Approximation and stability of Perron solutions in nonlinear potential theory

by

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Abstract

Behavior of Perron solutions of quasilinear elliptic equations is studied under perturbations of the equations.

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1. Introduction

The classical Perron–Wiener–Brelot method can be extended to a wide class of quasilinear elliptic second order equations

$$\nabla \cdot A(x, \nabla u) = 0 \tag{1.1}$$

where $A(x,\xi)\cdot\xi \approx |\xi|^p,\ 1 < p < \infty$. See [HKM] for this theory. If Ω is a bounded domain in $\mathbb{R}^n$, then this theory provides for each function $f : \partial\Omega \to [-\infty,\infty]$ an upper (Perron) solution $\overline{H}_f(A)$ and a lower (Perron) solution $\underline{H}_f(A)$ with "boundary values" f. The function $\overline{H}_f(A)$ is obtained as

$$\overline{H}_f(A) = \inf\{u : u \in U_f(A)\}$$

where $U_f(A)$ is the upper class for f, i.e. every $u \in U_f(A)$ is A–superharmonic in Ω, u is bounded from below and

$$\liminf_{y\to x} u(y) \geq f(x)$$

for each $x \in \partial\Omega$. The function $\underline{H}_f(A)$ is defined similarly using the lower class $L_f(A)$, see [HKM, p. 168]. The A–resolutivity of f means that $\overline{H}_f(A) = \underline{H}_f(A)$ and that $\overline{H}_f(A)$ is A–harmonic, i.e. $\overline{H}_f(A)$ is a solution to (1.1) in Ω.

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For the classical Laplace equation $\Delta u = 0$, i.e. $A(x,h) = h$, it is well–known that all bounded Borel functions f are resolutive. The result can be extended to linear equations of the type (1.1). Here the basic device is the representation

$$H_f(A)(x) = \int_{\partial\Omega} f d\mu_x$$

where μ_x is the harmonic measure associated with A. This method is not available for nonlinear equations (1.1). Because of this, resolutivity is largely open for nonlinear equations. For example it is not known if all bounded Borel functions $f : \partial B \to \mathbb{R}$ are resolutive for the p–Laplace equation

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 \tag{1.2}$$

for $p \neq 2$ in the unit ball B of $\mathbb{R}^n$. However, for some "regular" functions resolutivity is known: Continuous, and even semicontinuous functions are A–resolutive provided that $\partial\Omega$ is sufficiently regular.

In this paper we address the following problems associated with Perron solutions: Suppose that $A_i \to A_0$ in the sense which will be made precise later. (a) Does $\overline{H}_f(A_i) \to \overline{H}_f(A_0)$? (b) Is it possible to approximate $\overline{H}_f(A_0)$ with functions $H_{f_i}(A_i)$ where the functions f_i are "regular"? Note that the convergence $A_i \to A_0$ allows that the exponents p associated with the operators A_i and A_0 vary. For the problem (a) we have some partial answers and the answer to (b) is, in general, yes.

In [LM2] we used a particular case of the problem (a) to study the stability of nonlinear harmonic measures.

For the problem (a) the domain Ω has to be somewhat regular. For example let $\Omega = B(0,1) \setminus C$ where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ and C is a compact set in $B(0,1)$ such that the p–capacity of C is positive for some $p \in (1,n]$ but the q–capacity of C is zero for all $q \in (1,p)$. Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function such that $f|_{\partial B(0,1)} = 0$ and $f|_C = 0$. If now $p_i \nearrow p$, then $H_f(A_{p_i}) \equiv 1$ because sets of p_i–capacity zero are negligible for Perron solutions but $H_f(A_p)(x) < 1$ for all $x \in \Omega$. Thus no convergence $H_f(A_{p_i}) \to H_f(A_p)$ is possible. Our regularity condition of Section 2 for $\partial\Omega$ is, however, mild: All lipschitz domains satisfy our condition. To avoid technical difficulties we also assume that our boundary functions $f : \partial\Omega \to \mathbb{R}$ are bounded.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$. We assume that Ω satisfies an exterior corkscrew condition. This means that there are constants $c \geq 1$ and $r_0 > 0$ such that for each $x \in \partial\Omega$ and $0 < r < r_0$ the ball $B(x,r)$ contains a ball $B(y, r/c) \subset \mathbb{R}^n \setminus \Omega$. In Ω we consider the equation (1.1) where the mapping $A_p : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the assumptions

(a) the mapping $x \mapsto A_p(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$.

For all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$:

(b) $A_p(x,\xi) \cdot \xi \geq \alpha|\xi|^p$,

(c) $|A_p(x,\xi)| \le \beta|\xi|^{p-1}$,
(d) $(A_p(x,\xi_1) - A_p(x,\xi_2)) \cdot (\xi_1 - \xi_2) > 0$ whernever $\xi_1 \ne \xi_2$
(e) $A_p(x,\lambda\xi) = \lambda|\lambda|^{p-2}A_p(x,\xi)$, $\lambda \in \mathbb{R}$, $\lambda \ne 0$.

For the theory of solutions of (1.1), A_p–superharmonic and A_p–subharmonic functions we refer to [HKM]. We recall that a function u is called A_p–harmonic if $u \in C(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ and if u is a weak solution of (1.1).

We shall consider a sequence of mappings A_{p_i}, $i = 1, 2, \ldots$, and a mapping A_{p_0} satisfying (a)–(e) with fixed constants $0 < \alpha \le \beta < \infty$ and for a.e. $x \in \Omega$

$$A_{p_i}(x,\xi) \to A_{p_0}(x,\xi)$$

uniformly on compact subsets of $\mathbb{R}^n$. Note that then $p_i \to p_0$. We write $A_{p_i} \to A_{p_0}$ if this holds.

We recall some basic facts about the Perron solutions. Fix a mapping A satisfying (a)–(e). If $f : \partial\Omega \to \mathbb{R}$ is bounded, then the upper solution $\overline{H}_f(A)$ and the lower solution $\underline{H}_f(A)$ are always A–harmonic in Ω. If f is A–resolutive, then we write $H_f(A) = \overline{H}_f(A)$. Note that if $f \in C(\partial\Omega)$, then f is resolutive and

$$\lim_{y \to x} H_f(A)(y) = f(x) \tag{2.1}$$

for every $x \in \partial\Omega$. This is due to the exterior corkscrew condition which gives the solvability of the Dirichlet problem, see [HKM, p. 123]. If $f \in C(\partial\Omega)$, then we consider the function $H_f(A)$ to be defined in $\overline{\Omega}$ with $H_f(A)|_{\partial\Omega} = f$.

The following lemma is a basic stability result. This lemma is not true in irregular domains, see the end of Section 1.

2.2. **Lemma.** *Suppose that $f \in C(\partial\Omega)$ and $A_{p_i} \to A_{p_0}$. Then $H_f(A_{p_i}) \to H_f(A_{p_0})$ uniformly in $\overline{\Omega}$.*

Proof. Write $A_i = A_{p_i}$, $i = 0, 1, \ldots$. The sequence of functions $H_f(A_i)$ is bounded by $\max|f|$. This follows from the maximum principle. It then follows from the Hölder estimate [HKM, Theorem 6.6] that the functions $H_f(A_i)$ form an equicontinuous family at each point $x \in \Omega$. This family is also equicontinuous at every $x \in \partial\Omega$. This is due to (2.1) and to the Mazya estimate [HKM, Theorem 6.18]. Note that the estimate in [HKM, Theorem 6.18] holds by approximation for each function $f \in C(\partial\Omega)$, i.e. the condition $f \in C(\overline{\Omega}) \cap W^{1,p_i}(\Omega)$ is not needed, and that the corkscrew condition takes care of the uniform lower bound for the p_i–capacity density of $\mathbb{R}^n \setminus \Omega$ at each point of $\partial\Omega$, see [HKM, pp. 123–124].

By the Ascoli theorem there is a subsequence $H_f(A_{i_j})$ which converges uniformly on $\overline{\Omega}$ to $u_0 \in C(\overline{\Omega})$ and $u_0|\partial\Omega = f$. The function u_0 is a solution of $\nabla \cdot A_0(x, \nabla u) = 0$ in Ω, see [LM1, Remark 4.7] and since $u_0|\partial\Omega = f$, $u_0 = H_f(A_0)$. Since every subsequence of $H_f(A_i)$ has the above property, the whole sequence $H_f(A_i)$ converges to $H_f(A_0)$ uniformly on $\overline{\Omega}$ as required.

The next lemma concerns a fixed equation but varying boundary values. We suppose that the operator A satisfies (a)–(e).

2.3. **Lemma.** *Suppose that $f_i \in C(\partial\Omega)$, $i = 1, 2, \ldots$, is an increasing sequence and that $f = \lim f_i$ is bounded. Then $H_{f_i}(A) \to H_f(A)$ uniformly on compact subsets of Ω.*

Proof. Note first that f is lower semicontinuous and hence resolutive because f is bounded, see [HKM, Proposition 9.31], By the Harnack convergence theorem $u = \lim H_{f_i}(A)$ is A–harmonic and the convergence is uniform on compact subsets of Ω, see [HKM, p. 113]. Next for each $x \in \partial\Omega$

$$\liminf_{y\to x} u(y) \geq \lim_{y\to x} H_{f_i}(A)(y) = f_i(x)$$

and each i because the sequence $H_{f_i}(A)$ is increasing. Letting $i \to \infty$ we obtain

$$\liminf_{y\to x} u(y) \geq f(x)$$

and hence $u \in U_f(A)$ which gives $u \geq H_f(A)$. On the other hand $f_i \leq f$ and hence $H_{f_i}(A_i) \leq H_f(A)$ for ech i and so $u \leq H_f(A)$. This yields $u = H_f(A)$ as required.

The following result follows from the proof of [HKM, Proposition 9.31, p. 187].

2.4. **Lemma.** *Suppose that $f : \partial\Omega \to \mathbb{R}$ is bounded and lower semicontinuous. Then $H_f(A)$ belongs to the upper class $U_f(A)$.*

The next lemma shows that the function $\overline{H}_f(A)$ can be approximated from above by "regular" Perron solutions.

2.5. **Lemma.** *Suppose that $f : \partial\Omega \to \mathbb{R}$ is bounded. Then there is a decreasing sequence of lower semicontinuous bounded functions $f_j : \Omega \to \mathbb{R}$ such that $f_j \geq f$ for each j and*

$$H_{f_j}(A) \to \overline{H}_f(A)$$

unformly on compact subsets of Ω. Moreover, each $H_{f_j}(A)$ belongs to the upper class $U_f(A)$.

Proof. The Choquet topological lemma, see [HKM, p. 158], together with the Perron method gives a decreasing sequence $u_j \in U_f(A)$ such that the lower semicontinuous regularization of $u = \lim u_j$ and $\overline{H}_f(A)$ coincide in Ω. Note that $\overline{H}_f(A)$ is A–harmonic since f is bounded. By the same reason we can also assume that each u_j is bounded. Let $G_1 \subset G_2 \subset \ldots$ be a sequence of regular subdomains of Ω such that $\overline{G}_i \subset G_{i+1}$ and $\Omega = \cup\, G_i$. Consider the Poisson modifications $v_j = P_A(u_j, G_j)$, see [HKM, p. 135]. Now the sequence v_j is decreasing and $v_j \in U_f(A)$. By the Harnack convergence theorem $v_0 = \lim v_j$ is A–harmonic in Ω and $v_j \to v_0$ uniformly on compact subsets of Ω, see [HKM, p. 133]. Since $v_j \leq u_j$, this implies that $v_0 = \overline{H}_f(A)$.

Next define the functions $f_j : \partial\Omega \to \mathbb{R}$ as

$$f_j(x) = \liminf_{y\to x} v_j(y) = \liminf_{y\to x} u_j(y), \quad x \in \partial\Omega.$$

Then $f_1 \geq f_2 \geq \ldots$ and $f_j \geq f$ because $u_j \in U_f(A)$.

Each function f_j is lower semicontinuous and by Lemma 2.4, $H_{f_j}(A) \in U_{f_j}(A)$. Since $v_j \geq H_{f_j} \geq \overline{H}_f(A)$ and since $v_j \to \overline{H}_f(A)$, $H_{f_j}(A) \to \overline{H}_f(A)$ uniformly on compact subsets of Ω. This completes the proof.

2.6. **Remark.** The boundedness condition in Lemma 2.5 for f can be replaced by the condition that $\overline{H}_f(A)$ is A–harmonic. Then the lower semicontinuous functions $f_j : \partial\Omega \to (-\infty, \infty]$ need not be bounded but other conclusions of Lemma 2.5 remain true.

The next lemma is a counterpart of Lemma 2.3 for varying equations. We write $A_i = A_{p_i}$, $i = 0, 1, \ldots$.

2.7. **Lemma.** *Let $A_i \to A_0$ and suppose that $f_j \in C(\partial\Omega)$ is an increasing sequence of functions such that $\lim f_j = f$ is bounded. Then there is a subsequence A_{i_j} of A_i such that*
(i) $H_{f_j}(A_{i_j}) \leq H_{f_{j+1}}(A_{i_{j+1}}) + 2^{-j-1}$ *in* $\overline{\Omega}$ for $j = 1, 2, \ldots$,
(ii) $H_{f_j}(A_{i_j}) \to H_f(A_0)$ *uniformly on compact subsets of* Ω.

Proof. Since $f = \lim f_j$, f is lower semicontinuous and hence f is A_0–resolutive because f is bounded, see [HKM, Proposition 9.31]. Since the sequence f_j is increasing, the sequence $H_{f_j}(A_0)$ is increasing in $\overline{\Omega}$ as well. On the other hand $H_{f_j}(A_0) \to H_f(A_0)$ uniformly on compact subsets of Ω, see Lemma 2.3. By Lemma 2.2 we can inductively choose $i_1 < i_2 < \ldots$ so that

$$|H_{f_j}(A_{i_j}) - H_{f_j}(A_0)| < 2^{-j-2}$$

in $\overline{\Omega}$. Now

$$\begin{aligned} &H_{f_j}(A_{i_j}) - H_{f_{j+1}}(A_{i_{j+1}}) = H_{f_j}(A_{i_j}) - H_{f_j}(A_0) + H_{f_j}(A_0) - H_{f_{j+1}}(A_{i_{j+1}}) \\ &\leq 2^{-j-2} + H_{f_{j+1}}(A_0) - H_{f_{j+1}}(A_{i_{j+1}}) \leq 2^{-j-2} + 2^{-j-3} < 2^{-j-1} \end{aligned}$$

in $\overline{\Omega}$ and (i) follows.

3. Approximation

Here we show that the Perron solutions $\overline{H}_f(A_0)$ and $\underline{H}_f(A_0)$ can be approximated uniformly on compact subsets of Ω by functions $H_{g_i}(A_{p_i})$ where the functions g_j are smooth, at least in the case of bounded f. We assume that the bounded domain Ω satisfies the corkscrew condition of Section 2. We write $A_i = A_{p_i}$, $i = 0, 1, \ldots$.

3.1. **Theorem.** *Suppose that $A_i \to A_0$ and let $f : \partial\Omega \to \mathbb{R}$ be bounded. Then there is a sequence of continuous functions $g_j : \partial\Omega \to \mathbb{R}$ and a subsequence A_{i_j} of A_i such that $H_{g_j}(A_{i_j}) \to \overline{H}_f(A_0)$ uniformly on compact subsets of Ω.*

3.2. **Remark.** The proof shows that the functions g_j can be chosen to be C^∞–functions in $\mathbb{R}^n$ or, by the Weierstrass theorem, polynomials.

Proof for Theorem 3.1. By Lemma 2.5 there is a sequence of lower semicontinuous and bounded functions $f_j : \partial\Omega \to \mathbb{R}$ such that $H_{f_j}(A_0) \to \overline{H}_f(A_0)$ uniformly on compact subsets of Ω. Let $F_1 \subset F_2 \subset \ldots$ be compact sets such that $\cup\, F_j = \Omega$. Since each f_j can be approximated from below by continuous functions, Lemma 2.3 gives a continuous function $g_j : \partial\Omega \to \mathbb{R}$ with

$$|H_{g_j}(A_0) - H_{f_j}(A_0)| < 2^{-j} \tag{3.3}$$

on F_j. By Lemma 2.2 we can inductively choose a sequence $i_1 < i_2 < \ldots$ such that

$$|H_{g_j}(A_{i_j}) - H_{g_j}(A_0)| < 2^{-j} \tag{3.4}$$

in Ω for $j = 1, 2, \ldots$. From (3.3), (3.4) and from the fact that $H_{f_j}(A_0) \to \overline{H}_f(A_0)$ uniformly on compact subsets of Ω it follows that $H_{g_j}(A_{i_j}) \to \overline{H}_f(A_0)$ uniformly on each compact set of Ω.

Taking $A_i = A_0,\ i = 1, 2, \ldots$, in Theorem 3.1 we obtain

3.5. **Corollary.** *Let $f : \partial\Omega \to \mathbb{R}$ be bounded. Then there is a sequence of functions $f_j \in C(\partial\Omega)$ such that $H_{f_j}(A_0) \to \overline{H}_f(A_0)$ uniformly on compact subsets of Ω.*

3.6. **Remarks.** (a) A similar result holds for $\underline{H}_f(A_0)$.

(b) In Corollary 3.5 f need not be bounded. It is enough to assume that $\overline{H}_f(A_0)$ is A_0–harmonic. This is not enough for Theorem 3.1 because it is not known if $U_f(A_0) \neq \emptyset$ yields $U_f(A_i) \neq \emptyset$ for large i.

4. Stability of Perron solutions

Let Ω, A_i and A_0 be as in the previous section. Even for a bounded and lower semicontinuous function the authors do not know if $A_i \to A_0$ implies $H_f(A_i) \to H_f(A_0)$. However, something can be said in this case and this result has been used in [LM2] to study the stability of non–linear harmonic measures. In this application f was a characteristic function of an open set of $\partial\Omega$.

4.1. **Theorem.** *Let $A_i \to A_0$. Suppose that $f : \partial\Omega \to \mathbb{R}$ is bounded and lower semicontinuous. Then the sequence $H_f(A_i)$ has a subsequence $H_f(A_{i_j})$ such that*
(a) $H_f(A_{i_j}) \to u_0$ *uniformly on compact subsets of* Ω,
(b) u_0 *is A_0–harmonic.*
(c) $u_0 \in U_f(A_0)$, *in particular* $u_0 \geq H_f(A_0)$ *in* Ω.

Proof. The family $\{H_f(A_i)\}$ is equicontinuous and bounded in Ω. Hence there is a subsesquence $H_f(A_{i_j})$ such that $H_f(A_{i_j}) \to u_0 \in C(\Omega)$ uniformly on compact subsets of Ω. It follows from [LM1, Remark 2.9] that u_0 is A_0–harmonic.

Choosing an increasing sequence of functions $f_j \in C(\partial\Omega)$ such that $f_j \to f$ and passing to a subsequence, if necessary, we may assume that (i) and (ii) of Lemma 2.7 hold. From (i) it follows that for each j and $k \geq j$

$$H_{f_j}(A_{i_j}) \leq H_{f_k}(A_{i_k}) + 2^{-j}$$

in $\overline{\Omega}$. Since $f_k \leq f$, this gives

$$H_{f_j}(A_{i_j}) \leq H_f(A_{i_k}) + 2^{-j}$$

in Ω for each $j = 1, 2, \ldots$ and $k \geq j$. Letting $k \to \infty$ we have

$$H_{f_j}(A_{i_j}) \leq u_0 + 2^{-j}$$

in Ω for each j. This yields

$$f_j(x) = \lim_{y \to x} H_{f_j}(A_{i_j})(y) \leq \liminf_{y \to x} u_0(y) + 2^{-j}$$

for each $x \in \partial\Omega$ and for $j \to \infty$ we obtain $u_0 \in U_f(A_0)$ as required.

4.2. **Remark.** It would be interesting to know if Theorem 4.1 holds without the lower semicontinuity assumption on f and (c) replaced by (c') $u_0 \geq H_f(A_0)$ in Ω.

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