficient μ, i.e.,

$$\mathrm{Sing}(\mu) = \{z_0 \in \widehat{\mathbb{C}} : \|\mu\|_{L^\infty(B(z_0,r))} = 1 \quad \text{for any } r > 0\}.$$

Note that $\mathrm{Sing}(\mu)$ is a compact set in $\widehat{\mathbb{C}}$. Note also that $\mathrm{Sing}(\mu)$ can have positive area although $|\mu|$ is always assumed to be less than 1 almost everywhere. Let $\Omega = \widehat{\mathbb{C}} \setminus \mathrm{Sing}(\mu)$. Then, by construction, f is a locally quasiconformal in Ω whose complex dilatation agrees with the given μ a.e. and $f|_\Omega$ is unique up to the post-composition by a conformal map. Further discussions on the existence of μ-conformal homeomorphisms and their regularity will be made in the next section.

The proof of the theorem is based on Proposition 2.3 and the estimate for the change of moduli of ring domains under homeomorphisms in the Sobolev space $W^{1,1}_{\mathrm{loc}}$ stated in the lemmas below.

The first result is essentially due to [2] and [18] and is given now under the somewhat weaker assumptions.

2.19. **Lemma.** *Let μ be a Beltrami coefficient on a domain Ω in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a μ-conformal embedding. Suppose that $D_{\mu,z_0}(z)$ is locally integrable in the annulus $A = A(z_0, r_1, r_2) \subset \Omega$. Then*

$$\operatorname{mod} f(A) \geq \int_{r_1}^{r_2} \frac{dr}{r\psi_\mu(r, z_0)}, \tag{2.20}$$

where

$$\psi_\mu(r, z_0) = \frac{1}{2\pi} \int_0^{2\pi} D_{\mu,z_0}(z_0 + re^{i\theta}) d\theta. \tag{2.21}$$

Proof. We may assume that $z_0 = 0$ and $\Omega = A = A(0, 1, R)$. Composing a suitable conformal mapping, we may further assume that $A' = f(A) = A(0, 1, R')$. Let ν be the positive Borel measure on A defined by $\nu(E) = \mathcal{A}(f(E))$. By a usual argument (see, e.g., [1] or [14]), we obtain $J_f d\mathcal{A} \leq d\nu$ on A in the sense of measure, where $J_f = |f_z|^2 - |f_{\bar z}|^2$. In particular, $J_f \in L^1_{\mathrm{loc}}(A)$.

Denote by γ_r the circle $|z| = r$. Then the assumption $f \in W^{1,1}_{\mathrm{loc}}(A)$ together with the Gehring-Lehto theorem (see [14]) implies that f is absolutely continuous on γ_r and totally differentiable at every point in γ_r except for a set of linear measure 0 for almost all $r \in (1, R)$. By Fubini's theorem, we observe that $D_\mu := D_{\mu,z_0}$ and J_f are integrable on γ_r for almost all $r \in (1, R)$. For such an r, we have

$$2\pi \leq \int_{\gamma_r} |d \arg f| \leq \int_{\gamma_r} \frac{|df(z)|}{|f(z)|} = \int_0^{2\pi} \frac{|f_\theta(re^{i\theta})|}{|f(re^{i\theta})|} d\theta.$$

Taking (2.14) into account, we use Schwarz's inequality to obtain

$$(2\pi)^2 \leq r^2 \int_0^{2\pi} D_\mu(re^{i\theta}) d\theta \int_0^{2\pi} \frac{J_f}{|f|^2}(re^{i\theta}) d\theta,$$

and hence

$$\frac{2\pi}{r\psi_\mu(r)} \leq r \int_0^{2\pi} \frac{J_f}{|f|^2}(re^{i\theta}) d\theta$$

for almost all $r \in (1, R)$, where $\psi_\mu(r) = \psi_\mu(r, 0)$. Integrating both sides with respect to r from 1 to R, we obtain

$$\begin{aligned} 2\pi \int_1^R \frac{dr}{r\psi_\mu(r)} &\leq \int_1^R \int_0^{2\pi} \frac{J_f}{|f|^2} r\, d\theta dr = \int_A \frac{J_f d\mathcal{A}}{|f|^2} \\ &\leq \int_A \frac{d\nu}{|f|^2} = \int_{A'} \frac{d\mathcal{A}(w)}{|w|^2} = 2\pi \log R' = 2\pi \operatorname{mod} A' \end{aligned}$$

and thus arrive at the required inequality (2.20).

The following auxiliary result may be of independent interest. The basic idea is due to Blakalova-Jenkins [6], see also [16, p. 51].

2.22. **Lemma.** *Let f be a μ-conformal embedding of $A = A(z_0, r_0e^{-m}, r_0)$ into $\mathbb{C}$. Suppose that a dominating factor H satisfies*

$$\int_A e^{H(D_{\mu,z_0}(z))} d\mathcal{A}(z) \le M, \quad \text{if } z_0 \in \mathbb{C}, \text{ and}$$

$$\int_A e^{H(D_{\mu,0}(z))} \frac{d\mathcal{A}(z)}{|z|^4} \le M, \quad \text{if } z_0 = \infty. \tag{2.23}$$

Then we have

$$\operatorname{mod} f(A(z_0, r_0e^{-m}, r_0)) \ge \int_{1/2}^{m} \frac{dt}{H^{-1}(2t + \log(M/\pi r_0^2))} \tag{2.24}$$

Proof. Let first $z_0 \ne \infty$. Setting

$$h(r) = \frac{r^2}{2\pi} \int_0^{2\pi} e^{H(D_{\mu,z_0}(z_0 + re^{i\theta}))} d\theta,$$

we rewrite inequality (2.23) in the form

$$2\pi \int_{r_0e^{-m}}^{r_0} h(r)\frac{dr}{r} = 2\pi \int_0^m h(r_0e^{-t})dt \le M.$$

By Chebyshev's inequality, the set $T = \{t \in (0,m) : h(r_0e^{-t})) > L\}$ has the length

$$|T| = \int_T dt \le \frac{M}{2\pi L}.$$

Since e^H is a convex function, Jensen's inequality yields $e^{H(\psi(r))} \le h(r)/r^2$ where $\psi(r) = \psi_\mu(r, z_0)$. This implies the inequality $\psi(r_0e^{-t}) \le H^{-1}(2t + \log(L/r_0^2))$ for $t \in (0,m) \setminus T$. By Lemma 2.19 we get

$$\begin{aligned}\operatorname{mod} f(A) \;&\ge\; \int_0^m \frac{dt}{\psi(r_0e^{-t})} \ge \int_{(0,m)\setminus T} \frac{dt}{H^{-1}(2t + \log(L/r_0^2))} \\ &\ge\; \int_{|T|}^m \frac{dt}{H^{-1}(2t + \log(L/r_0^2))} \ge \int_{M/2\pi L}^m \frac{dt}{H^{-1}(2t + \log(L/r_0^2))}.\end{aligned}$$

Finally, letting $L = M/\pi$, we obtain (2.24).

The remaining case is that $z_0 = \infty$. Let $\varphi(z) = 1/z$ be the inversion. Let $\hat\mu$ be the complex dilatation of the map $g = \varphi \circ f \circ \varphi$. Then $D_{\hat\mu,0}(z) = D_{\mu,0}(1/z)$. Now the required inequality immediately follows from the previous one.

Proof of Theorem 2.15**.** Let f_n be the canonical approximating sequence for μ. Suppose that μ satisfies assumption 2) at $z_0 \in \mathbb{C}$. Since

$$H \circ D_{\mu_n,z_0}(z) \le \max\{H \circ D_{\mu,z_0}(z),\ H(1)\} \tag{2.25}$$

for the dominating factor H, the sequence

$$M_n = \int_{B(z_0,r_0)} e^{H(D_{\mu_n,z_0}(z))} d\mathcal{A}(z) \tag{2.26}$$

satisfies

$$\lim_{n\to\infty} M_n = \int_{B(z_0,r_0)} e^{H(D_{\mu,z_0}(z))} d\mathcal{A}(z) \le M = M(z_0)$$

by Lebesgue's dominating convergence theorem. We now see by Lemma 2.22 that

$$\operatorname{mod} f_n(A(z_0, re^{-m}, r)) \ge \int_{1/2}^{m} \frac{dt}{H^{-1}(2t + \log(M_n/\pi r^2))} =: \rho_n(z_0, re^{-m}, r)$$

for $0 < r \le r_0$ and $m > 1/2$. Set $\hat\rho_n = \inf\{\rho_k; k \ge n\}$. Then, by virtue of Lemma 2.11, $\hat\rho_n(z_0, re^{-m}, r) \to \infty$ as $m \to +\infty$. For $z_0 = \infty$ the same conclusion follows.

In the case when μ satisfies assumption 1), the situation is simpler. Since $\psi_\mu(r, z_0) \le M$ for a.e. $0 < r < r_0$, by Lemma 2.19, we obtain $\operatorname{mod} f_n(A(z_0, re^{-m}, r)) \ge m/M$ for $0 < r \le r_0$ and $m > 0$, so we just set $\hat\rho_n(z_0, re^{-m}, r) = m/M$ in this case. For the other points (z_0, r, R), we simply set $\hat\rho_n(z_0, r, R) = 0$.

The sequence $f_k,\ k \ge n$, then belongs to the class $\mathcal{H}_{\hat\rho_n}$ for each n, and hence is normal by Proposition 2.3. Let f be a limit function of the sequence f_k. We show the estimates of modulus of continuity of f. Since case 1) is easier to treat, we consider only case 2). Letting $n \to \infty$, we obtain $f \in \mathcal{H}_{\rho_\infty}$, where $\rho_\infty = \liminf_{n\to\infty} \rho_n$. By Proposition 2.3, we have

$$|f(z) - f(z_0)| \le C \exp\big(-\rho_\infty(z_0, r_1e^{-m}, r_1)\big)$$

for $|z - z_0| \le r_1e^{-m}$, $|z_0| \le R_0$ and $r_1 \le \min\{r_0, R_0\}$, where C is a constant depending only on μ and R_0. Letting $|z - z_0| = r_1e^{-m}$, we obtain (2.17) by

$$\operatorname{mod} f(A(z_0, r_1e^{-m}, r_1)) \ge \rho_\infty(z_0, r_1e^{-m}, r_1) \ge \int_{1+c}^{2m+c} \frac{dt}{H^{-1}(t)},$$

where $c = \log(M/\pi r_1^2)$.

3. Existence Theorems

In order to give some applications of Theorem 2.15 to the study of the degenerate Beltrami equation we need the well-known regularity results, see, e.g., [6], Lemmas 4 – 6 and Proposition 10, and also [17], [16].

3.1. **Proposition.** *Let μ be a Beltrami coefficient in $\mathbb{C}$ and suppose that a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a uniform limit of the canonical approximating sequence f_n for μ. If $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 1$, then $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$, $q = 2p/(1+p)$, and f satisfies the Beltrami equation with μ. Moreover, $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.*

3.2. **Remark.** It is shown in [6, Lemma 3] that $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$ implies the ACL property of f.

3.3. **Remark.** The Sobolev embedding theorem for spheres [15] or Gehring's oscillation inequality, see, e.g., [10, Lemma 5.2], implies that if $f : \mathbb{C} \to \mathbb{C}$ is a μ-conformal homeomorphism with $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$, then for every compact set E in $\mathbb{C}$ there exist constants C and a such that

$$|f(z_1) - f(z_2)| \geq C\, e^{-a/|z_1 - z_2|^2} \tag{3.4}$$

for $z_1, z_2 \in E$ with $z_1 \neq z_2$. The same inequality is also obtained in [20, p. 75] as a consequence of the Length-Area principle.

Proposition 2.1 together with Theorem 2.15 yields the following statement.

3.5. **Theorem.** *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that:*

1) *$K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 1$; and*
2) *for each point $z_0 \in \widehat{\mathbb{C}}$ either $D_{\mu,z_0} \leq M$ a.e. in $B(z_0, r_0)$ or*

$$\begin{aligned} &\int_{B(z_0,r_0)} e^{H(D_{\mu,z_0}(z))} d\mathcal{A}(z) \leq M, \quad \text{if } z_0 \in \mathbb{C}, \text{ and} \\ &\int_{B(\infty,r_0)} e^{H(D_{\mu,0}(z))} \frac{d\mathcal{A}(z)}{|z|^4} \leq M, \qquad \text{if } z_0 = \infty \end{aligned} \tag{3.6}$$

   for some dominating factor $H = H_{z_0}$ of divergence type and positive constants $M = M(z_0)$ and $r_0 = r_0(z_0)$.

Then there exists a normalized homeomorphic solution $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of (1.1) such that $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$, $q = 2p/(1+p)$, and $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. This homeomorphism admits the modulus of continuity estimate $|f(z) - f(z_0)| \leq C|z - z_0|^{1/M}$ or

$$|f(z) - f(z_0)| \leq C \exp\left\{-\frac{1}{2}\int_{1+c}^{2m+c} \frac{dt}{H_{z_0}^{-1}(t)}\right\}, \tag{3.7}$$

respectively, for $|z - z_0| < r_1$, $|z_0| \leq R_0$ and $r_1 \leq \min\{r_0(z_0), R_0\}$, where $m = \log(r_1/|z - z_0|)$, $c = \log(M(z_0)/\pi r_1^2)$, R_0 is a fixed number and C is a constant depending only on μ and R_0.

Proof. Let f_n be the canonical approximating sequence corresponding to the Beltrami coefficient μ. From Theorem 2.15 we see that f_n forms a pre-compact family with respect to the uniform convergence in $\widehat{\mathbb{C}}$ and every limit function f of this family is a self-homeomorphism of $\widehat{\mathbb{C}}$. Passing to subsequence, we may assume that $f_n \to f$ uniformly in $\widehat{\mathbb{C}}$ and $\mu_n \to \mu$ a.e. as $n \to \infty$. Since $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for some $p > 1$, we see by Proposition 3.1 that $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$, $q = 2p/(1+p)$ and f satisfies (1.1). Moreover, $f_n^{-1} \to f^{-1}$ uniformly in $\widehat{\mathbb{C}}$ as $n \to \infty$ and $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. The modulus continuity estimate (3.7) follows from Theorem 2.15.

Specifying the dominating factor $H(x)$ we obtain more concrete consequences from Theorem 3.5. Typical examples are ηx and $\eta x/(1+\log^+ x)$ for a positive constant η. The corresponding assumptions to (3.6) can be described by the local exponential integrability condition

$$\int_{B(z_0,r_0)} e^{\eta D_{\mu,z_0}(z)} d\mathcal{A}^\sharp(z) < +\infty \tag{3.8}$$

and by the local subexponential integrability condition

$$\int_{B(z_0,r_0)} \exp\left\{\frac{\eta D_{\mu,z_0}(z)}{1+\log^+ D_{\mu,z_0}(z)}\right\} d\mathcal{A}^\sharp(z) < +\infty \tag{3.9}$$

for the angular dilatation coefficient $D_{\mu,z_0}(z)$. More examples will be given in Section 5.

The following result can be viewed as an extended version of the corresponding existence theorems from [6] and [10].

3.10. **Theorem.** *Let H be a dominating factor of divergence type. Suppose that a Beltrami coefficient μ on $\mathbb{C}$ with $\|\mu\|_\infty \le 1$ satisfies*

$$\int_{\mathbb{C}} e^{H(K_\mu(z))} d\mathcal{A}^\sharp(z) < +\infty, \tag{3.11}$$

where $d\mathcal{A}^\sharp(z) = (1+|z|^2)^{-2} d\mathcal{A}(z)$ denotes the spherical area element. Then there exists a normalized homeomorphic solution f to the Beltrami equation with μ such that $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ for every $1 \le q < 2$ and that $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

Proof. Since H is of divergence type, assumption (3.11) implies that $K_\mu \in L^p_{\mathrm{loc}}(\mathbb{C})$ for every $1 < p < \infty$. On the other hand, the inequality $D_{\mu,z_0}(z) \le K(z)$ a.e. and the convergence of the integral (3.11) imply the local assumption (3.6). Hence, by Theorem 3.5, we have the required conclusion.

3.12. **Remark.** Condition (3.11) is optimal for the solvability of Beltrami equations in the following sense.

Assume that H is a dominating factor of convergence type. We may assume that H is smooth enough and $H(1) = 1$. Then, by Theorem 3.1 of [10], there exists a μ satisfying (3.11) for which the following holds:

1. $\mu = 0$ off the unit disk B,
2. K_μ is locally essentially bounded in $\mathbb{C} \setminus \{0\}$,
3. there are no $W^{1,1}_{\text{loc}}$-solutions to the Beltrami equation with μ in the unit disk which are continuous at the origin other than constant functions, and
4. there is a solution f to the Beltrami equation in the weak-$W^{1,2}(B)$ Sobolev space, where weak-$W^{1,2}(B) = \bigcap\limits_{1\le q<2} W^{1,q}(B)$, which maps the punctured disk $B \setminus \{0\}$ homeomorphically onto the annulus $A(0,1,R)$ for some $1 < R < +\infty$.

3.13. **Corollary.** *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that for some $\eta > 0$*

$$\int_{\mathbb{C}} e^{\eta K_\mu(z)} d\mathcal{A}^\sharp(z) < +\infty. \tag{3.14}$$

Then there exists a μ-conformal homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f \in W^{1,q}_{\text{loc}}(\mathbb{C})$ for every $q < 2$ and $f^{-1} \in W^{1,2}_{\text{loc}}(\mathbb{C})$. Moreover, for every compact set $E \subset \mathbb{C}$ there are positive constants C, C' and a such that

$$C \exp\left(-\frac{a}{|z_1 - z_2|^2}\right) \le |f(z_1) - f(z_2)| \le C' \left|\log \frac{1}{|z_1 - z_2|}\right|^{-\eta/2} \tag{3.15}$$

for $z_1, z_2 \in E$ with $0 < |z_1 - z_2| < 1/e$. The exponent $-\eta/2$ is sharp.

The right hand side of (3.15) can be written in the more precise form

$$|f(z_1) - f(z_2)| \le C \operatorname{dist}(E_0, E_1) \left(\frac{1 + \log(M/\pi R^2)}{\log 1/|z_1 - z_2|}\right)^{\eta/2}$$

for $z_1, z_2 \in B(0, R/2)$ with $|z_1 - z_2| < 1/e$ where

$$M = \int_{B(0,R)} e^{\eta K_\mu} d\mathcal{A},$$

$E_0 = f(\overline{B}(0, R/2))$, $E_1 = f(\widehat{\mathbb{C}} \setminus B(0, R))$ and C is a constant depending only on η.

The sharpness can be seen by the following examples. Let $f : \mathbb{C} \to \mathbb{C}$ be the radial stretching defined by

$$f(z) = \frac{z}{|z|}\left(1 + \frac{2}{\beta}\log\frac{1}{|z|}\right)^{-\beta/2} \tag{3.16}$$

for $|z| \le 1$ and $f(z) = z$ for $|z| > 1$. (This example was given in [10, §11.1].) Then $K(z) = 1 + (2/\beta)\log(1/|z|)$ for $|z| < 1$. Hence integrability condition (3.14) holds for $\beta > \eta$ but not for $\beta \le \eta$. Since $|f(z) - f(0)| = (1+2\beta^{-1}\log 1/|z|)^{-\beta/2} \sim (\log 1/|z|)^{-\beta/2}$ as $z \to 0$, the exponent $\eta/2$ in (3.15) cannot be replaced by any larger number.

3.17. **Remark.** It is noted in [3] that a necessary and sufficient condition for a measurable function $K(z) \ge 1$ to be majorized in $\Omega \subset \mathbb{C}$ by a function $M \in BMO(\mathbb{C})$ is that

$$\int_\Omega e^{\eta K(z)} d\mathcal{A}^\sharp(z) < +\infty \tag{3.18}$$

for some positive number η. Moreover, M can be chosen so that $\|M\|_{BMO} \le C/\eta$ where C is an absolute constant. For mappings of BMO-bounded distortion David [7] has proved the estimate

$$|f(z_1) - f(z_2)| \le A\left|\log\frac{1}{|z_1 - z_2|}\right|^{-b/\|M\|_{BMO}} \tag{3.19}$$

for some positive constant b, which agrees with our modulus estimate (3.15).

3.20. **Remark.** It is shown in [10] that if μ has a compact support, then there exists a number $\eta_0 > 1$ such that the Beltrami equation with μ satisfying (3.14) with $\eta \ge \eta_0$ admits a unique principal solution f with $f(z) - z \in W^{1,2}(\mathbb{C})$.

The following consequence is due to [10, Theorem 14.2] except for the modulus of continuity estimate. The almost same result has been obtained by [6] earlier (see the remark below).

3.21. **Corollary.** *Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ such that*

$$\int_{\mathbb{C}} \exp\left\{\frac{\eta K(z)}{1 + \log K(z)}\right\} d\mathcal{A}^\sharp(z) < \infty \tag{3.22}$$

for some $\eta > 0$. Then there exists a homeomorphic solution f in $\widehat{\mathbb{C}}$ to (1.1) such that $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{C})$ for every $q < 2$ and $f^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Moreover, for every compact set $E \subset \mathbb{C}$ there are constants C, C' and a such that

$$(3.23)\quad C\exp\left\{-\frac{a}{|z_1-z_2|^2}\right\} \le |f(z_1)-f(z_2)| \le C'\left(\log\log\frac{1}{|z_1-z_2|}\right)^{-\eta/2}$$

for $z_1, z_2 \in E$ with $0 < |z_1 - z_2| < e^{-e}$. The exponent $-\eta/2$ is sharp.

The modulus of continuity follows from the fact that $\eta^{-1} y \log y < H^{-1}(y)$ for sufficiently large y where $H(x) = \eta x/(1+\log^+ x)$. More precisely, we have an estimate in the following form:

$$|f(z_1)-f(z_2)| \le C\,\mathrm{dist}(E_0, E_1)\left(\frac{\log\log(M/\pi R^2)}{\log\log(1/|z_1-z_2|)}\right)^{\eta/2}$$

for $z_1, z_2 \in B(0, R/2)$ with $|z_1 - z_2| < e^{-e}$, where

$$M = \int_{B(0,R)} \exp\left\{\frac{\eta K(z)}{1+\log K(z)}\right\} d\mathcal{A}(z),$$

$E_0 = f(\overline{B}(0, R/2))$, $E_1 = f(\widehat{\mathbb{C}} \setminus B(0,R))$ and C is a constant depending only on η.

Next we show the sharpness by examples. Let $f : \mathbb{C} \to \mathbb{C}$ be the radial stretching defined by

$$(3.24)\qquad f(z) = \frac{z}{|z|}\left(1 + \frac{2}{\beta}\log\log\frac{e}{|z|}\right)^{-\beta/2}$$

for $|z| \le 1$ and $f(z) = z$ for $|z| > 1$. Then $K(z) = \log\frac{e}{|z|}\left(1 + \frac{2}{\beta}\log\log\frac{e}{|z|}\right)$ for $|z| < 1$. (This example appeared in [10, §12.1], however, there seems to be an error in the formula for $K(z)$.) Hence

$$\frac{K(z)}{1+\log K(z)} = \frac{2}{\beta}\log\frac{e}{|z|}\left(1 - \frac{\log_3 e/|z|}{\log_2 e/|z|} + O\left(\frac{1}{\log_2 e/|z|}\right)\right)$$

as $z \to 0$ and, in particular, we can see that $\exp\{\eta K(z)/(1+\log K(z))\}$ is integrable in $|z| < 1$ if and only if $\beta \ge \eta$. Therefore the exponent $\eta/2$ in (3.23) cannot be replaced by any larger number in general.

3.25. **Remark.** The first proof of the above statement was given in [6] under the assumption of special behavior of $K_f(z)$ around the point at infinity:

$$\int_{B(0,R)} K(z)d\mathcal{A}(z) = O(R^2).$$

This assumption says that K is bounded in the sense of the mean, and hence it is different from (3.22). Theorem 14.2 in [10] contains also the quite accurate regularity assertion

$$\int_{\mathbb{C}} \frac{|\Psi(z)|^2}{\log(e+\Psi(z))\log_2(3+\Psi(z))} d\mathcal{A}^\sharp(z) < +\infty$$

where $\Psi(z)$ stands for the spherical derivative $\frac{1+|z|^2}{1+|f(z)|^2}|Df(z)|$ of f.

Next we consider the singular set $\mathrm{Sing}(\mu)$ of μ in $\widehat{\mathbb{C}}$. The following result shows that if $\mathrm{Sing}(\mu)$ is a finite set then μ can be modified without changing its absolute value and the modified μ admits a "good" solution.

3.26. **Proposition.** *Let $\mu(z)$ be a Beltrami coefficient in $\widehat{\mathbb{C}}$ such that $\mathrm{Sing}(\mu)$ consists of finitely many points $\{a_1, ..., a_n\} \in \widehat{\mathbb{C}}$. Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\widehat{\mathbb{C}} \setminus \{a_1, ..., a_n\}$ and whose complex dilatation μ_f satisfies*

$$|\mu_f(z)| = |\mu(z)| \quad a.e. \tag{3.27}$$

Proof. Without loss of generality we will assume that all a_k's are finite. Let δ be a positive number such that the disks $|z - a_k| \le \delta$, $k = 1, ..., n$, are disjoint. Setting

$$\mu'(z) = |\mu(z)| \cdot \frac{z - a_k}{\bar{z} - \bar{a}_k}, \quad k = 1, ..., n,$$

for $|z - a_k| \le \delta$ and $\mu'(z) = \mu(z)$ otherwise, we see that $D_{\mu,z_0}(z)$ is essentially bounded in a sufficiently small disk $B(z_0, r_0)$ for each $z_0 \in \widehat{\mathbb{C}}$. By Theorem 2.15 there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus \{a_1, ..., a_n\}$ and $f|_\Omega$ satisfies the Beltrami equation with μ'. The condition (3.27) is immediate by construction.

3.28. **Remark.** For every $H \in \mathcal{D}$ there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ locally quasiconformal in $\widehat{\mathbb{C}} \setminus \{a_1, ..., a_n\}$ with complex dilatation μ such that

$$\int_{\mathbb{C}} e^{H(K_\mu(z))} d\mathcal{A}^\sharp(z) = +\infty$$

but weaker directional condition (3.6) holds.

In the above examples the singular set of μ consists of isolated points only. In the following examples the singular sets of μ are the whole extended real axis $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Define the Beltrami coefficient μ in $\widehat{\mathbb{C}}$ for any α, $1 \le \alpha < 2$, by

$$\mu(z) = 1 - \frac{2}{\alpha \log \frac{1}{|y|}}, \quad z = x + iy,$$

for $|y| \le y_0$ and $\mu(z) = 0$ for $|y| > y_0$, where $0 < y_0 \le e^{-1/\alpha}$ is a constant. Then integrability assumption (3.14) with $\eta = 1$ does not hold at any neighborhood of each point on the real axis, whereas assumption (3.8) with $\eta = 1$ still holds. Indeed, since $K_\mu(z) = 1 + \alpha \log(1/|y|)$ for $|y| < y_0$, we see that $e^{K_\mu(z)} = e|y|^{-\alpha}$. Now the assumption $\alpha \ge 1$ implies that e^{K_μ} is not locally integrable at any point on the real axis. On the other hand, writing $z = re^{i\theta}$, we compute for any $a \in \mathbb{R}$,

$$D_{\mu,a}(z) = D_{\mu,0}(z) = \frac{1-\mu}{1+\mu} + \frac{4\mu}{1+\mu} \cdot \frac{\sin^2\theta}{1-\mu} \le 1 + \alpha \sin^2\theta \left(\log \frac{1}{|y|}\right), \tag{3.29}$$

where the inequality $4\mu/(1+\mu) \le 2$ has been used. Since the function $x \log x$ is bounded in $(0,1)$, we obtain

$$\exp(D_{\mu,a}(z)) \le e|y|^{-\alpha \sin^2\theta} \le e r^{-\alpha} e^{-\frac{\alpha}{2}\sin^2\theta \log(\sin^2\theta)} \le C r^{-\alpha} \tag{3.30}$$

for some constant C. Hence, for sufficiently small $r_0 > 0$, we have

$$\int_{B(a,r_0)} e^{D_{\mu,a}(z)} d\mathcal{A}^\sharp(z) \le 2\pi C \int_0^{r_0} r^{1-\alpha} dr < +\infty \tag{3.31}$$

and the claim follows.

Another example of Beltrami coefficient μ in $\mathbb{C}$, for which $\mathrm{Sing}(\mu) = \widehat{\mathbb{R}}$, is provided by

$$\mu(z) = 1 - \frac{2}{\alpha \log \frac{1}{|y|} \log\log \frac{1}{|y|}}, \quad z = x + iy,$$

for $|y| \le y_0$ and $\mu(z) = 0$ for $|y| > y_0$, where α is a constant with $1 < \alpha < 2e/(e+1) = 1.462\ldots$ and y_0 is a positive constant with $y_0 \le e^{-e^{1/\alpha}}$. An elementary computation shows that integrability assumption (3.22) does not hold at any neighborhood of each point lying on the real axis. On the other hand, assumption (3.9) still holds. Indeed, since $K_\mu(z) = 1 + \alpha \log(1/|y|)\ \log_2(1/|y|)$, $|y| < y_0$, we obtain

$$\frac{K_\mu(z)}{1 + \log K_\mu(z)} = \frac{\alpha \log \frac{1}{|y|} \log_2 \frac{1}{|y|}}{\log_2 \frac{1}{|y|} + \log_3 \frac{1}{|y|} + O(1)} = \alpha \log \frac{1}{|y|} \left[1 + O\left(\frac{\log_3 \frac{1}{|y|}}{\log_2 \frac{1}{|y|}}\right)\right]$$

as $y \to 0$. In particular, we see that $e^{K_\mu(z)/(1+\log K_\mu(z))}$ is not locally integrable at every point on the real axis. On the other hand, as above, we obtain

$$D_{\mu,0}(z) \le 1 + \alpha \sin^2\theta \log\frac{1}{|y|}\log_2\frac{1}{|y|}.$$

Hence, under the assumption that $\alpha\sin^2\theta\log(1/|y|)\log_2(1/|y|)$ is large enough, we have

$$\begin{aligned}
&\frac{D_{\mu,0}(z)}{1+\log^+ D_{\mu,0}(z)} \\
\le\ & \frac{1+\alpha\sin^2\theta\log\frac{1}{|y|}\left[\log(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|}) - \log(\alpha\sin^2\theta\log_2\frac{1}{|y|})\right]}{1+\log(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|})} \\
=\ & 1+\alpha\sin^2\theta\log\frac{1}{|y|} - \frac{\alpha\sin^2\theta\log\frac{1}{|y|}\left[\log(\alpha\sin^2\theta)+\log_3\frac{1}{|y|}\right]}{1+\log(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|})} \\
\le\ & \alpha\sin^2\theta\log\frac{1}{|y|} + \frac{\alpha e^{-1}\log\frac{1}{|y|}}{1+\log(\alpha\sin^2\theta\log\frac{1}{|y|}\log_2\frac{1}{|y|})} \\
\le\ & \alpha(\sin^2\theta+e^{-1})\log\frac{1}{|y|},
\end{aligned}$$

where we used the inequality $-x\log x \le e^{-1}$ for $0 < x$. This yields

$$\exp\left(\frac{D_{\mu,0}(z)}{1+\log^+ D_{\mu,0}(z)}\right) \le |y|^{-\alpha(\sin^2\theta+e^{-1})} \le Cr^{-\alpha(1+e^{-1})}(\sin\theta)^{-\alpha/e}$$

for some positive constant C. The last function in the polar coordinates (r,θ) is integrable over $0<r<1, -\pi/2<\theta<\pi/2$ because $\alpha(1+e^{-1})-1<1$ and $\alpha/e<1$.

3.32. **Remark.** For the above examples, whose Beltrami coefficients $\mu(z) = \mu(x+iy)$ depend on y only, we can give normalized μ-conformal homeomorphisms $f:\mathbb{C}\to\mathbb{C}$ in the explicit form

$$f(z) = x + i\int_0^y \frac{1-\mu(it)}{1+\mu(it)}dt.$$

4. Uniqueness

The following remark is a simple consequence of a well-known removability theorem for analytic functions.

4.1. **Proposition.** *Let μ be a Beltrami coefficient in a domain Ω such that the singular set $E = \mathrm{Sing}(\mu)$ in Ω is countable. Suppose that a topological embedding $f : \Omega \to \widehat{\mathbb{C}}$ is locally quasiconformal in $\Omega \setminus E$ and satisfies (1.1) with μ there. Then f has the property that for any homeomorphic solution $\hat{f}$ of (1.1) in $\Omega \setminus E$ there exists a conformal map h in $\Omega' = f(\Omega)$ such that $\hat{f} = h \circ f$ in $\Omega \setminus E$. In particular, $\hat{f}$ extends to an embedding of Ω.*

Indeed, let us assume that $\hat{f}$ is another solution to the Beltrami equation (1.1) with μ in $\Omega \setminus E$. Then the function $h = \hat{f} \circ f^{-1}$ is an injective holomorphic function in $\Omega' \setminus f(E)$. Since $f(E)$ is closed and countable in Ω', it is removable for such a function and we conclude that h can be extended to a conformal map on Ω'.

In order to apply the removability arguments for the uniqueness problem it seems necessary to have information on the singular set E and its image $E' = f(E)$, simultaneously, see, e.g., [11] and [3].

As Lehto [13] noted, it is reasonable to restrict ourselves to the case when E is totally disconnected. However, this condition does not imply uniqueness in general. We will introduce a geometric condition on totally disconnected compact sets E and show that this condition can be combined with integrability conditions involving the angular dilatation to guarantee the uniqueness.

A positive function $m(r)$ defined on the interval $(0, \delta)$ for small $\delta > 0$ is said to be a *modulus bound* for a dominating factor H of divergence type if it satisfies the condition

$$\liminf_{r \to 0} \int_0^{m(r)} \frac{dt}{H^{-1}(2t - 2\log r)} > 0. \tag{4.2}$$

Note that $m(r) \to +\infty$ as $r \to 0$.

For example, if $H(x) = \eta x$, then $m(r) = \varepsilon \log(1/r)$ is a modulus bound for H, where ε is an arbitrary positive constant. If $H(x) = \eta x/(1 + \log^+ x)$, then $m(r) = \varepsilon(\log(1/r))^C$ is a modulus bound for H, where $\varepsilon > 0$ and $C > 1$ are arbitrary constants. More examples will be given in Section 5. Each modulus bound $m(r)$ generates a family of annuli $A_m(z_0, r) = A(z_0, re^{-m(r)}, r)$ around every point $z_0 \in \widehat{\mathbb{C}}$.

The following notions describe the thinness of the boundary. Let H be a dominating factor. A compact subset E of $\widehat{\mathbb{C}}$ is said to be *H-coarse* at $z_0 \in E$ if there exists a modulus bound $m(r)$ for H such that for any small number $\delta > 0$ there is an r with $0 < r < \delta$ such that $A_m(z_0, r) \subset \Omega$. The set E is said to be *H-coarse* if E is H-coarse at each point $z_0 \in E$. We will also say that E is *radially coarse* at $z_0 \in E$ if a positive constant function can be chosen as $m(r)$ above.

It is clear that z_0 forms a degenerate boundary component of $\Omega = \widehat{\mathbb{C}} \setminus E$ if E is H-coarse at the point.

4.3. **Theorem.** *Let μ be a Beltrami coefficient in $\widehat{\mathbb{C}}$ such that $E = \mathrm{Sing}(\mu)$ is a totally disconnected compact subset of $\widehat{\mathbb{C}}$. Assume that one of the following conditions holds for each $z_0 \in E$:*

1) *$D_{\mu,z_0}(z)$ is essentially bounded in a neighborhood of z_0 and E is radially coarse at z_0;*
2) *There is a dominating factor $H = H_{z_0}$ of divergence type for which E is H-coarse at z_0 and*

$$\int_V e^{H(D_{\mu,z_0}(z))} d\mathcal{A}^\sharp(z) < +\infty \tag{4.4}$$

for some open neighborhood V of z_0.

Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is locally quasiconformal in $\Omega = \widehat{\mathbb{C}} \setminus E$ and satisfies the Beltrami equation with μ in Ω. If $\hat{f}$ is another homeomorphic solution to the Beltrami equation in Ω with the same μ, then $\hat{f} = h \circ f$ for a Möbius transformation h. In particular, $\hat{f}$ can be extended to a homeomorphism of $\widehat{\mathbb{C}}$.

For the proof of this theorem we need the following auxiliary lemma which is due to Gotoh and Taniguchi. A compact set E is said to be *annularly coarse* at $z_0 \in E$ if we can find a nesting sequence of disjoint ring domains A_n around z_0 in $\widehat{\mathbb{C}} \setminus E$ with $\inf_n \operatorname{mod} A_n > 0$. Here a sequence A_n of disjoint ring domains is said to be nesting if each A_n separates A_{n-1} from A_{n+1}. A compact set E is said to be *annularly coarse* if E is annularly coarse at every point in E.

4.5. **Lemma** (Gotoh-Taniguchi [8]). *If a compact subset E of the Riemann sphere is annularly coarse, then E is removable for conformal mappings off E.*

Proof of Theorem 4.3. Let f be a uniform limit of the canonical approximating sequence for μ. Note that the existence of such an f is guaranteed by Theorem 2.15. Set $E' = f(E)$. We show that E' is annularly coarse. We may assume that $E \subset \mathbb{C}$. Let $z_0 \in E$. Suppose first that case 2) occurs for z_0. Then, by assumption, there is an $H \in \mathcal{D}$ and a constant M with

$$\int_{B(z_0,r_0)} e^{H(D_{\mu,z_0}(z))} d\mathcal{A}(z) \le M$$

for which E is H-coarse at z_0 with modulus bound $m : (0,\delta) \to (0,+\infty)$. We then take arbitrarily small $r > 0$ so that $A_m(z_0, r) \subset \Omega$. By Lemma 2.22, we estimate

$$\operatorname{mod} f(A_m(z_0,r)) \ge \int_{1/2}^{m(r)} \frac{dt}{H^{-1}(2t + \log(M/\pi r^2))}.$$

It is easy to see that

$$\int_a^b \frac{dt}{H^{-1}(2t + c - 2\log r)} \to 0$$

as $r \to 0$ for any fixed a, b and c. Hence, by the definition of the modulus bound, we obtain

$$\liminf_{r\to 0} \int_{1/2}^{m(r)} \frac{dt}{H^{-1}(2t + \log(M/\pi r^2))} = \liminf_{r\to 0} \int_0^{m(r)} \frac{dt}{H^{-1}(2t - 2\log r)} > 0.$$

We can now find a sequence r_n with $0 < r_{n+1} < r_n e^{-m(r_n)}$ such that $A_m(z_0, r_n) \subset \Omega$ for $n = 1, 2, \ldots$ and that $\inf_n \operatorname{mod} f(A_m(z_0, r_n)) > 0$. Hence, we conclude that E' is annularly coarse at $f(z_0)$.

If case 2) occurs for z_0, by assumption, z_0 is radially coarse in E. Then it is easier to see that E' is annularly coarse at $f(z_0)$ than the above.

Assume that $\hat{f}$ is another homeomorphic solution to the Beltrami equation (1.1) with μ in Ω. Then the function $h = \hat{f} \circ f^{-1}$ is an injective holomorphic function in $\widehat{\mathbb{C}} \setminus E'$. Since $E' = f(E)$ is removable for such a function by Lemma 4.5, we conclude that h can extends to a Möbius transformation. Thus, the proof is complete.

Specifying $H \in \mathcal{D}$, we may obtain some consequences of Theorem 4.3. Let us restrict ourselves to the dominating factors $H(x) = \eta x$ and $H(x) = \eta x/(1 + \log^+ x)$ corresponding to exponential and subexponential integrability assumptions on $K_\mu(z)$, respectively.

4.6. **Corollary.** *Let E be a totally disconnected, x-coarse compact subset of $\widehat{\mathbb{C}}$. Suppose that μ is a Beltrami coefficient in $\mathbb{C}$ with* $\operatorname{Sing}(\mu) \subset E$ *such that*

$$\int_{\mathbb{C}} e^{\eta K_\mu(z)} d\mathcal{A}^\sharp(z) < \infty \tag{4.7}$$

holds for a positive constant η. *Then there exists a homeomorphism* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *which is locally quasiconformal in* $\Omega = \widehat{\mathbb{C}} \setminus E$ *and satisfies the Beltrami equation with* μ *in* Ω. *If* $\hat{f} : \Omega \to \widehat{\mathbb{C}}$ *is a locally quasiconformal embedding whose Beltrami coefficient agrees with* μ *a.e. then* $\hat{f} = h \circ f$ *for a Möbius transformation* h. *In particular,* $\hat{f}$ *extends to a homeomorphism of the Riemann sphere.*

4.8. **Corollary.** *Let* E *be a totally disconnected compact subset of* $\widehat{\mathbb{C}}$ *which is* $x/(1+\log^+ x)$*-coarse in* E. *Suppose that* μ *is a Beltrami coefficient in* $\mathbb{C}$ *with* $\mathrm{Sing}(\mu) \subset E$ *such that*

$$\int_{\mathbb{C}} \exp\left\{\frac{\eta K_\mu(z)}{1+\log K_\mu(z)}\right\} d\mathcal{A}^\sharp(z) < \infty. \tag{4.9}$$

Then there exists a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *which is locally quasiconformal in* $\Omega = \widehat{\mathbb{C}} \setminus E$ *and satisfies the Beltrami equation with* μ *in* Ω. *If* $\hat{f} : \Omega \to \widehat{\mathbb{C}}$ *is a locally quasiconformal embedding whose Beltrami coefficient agrees with* μ *a.e. then* $\hat{f} = h \circ f$ *for a Möbius transformation* h. *In particular,* $\hat{f}$ *extends to a homeomorphism of the Riemann sphere.*

5. Dominating factors and Modulus bounds

In this section, we provide critical dominating factors H of divergence type and their modulus bounds. The functions presented below are more or less standard. For example, similar examples can be found in [10].

We define the functions $\log_n, \exp_n, \Pi_{n,\alpha}$ and Π_n for $n = 0, 1, 2, \ldots$ and for $\alpha > 0$ by

$$\log_0 x = x, \quad \log_n x = \log(\log_{n-1} x) \quad (n = 1, 2, \ldots),$$
$$\exp_0 x = x, \quad \exp_n = \exp(\exp_{n-1} x) \quad (n = 1, 2, \ldots),$$
$$\Pi_{n,\alpha}(x) = x(\log_1 x) \cdots (\log_{n-1} x)(\log_n x)^\alpha, \text{ and}$$
$$\Pi_n(x) = \Pi_{n,1}(x) \quad (n = 0, 1, 2, \ldots).$$

In particular, $\log_1$ and $\exp_1$ coincide with the usual log and exp respectively. Note that $\log_n$ is the inverse function of $\exp_n$ for each n. We also define the numbers e_n

by $e_n = \exp_n 0$. Then for each $n > 0$ the functions $\log_n$, $\Pi_{n,\alpha}$ and Π_n are defined on $(e_{n-1}, +\infty)$, positive on $(e_n, +\infty)$, and greater than 1 on $(e_{n+1}, +\infty)$.

We consider the function $H(x) = x^2/\Pi_{n,\alpha}(x)$ for fixed $n \geq 0$ and $\alpha > 0$. Then, $H'(x) = (1 + o(1))H(x)/x$ and $H''(x) = (1 + o(1))H(x)/\log x$ when $x \to +\infty$. In particular, $(e^H)''/e^H = H'' + (H')^2 = (1 + o(1))(H')^2$. Hence, we can choose a sufficiently large number $x_{n,\alpha} > e_n$ so that the function $H_{n,\alpha}$ defined by

$$H_{n,\alpha}(x) = \begin{cases} H(x) = \dfrac{x^2}{\Pi_{n,\alpha}(x)} & \text{if } x > x_{n,\alpha}, \\ H(x_{n,\alpha}) & \text{if } 0 \leq x \leq x_{n,\alpha}, \end{cases}$$

is a dominating factor. Furthermore, $H_{n,\alpha}$ is of divergence type if and only if $\alpha \leq 1$ since the integral

$$\int_{x_{n,\alpha}}^{\infty} \frac{H_{n,\alpha}(x)dx}{x^2} = \int_{x_{n,\alpha}}^{\infty} \frac{dx}{\Pi_{n,\alpha}(x)} = \int_{x_{n,\alpha}}^{\infty} \frac{dx}{x(\log x)\dots(\log_{n-1} x)(\log_n x)^{\alpha}}$$

is divergent if and only if $\alpha \leq 1$. Note that the choice of $x_{n,\alpha}$ does not effect on the integrability condition such as $\displaystyle\int_V e^{H(D_{\mu,z_0})} d\mathcal{A}^{\sharp} < +\infty$. We also write $H_n = H_{n,1}$.

5.1. **Remark.** The dominating factor $H_0 = H_{0,1}$ is essentially same as x. The dominating factor $H_1 = H_{1,1}$ is equivalent to $H(x) = x/(1 + \log^+ x)$ in the sense that for any $\eta > 1$ we have $H(x) < H_1(x) < \eta H(x)$ for sufficiently large x.

First we give information on the behavior of the inverse of $H_{n,\alpha}$, which is useful to derive a modulus of continuity estimate in connection with the previous theorems.

5.2. **Lemma.** *For each $n \geq 0$ and $\alpha > 0$, we have*

$${H_{n,\alpha}}^{-1}(y) = (1 + o(1))\Pi_{n,\alpha}(y) \quad (y \to +\infty)$$

Furthermore if $\alpha \leq 1$, we have

$$\int_{t_1}^{t} \frac{dy}{{H_{n,\alpha}}^{-1}(y)} = \begin{cases} \dfrac{1}{1-\alpha}(\log_n t)^{1-\alpha} + O(1) & \textit{if } \alpha < 1, \\ \log_{n+1} t + O(1) & \textit{if } \alpha = 1 \end{cases}$$

as $t \to +\infty$ for a sufficiently large t_1.

Proof. We first observe

$$\begin{aligned}\log(\eta\Pi_{n,\alpha}(y)) &= \log\eta + \log y + \log_2 y + \cdots + \log_n y + \alpha\log_{n+1} y \\ &= (1+o(1))\log y \quad (y\to+\infty)\end{aligned}$$

for a positive constant η. We can then see inductively that $\log_k(\eta\Pi_{n,\alpha}(y)) = (1+o(1))\log_k y$ for every integer $k\geq 1$. Therefore,

$$H_{n,\alpha}(\eta\Pi_{n,\alpha}(y)) = \frac{\eta\Pi_{n,\alpha}(y)}{\log(\eta\Pi_{n,\alpha}(y))\cdots(\log(\eta\Pi_{n,\alpha}(y)))^{\alpha}} = (\eta+o(1))y$$

as $y\to+\infty$. Since if $\eta>1$, $\eta\Pi_{n,\alpha}(y) > {\Pi_{n,\alpha}}^{-1}(y)$ for sufficiently large y, and if $\eta<1$, $\eta\Pi_{n,\alpha}(y) < {\Pi_{n,\alpha}}^{-1}(y)$ for sufficiently large y, the asymptotic formula for the inverse function follows. The second assertion is immediate once if we know the first one. Thus we have shown the lemma.

To state results on modulus bounds of the above dominating factors, we prepare some auxiliary functions. For constants $C>1$ and $\beta,\delta>0$ and a non-negative integer n, we set

$$\begin{aligned}\varphi_{n,C}(x) &= \exp_n(\log_n x + \log C), \text{ and} \\ \psi_{n,\beta,\delta}(x) &= \exp_n\left(\left[(\log_n x)^{\beta}+\delta\right]^{1/\beta}\right)\end{aligned}$$

for $x\geq e_n$. By definition, $\varphi_{n,C}=\psi_{n,1,\log C}$. For example, we see that

$$\begin{aligned}\varphi_{0,C}(x) &= x+\log C, \\ \varphi_{1,C}(x) &= Cx, \quad \text{and} \\ \varphi_{2,C}(x) &= x^C.\end{aligned}$$

5.3. **Proposition.** *For a positive integer n a modulus bound for H_n can be given by*

$$m(r) = \varepsilon\varphi_{n+1,C}(\log 1/r)$$

for r small enough, where ε and C are arbitrary constants with $\varepsilon>0$ and $C>1$.

Proof. From Lemma 5.2, we see that

$$\begin{aligned}&\liminf_{r\to 0}\int_0^{m(r)}\frac{dt}{{H_n}^{-1}(2t-2\log r)} \\ &= \frac{1}{2}\liminf_{r\to 0}\left[\log_{n+1}(2m(r)-2\log r) - \log_{n+1}(-2\log r)\right].\end{aligned}$$

Noting that $\log r = o(m(r))$ as $r \to 0$, we obtain

$$\begin{aligned}\log_{n+1}(2m(r) - 2\log r) &= \log_{n+1}(\varphi_{n+1,C}(\log 1/r)) + o(1) \\ &= \log_{n+1}(\log 1/r) + \log C + o(1)\end{aligned}$$

and $\log_{n+1}(c(r)) = \log_{n+1}(\log 1/r) + o(1)$ as $r \to 0$. Hence, we have confirmed that $m(r)$ is a modulus bound for H_n.

In the same way, we can also show the following.

5.4. **Proposition.** *For a positive integer n and $\alpha \in (0,1)$, a modulus bound for $H_{n,\alpha}$ can be given by*

$$m(r) = \varepsilon\psi_{n,1-\alpha,\delta}(\log 1/r)$$

for r small enough, where ε and δ are arbitrary positive constants.

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