On Conformal Differentiation in Space

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October 9, 2001

Abstract

We study the local conformality problem associated with quasireqular mappings in space. The solution of the conformal differentiation conjecture for quasiconformal rotations in space is given.

1991 *Mathematics Subject Classification.* AMS (MOS) 30C55,60.

1. Introduction

In this paper we continue to study the local conformality problem associated with quasireqular mappings in space started in [1]. The infinitesimal space in [4] and the rotation estimates in [3] provide the most important tools.

Let G be an open set in $\mathbb{R}^n$, $n \geq 2$. A nonconstant K - quasiregular mapping $f : G \to \mathbb{R}^n$ is said to be *conformally differentiable* at a point $x_0 \in G$ if there exist a constant $\lambda > 0$ and a linear isometric mapping $\mathcal{O}$ such that

$$|f(x) - f(x_0) - \lambda\mathcal{O}(x - x_0)|/|x - x_0| \to 0 \text{ as } x \to x_0. \tag{1.1}$$

For $n = 2$ it is known, see e.g. [6], p. 232, that the condition

$$\int\limits_{|x|<R} \frac{D_f(x) - 1}{|x|^n} dx < \infty \tag{1.2}$$

implies that a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, is conformally differentiable at $x_0 = 0$. Here $D_f(x)$ stands for one of the local distortion coefficients defined a.e. as

$$K_f(x) = \frac{||f'(x)||^n}{J_f(x)}, \;\; L_f(x) = \frac{J_f(x)}{\ell(f'(x))^n}, \;\; H_f(x) = \frac{\|f'(x)\|}{\ell(f'(x))}, \tag{1.3}$$

that are called the outer, inner and linear dilatation of f, respectively, and $J_f(x)$ is the Jacobian determinant of $f'(x)$ and $||f'(x)|| = \sup |f'(x)h|$, $\ell(f'(x)) = \inf |f'(x)h|$, where the supremum and the infimum are taken over all unit vectors $h \in \mathbb{R}^n$. Later on, for convenience, we will prove the main statements only for the inner dilatation coefficient $L_f(x)$ because for the other dilatation coefficients the result will follow from the relations

$$L_f(x) \leq K_f^{n-1}(x), \;\; K_f(x) \leq L_f^{n-1}(x), \;\; H_f^n(x) = K_f(x)L_f(x), \tag{1.4}$$

that hold for every $n \geq 2$, see, e.g., [8], p. 44.

It has been a long standing conjecture to prove that for a quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, (1.2) implies the conformal differentiability of f at $x_0 = 0$. The motivation for the conjecture comes from the well known Liouville theorem in space. It states that for $n \geq 3$ there exists $\varepsilon > 0$ such that if $L_f(x) < 1 + \varepsilon$ for all $x \in G$, then f can be approximated by a Möbius transformation in the sense of [7], p. 180. However, the assumption $L_f(x) \to 1$ as $x \to 0$, $0 \in G$, does not imply that f is conformally differentiable at 0. The mapping, defined as

$$f(x) = x(1 - \log|x|), \;\; f(0) = 0, \tag{1.5}$$

at a neighborhood of the origin, satisfies $L_f(x) = (1 - 1/\log|x|)^{n-1} \to 1$ as $x \to 0$ but f is not differentiable at 0. Thus a stronger convergence of $L_f(x)$ to 1 is needed for the conformality at a point. To this direction Yu. Reshetnyak [7] proved that the Dini condition

$$\int\limits_0 \frac{\omega(t)}{t} dt < \infty \tag{1.6}$$

for the essential modulus of continuity $\omega(t) = \operatorname*{ess\,sup}_{|x|<t}(K_f(x) - 1)$ implies the conformal differentiability of f at the origin. However, the assumption (1.6) is rather restrictive. For instance, simple differentiable radial mappings fail to satisfy (1.6). To see this, consider the mappings $f : B \to B$ defined on the unit ball B in $\mathbb{R}^n$ as follows: fix an arbitrary locally integrable function g on $[0, 1]$ with $g(t) \geq 1$ for a.e. t and let

$$f(x) = x \exp\{-\alpha(|x|)\}, \;\; \alpha(|x|) = \int\limits_{|x|}^{1} \frac{g(t) - 1}{t} dt, \;\;\; f(0) = 0. \tag{1.7}$$

It follows from (1.7) that $g(|x|) = J_f(x)/\ell(f'(x))^n$ a.e. and therefore $g(|x|)$ agrees with the inner dilatation coefficient of f at x. On the other hand, the mappings (1.7) are conformally differentiable at 0 if and only if there exists a finite limit $\alpha(t) \to \alpha_0$ as $t \to 0$. If we now set $g(t) = 2$ for $t \in [1/n, 1/n + 1/n^3]$, $n = 2, 3, ...,$ and $g(t) = 1$ for the rest $t \in [0,1]$, then we see that $\omega(t) = 1$. Hence the condition (1.6) does not hold. At the same time the mapping f is conformally differentiable at the origin since

$$\int_0^1 \frac{g(t)-1}{t} dt \le \sum_{n=2}^{\infty} 1/n^2 < \infty. \tag{1.8}$$

For an arbitrary quasiregular mapping $f : B \to B$, $f(0) = 0$, the convergence assumption (1.8) is of the form (1.2) and we arrive at the aforementioned conjecture.

This paper is devoted to prove the conjecture in a subclass, called quasiconformal rotations, of quasiregular mappings in $\mathbb{R}^3$. We emphasise that, although some auxiliary results hold in $\mathbb{R}^n$, $n \ge 3$, the final conclusions apply to $\mathbb{R}^3$ only. For example, we make use of the special character of orthogonal mappings in $\mathbb{R}^3$. The integral (1.2) takes care of the radial behavior of the mapping f in all dimensions $n \ge 3$ but the rotation control seems to be more difficult.

2. Main Results

Before the main statements we recall some basic notions from the infinitesimal geometry of quasiregular mappings in space [4]. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a nonconstant K – quasiregular mapping. For $x_0 \in \mathbb{R}^n$ we set

$$F_t(x; f) = \frac{f(tx + x_0) - f(x_0)}{\tau(x_0, f, t)}, \tag{2.1}$$

where

$$\tau(x_0, f, t) = \left(\frac{\operatorname{meas} f(B(x_0, t))}{\Omega_n} \right)^{\frac{1}{n}}. \tag{2.2}$$

Here Ω_n denotes the volume of the unit ball B in $\mathbb{R}^n$. Let $T(x_0, f)$ be the set of all the limit functions for the family of the mappings F_t as $t \to 0$, where the limit is taken in terms of the locally uniform convergence. This set, which we call the *infinitesimal space* for the mapping f at the point x_0, is not empty and consists only of K – quasiregular mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ normalized by $F(0) = 0$, $F(\infty) = \infty$, $\operatorname{meas} F(B) = \Omega_n$, see [4], Th. 2.7. The infinitesimal space $T(x_0, f)$ is called *simple* if it consists of one mapping F only.

2.3. **Theorem.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \ge 3$, be a nonconstant K – quasiregular mapping such that*

$$I(r) = \frac{1}{\omega_{n-1}} \int\limits_{|x|<r} \frac{L_f(x) - 1}{|x|^n} dx < \infty \tag{2.4}$$

for some $r > 0$. Then the following assertions hold:

1. *The radius of injectivity $R_f(0)$ of f at 0 satisfies $R_f(0) > 0$ and there exists a constant λ,*

$$\min_{|x|=R} |f(x)| \frac{e^{-I(R)}}{R} \leq \lambda \leq \max_{|x|=R} |f(x)| \frac{e^{I(R)}}{R}, \quad 0 < R \leq R_f(0), \tag{2.5}$$

such that

$$\frac{|f(x)|}{|x|} \to \lambda \quad \text{as } x \to 0; \tag{2.6}$$

2. *The infinitesimal space $T(0, f)$ consists of linear isometric mappings only and their inverses form the space $T(0, f^{-1})$. Moreover, each pair $F \in T(0, f)$ and $F^{-1} \in T(0, f^{-1})$ is generated by the same sequence $t_n \to 0$ as $n \to \infty$;*

3. *f is conformally differentiable at 0 iff either $T(0, f)$ or $T(0, f^{-1})$ is simple.*

2.7. **Remark.** The first statement has been proved in [1], Theorem 3.1.

Theorem 2.3 shows that one has to eliminate the rotation of f at 0 in order to solve the conformal differentiation problem. We expect that it could be done by means of the estimate technique developed in [3] where integrals similar to (2.4) control the rotation of radial segments under quasiconformal mapping in space. The following example can be viewed as an evidence to support this expectation.

Let us consider K - quasiconformal mappings in 3 – space satisfying

$$\frac{\langle f(x), \mu \rangle}{|f(x)|} = \frac{\langle x, \mu \rangle}{|x|} \tag{2.8}$$

for some unit vector $\mu \in \mathbb{R}^3$ and all $x \in \mathbb{R}^3 \setminus \{0\}$. We call such mappings *quasiconformal rotations* about fixed axis μ, see [3].

2.9. **Theorem.** *Let f be a K – quasiconformal rotation of the unit ball B in $\mathbb{R}^3$ onto itself such that $f(0) = 0$ and*

$$I(r) = \frac{1}{4\pi} \int\limits_{|x|<r} \frac{L_f(x) - 1}{|x|^3} dx < \infty \tag{2.10}$$

for some $0 < r < 1$. Then there exists a constant λ,

$$e^{-I(1)} \leq \lambda \leq e^{I(1)}, \tag{2.11}$$

and a linear isometric mapping $\mathcal{O}$, that generates the rotation of $\mathbb{R}^3$ about μ - axis by the angle θ, such that

$$f'(0) = \lambda\mathcal{O}, \tag{2.12}$$

i.e. f is conformally differentiable at 0.

2.13. **Theorem.** *Let f be a K – quasiconformal rotation of the unit ball B in $\mathbb{R}^3$ onto itself such that $f(0) = 0$, $f(x) = x$ for $|x| = 1$, and (2.10) holds. Then the rotation angle θ admits the estimate*

$$|\theta| \leq CI(1) \tag{2.14}$$

where C is a constant. The inequality (2.14) is true with $C = 32\sqrt{2}/9\pi$.

3. Auxiliary Lemmas

The following statement describes qualitative structure of the infinitesimal space $T(x_0, f)$ in terms of some integral means of the dilatation coefficients.

3.1. **Lemma.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 3$, be a nonconstant K – quasiregular mapping such that*

$$\frac{1}{r^n} \int\limits_{|x|<r} (D_f(x) - 1)dx \to 0 \;\; as \;\; r \to 0. \tag{3.2}$$

Then $T(0, f)$ consists of the linear isometric mappings only. Here $D_f(x)$ stands for one of the dilatation coefficients defined by (1.3).

Proof. For convenience we will prove the statement only for the outer dilatation $K_f(x)$. Let $F(x) \in T(0, f)$. Then there exists a sequence $t_j \to 0$ as $j \to \infty$ such that

$$F_{t_j}(x; f) = \frac{f(t_j x)}{\tau(0, f, t_j)} \to F(x) \tag{3.3}$$

locally uniformly in $x \in \mathbb{R}^n$. By formula (2.1) we obtain that

$$K_{F_{t_j}}(x) = K_f(t_j x) \;\; \text{a.e.} \tag{3.4}$$

and hence the assumption (3.2) with $D_f(x) = K_f(x)$ and $r = Rt_j$ can be written as

$$(3.5) \qquad \int\limits_{|x|<R} (K_{F_{t_j}}(x) - 1)dx \to 0 \text{ as } j \to \infty$$

for every positive constant R. Next, the condition (3.5) implies that $K_{F_{t_j}}(x) \to 1$ as $t \to 0$ in measure in $\mathbb{R}^n$. Without loss of generality we may assume that $K_{F_{t_j}}(x) \to 1$ almost everywhere. This can be achieved by passing to a subsequence. Since $F_{t_j}(x; f) \to F(x)$ locally uniformly as $j \to \infty$ then, by Theorem 3.1 from [5], the limit $F(x)$ is a nonconstant 1 – quasiregular mapping. Applying Liouville's theorem we see that $F(x)$ is a Möbius mapping. Because of the above normalization, $F(x)$ is a linear isometry.

Let γ be a rectifiable curve in $\mathbb{R}^n \setminus \{0\}$ and let $\eta(t) : [a, b] \to \mathbb{R}^n$, be its absolutely continuous parametric representation. We call the quantity

$$(3.6) \qquad \delta_\gamma = \int\limits_a^b \left| \frac{|\eta'(t)|^2 - |\eta(t)|'^2}{|\eta(t)|^2} \right|^{\frac{1}{2}} dt,$$

the *total rotation* of the rectifiable curve γ. Note that $\delta_\gamma = 0$ iff γ is a radial segment. The total rotation can be considered as a space version of the total variation for a continuous branch of $\arg \eta(t)$.

We will use also the quasihyperbolic length $\ell(\gamma)$ of $\gamma \in \mathbb{R}^n \setminus \{0\}$ that is defined as (see, e.g., [9], p. 33)

$$(3.7) \qquad \ell(\gamma) = \int\limits_\gamma \frac{|dx|}{|x|}.$$

Observe that for the radial segment $\gamma_{ab}^\nu : \eta(t) = t\nu$, $0 < a \le t \le b$, $|\nu| = 1$, we have $\ell(\gamma_{ab}^\nu) = \log(b/a)$.

For the remainder of the section we will assume that f is a K – quasiconformal rotation of $\mathbb{R}^3$.

Denote by e_1, e_2, e_3 the standard unit vectors of $\mathbb{R}^3$ and provide $\mathbb{R}^3$ with the spherical coordinates r, ω, α in such a way that α stands for an angle between the radius vector and e_3. We will identify the $e_1 e_2$ – plane with the complex plane $\mathbb{C}$, so that $z = x_1 + ix_2 = re^{i\omega} \sin \alpha$. Note that $\sin \alpha > 0$ except when the point lies on the e_3 - axis. In the sequel for convenience we have chosen the rotation axis to be the third coordinate axis.

Let $f(x) = (f_1(x), f_2(x), f_3(x))$ be a quasiconformal rotation which fixes the points 0, e_1 and ∞, and

$$(3.8) \qquad \Phi_f(x) = f_1(x) + if_2(x) = R_f(r, \omega, \alpha)e^{i\Omega_f(r,\omega,\alpha)}$$

be its orthogonal projection onto the e_1e_2 – plane. We specify the continuous branch of the argument of Φ_f by the assumption $\Omega(1,0,\pi/2) = 0$. Since f is a normalized sense–preserving homeomorphism of the space fixing the e_3 -axis, we obtain that

$$\Omega(r,\omega+2\pi k,\alpha) = \Omega(r,\omega,\alpha) + 2\pi k$$

for all $k = \pm 1, \pm 2, \ldots$. The latter allows us to introduce the following continuous single–valued function

$$\mathcal{A}_f(x) \equiv \arg \frac{\Phi_f(x)}{z} = \Omega(r,\omega,\alpha) - \omega, \ \ \mathcal{A}_f(e_1) = 0 \tag{3.9}$$

defined in $\mathbb{R}^3 \setminus \{e_3\}$, which we will call the *angular displacement* of the point x under the mapping f.

If $0 < a < b < \infty$, we will denote by $R(a,b)$ the spherical annulus $x : a < |x| < b$.

3.10. **Lemma.** *Let f be a K – quasiconformal rotation of a space ring $\mathcal{R}$ onto the spherical annulus $R(a,b)$ with the dilatation coefficient $L_f(x)$ and let $\mathcal{R}$ contain $R(\tau a(1+\varepsilon), \tau b/(1+\varepsilon))$ and be contained in $R(\tau a/(1+\varepsilon), \tau b(1+\varepsilon))$ for some $0 < \tau < \infty$. If h is a volume preserving automorphism of $R(a,b)$ of the form*

$$h(x) = \left((x_1 + ix_2)e^{\pm i \log \frac{b}{|x|}}\ ,\ x_3\right), \quad x = (x_1, x_2, x_3), \tag{3.11}$$

then

$$\int\limits_{S^2} \left(\delta_{\varphi\circ h(\gamma^{\nu}_{ab})} - \delta_{h(\gamma^{\nu}_{ab})}\right) w(\nu) dm(\nu) \le c_1 \int\limits_{\mathcal{R}} \frac{L_f(x) - 1}{|x|^3} dx + c_2\varepsilon \tag{3.12}$$

where

$$w(\nu) = \frac{3\sin\alpha \left[(\log^2(b/a))\sin^2\alpha + (\log(b/a) - 2\varepsilon)^2\right]^{1/2}}{\log(b/a)}. \tag{3.13}$$

Here $\nu = (e^{i\omega}\sin\alpha,\ \cos\alpha)$ is a unit vector in $\mathbb{R}^3$, c_1, c_2 are constants and φ stands for the inverse of f.

3.14. **Remark.** The inequality (3.12) can be viewed as a variational inequality that controls the rotation of the radial segments under K - quasiconformal rotations of $\mathbb{R}^3$ where volume preserving automorphisms (3.11) of the spherical annulus $R(a,b)$ form the test class.

3.15. **Remark.** The inequality (3.12) holds with $c_1 = 2\sqrt{2}$ as we shall prove below.

Proof of Lemma 3.10. The proof consists of three steps. Assuming that f is a K – quasiconformal rotation about the e_3 - axis we first derive the following estimate

$$\int_{S^2} \left(\ell^3(\varphi \circ h(\gamma^\nu_{ab})) - \ell^3(h(\gamma^\nu_{ab}))\right) dm(\nu) \leq c_1 \log^2(b/a) \int_{\mathcal{R}} \frac{L_f(x)-1}{|x|^3} dx + C_2 \varepsilon \log^2(b/a). \tag{3.16}$$

Then we show that the left - hand integrand in (3.16) can be estimated in terms of the total rotations of the curves $\varphi \circ h(\gamma^\nu_{ab})$ and $h(\gamma^\nu_{ab})$ as

$$\ell^3(\varphi \circ h(\gamma^\nu_{ab})) - \ell^3(h(\gamma^\nu_{ab})) \geq \left[\delta^2_{\varphi\circ h(\gamma^\nu_{ab})} + (\log(b/a) - 2\varepsilon)^2\right]^{3/2} - (1+\sin^2\alpha)^{3/2}\log^3(b/a). \tag{3.17}$$

Combining inequalities (3.16) and (3.17) we see that

$$\int_{S^2} \left(\frac{\left[\delta^2_{\varphi\circ h(\gamma^\nu_{ab})} + (\log(b/a) - 2\varepsilon)^2\right]^{3/2} - (1+\sin^2\alpha)^{3/2}\log^3(b/a)}{\log^2(b/a)} \right) dm(\nu) \leq \leq c_1 \int_{\mathcal{R}} \frac{L_f(x)-1}{|x|^3} dx + C_2\varepsilon. \tag{3.18}$$

Finally, making use of the elementary inequality

$$\frac{\left[\delta^2_{\varphi\circ h(\gamma^\nu_{ab})} + (\log(b/a) - 2\varepsilon)^2\right]^{3/2}}{\log^2(b/a)} - (1+\sin^2\alpha)^{3/2}\log(b/a) \geq \geq 3w(\nu)\left(\delta_{\varphi\circ h(\gamma^\nu_{ab})} - (\sin\alpha)\log(b/a)\right) - 6(1+\sin^2\alpha)^{1/2}\varepsilon \tag{3.19}$$

we can replace (3.18) by

$$\int_{S^2} \left(\delta_{\varphi\circ h(\gamma^\nu_{ab})} - (\sin\alpha)\log(b/a)\right) w(\nu) dm(\nu) \leq c_1 \int_{\mathcal{R}} \frac{L_f(x)-1}{|x|^3} dx + C_2\varepsilon + 6\varepsilon \int_{S^2} (1+\sin^2\alpha)^{1/2} dm(\nu). \tag{3.20}$$

Since, by (3.6),

$$\delta_{h(\gamma^\nu_{ab})} = \int_a^b \sin\alpha \frac{dt}{t} = (\sin\alpha)\log(b/a), \tag{3.21}$$

we arrive at (3.12) with

$$c_2 = C_2 + 6\int (1+\sin^2\alpha)^{1/2} dm(\nu).$$

We first prove the inequality (3.16). For this, let us fix a radial segment $\{t\nu : a \leq t \leq b\}$, $\nu \in S^2$, in $R(a,b)$ and estimate the quasihyperbolic length of its image under the mapping $\varphi \circ h(x)$, where φ denotes the inverse of f. We assume that $t \mapsto \varphi \circ h(t\nu)$ provides an absolutely continuous parameterization. Thus

$$\ell(\varphi \circ h(\gamma_{ab}^{\nu})) = \int_a^b \frac{\left|\frac{d}{dt}(\varphi \circ h(t\nu))\right|}{|\varphi(h(t\nu))|} dt \leq \int_a^b \frac{||\varphi'(h(t\nu))|| (1+\sin^2\alpha)^{1/2}}{|\varphi(h(t\nu))|} dt \tag{3.22}$$

since

$$\left|\frac{d}{dt} h(t\nu)\right| = (1+\sin^2\alpha)^{1/2} = v(\nu). \tag{3.23}$$

Using Hölder's inequality we get that

$$\int_a^b \frac{||\varphi'(h(t\nu))||}{|\varphi(h(t\nu))|} v(\nu) dt \leq \tag{3.24}$$

$$\leq \left(\int_a^b \frac{||\varphi'(h(t\nu))||^3}{|\varphi(h(t\nu))|^3} v^3(\nu) t^2 dt \right)^{\frac{1}{3}} \log^{2/3}(b/a).$$

Therefore

$$\ell^3(\varphi \circ h(\gamma_{ab}^{\nu})) \leq \log^2(b/a) \int_a^b \frac{||\varphi'(h(t\nu))||^3}{|\varphi(h(t\nu))|^3} v^3(\nu) t^2 dt. \tag{3.25}$$

Now (3.25) holds for a.e. $\nu \in S^2$ and if we integrate both sides of (3.25) with respect to the parameter ν over S^2, then Fubini's theorem yields

$$\int_{S^2} \ell^3(\varphi \circ h(\gamma_{ab}^{\nu})) dm(\nu) \leq \log^2(b/a) \int_{R(a,b)} \frac{||\varphi'(h(y))||^3}{|\varphi(h(y))|^3} v^3(\nu) dy. \tag{3.26}$$

Let us remark that $v(\nu)$ depends on α only and hence it is invariant under the quasiconformal rotations of 3 – space about e_3 - axis. Since h possesses this property and, moreover, it is a volume preserving automorphism of $R(a,b)$ and therefore, its Jacobian is one a.e., we get

$$\int_{S^2} \ell^3(\varphi \circ h(\gamma_{ab}^{\nu})) dm(\nu) \leq \log^2(b/a) \int_{R(a,b)} \frac{||\varphi'(x)||^3}{|\varphi(x)|^3} v^3(\nu) dx. \tag{3.27}$$

Performing the change of variable $x \mapsto f(x)$ and again making use of the fact that $v(\nu)$ is invariant under the quasiconformal rotations about the e_3 axis, we obtain the inequality

$$\int_{S^2} \ell^3(\varphi \circ h(\gamma_{ab}^\nu))dm(\nu) \le \log^2(b/a) \int_{\mathcal{R}} \frac{L_f(x)-1}{|x|^3} v^3(\nu)dx \tag{3.28}$$
$$+ \log^2(b/a) \int_{\mathcal{R}} \frac{v^3(\nu)}{|x|^3}dx.$$

Taking into account that $\mathcal{R}$ is contained in the annulus $R(\tau a/(1+\varepsilon), \tau b(1+\varepsilon))$ and applying the formula

$$\ell(h(\gamma_{ab}^\nu)) = \int_a^b (1+\sin\alpha)^{1/2}\frac{dt}{t} = (1+\sin\alpha)^{1/2}\log(b/a) \tag{3.29}$$

we see that

$$\log^2(b/a)\int_{\mathcal{R}} \frac{v^3(\nu)}{|x|^3}dx \le \left(\log^2(b/a)\right)(\log(b/a)+2\varepsilon)\int_{S^2} v^3(\nu)dm(\nu) = \tag{3.30}$$
$$\int_{S^2} \ell^3(h(\gamma_{ab}^\nu))dm(\nu) + C_2\varepsilon\log^2(b/a).$$

Combining (3.28) and (3.30) and noting that, by (3.23), $v(\nu) \le \sqrt{2}$, we arrive at the inequality (3.16) with $c_1 = 2\sqrt{2}$.

In order to prove (3.17) we make use of the following estimate

$$\ell^2(\gamma) \ge \left(\delta_\gamma^2 + \log^2\frac{|y|}{|x|}\right), \tag{3.31}$$

that holds for every rectifiable curve γ joining x and y in $\mathbb{R}^3 \setminus \{0\}$. Indeed, let $\eta(t)$, $a \le t \le b$, be its absolutely continuous parameterization. Then

$$\ell^2(\gamma) = \left(\int_a^b \frac{|\eta'(t)|}{|\eta(t)|}dt\right)^2 = \left(\int_a^b \left(\frac{|\eta'(t)|^2 - |\eta(t)|'^2}{|\eta(t)|^2} + \frac{|\eta(t)|'^2}{|\eta(t)|^2}\right)^{\frac{1}{2}}dt\right)^2 \tag{3.32}$$
$$\ge \left(\int_a^b \left|\frac{|\eta'(t)|^2 - |\eta(t)|'^2}{|\eta(t)|^2}\right|^{\frac{1}{2}}dt\right)^2 + \left(\int_a^b \frac{|\eta(t)|'}{|\eta(t)|}dt\right)^2.$$

The inequality (3.32) is (3.31) because the first integral in (3.32) is δ_γ and

$$\int_a^b \frac{|\eta(t)|'}{|\eta(t)|}dt = \log\frac{|y|}{|x|}. \tag{3.33}$$

So, by (3.31),

$$\ell(\varphi\circ h(\gamma_{ab}^\nu)) \ge \left(\delta^2_{\varphi\circ h(\gamma_{ab}^\nu)} + \log^2\frac{|\varphi\circ h(b\nu)|}{|\varphi\circ h(a\nu)|}\right)^{1/2}. \tag{3.34}$$

Since $\mathcal{R}$ contains the annulus $R(\tau a(1+\varepsilon), \tau b/(1+\varepsilon))$, we see that

$$\log\frac{|\varphi\circ h(b\nu)|}{|\varphi\circ h(a\nu)|}\geq\log(b/a)-2\log(1+\varepsilon)\geq\log(b/a)-2\varepsilon. \tag{3.35}$$

Hence

$$\ell(\varphi\circ h(\gamma_{ab}^{\nu}))\geq\left(\delta^2_{\varphi\circ h(\gamma^{\nu}_{ab})}+(\log(b/a)-2\varepsilon)^2\right)^{1/2} \tag{3.36}$$

and we come to the inequality (3.17).

In order to complete the proof of Lemma 3.10 it remains to verify the inequality (3.19). Let us denote $\log(b/a)$, $\sin\alpha$, and $\delta_{\varphi\circ h(\gamma^{\nu}_{ab})}$ by M, β and $x+\beta M$, respectively, and introduce the function

$$g(x)=\frac{\left[(x+\beta M)^2+(M-2\varepsilon)^2\right]^{3/2}}{M^2}-M(1+\beta^2)^{3/2}. \tag{3.37}$$

In these notations we have to show that

$$g(x)\geq 3w(\nu)x-6(1+\beta^2)^{1/2}\varepsilon. \tag{3.38}$$

Since

$$g'(x)=\frac{3(x+\beta M)\left[(x+\beta M)^2+(M-2\varepsilon)^2\right]^{1/2}}{M^2}, \tag{3.39}$$

we obtain that $g'(x)$ and $g''(x)$ are positive. Hence

$$g(x)\geq g'(0)x+g(0). \tag{3.40}$$

Formula (3.39) implies that

$$g'(0)=\frac{3\beta\left[\beta^2M^2+(M-2\varepsilon)^2\right]^{1/2}}{M} \tag{3.41}$$

and therefore $3\beta^2\leq g'(0)\leq 3\beta(1+\beta^2)^{1/2}$. In order to estimate

$$g(0)=\frac{\left[\beta^2M^2+(M-2\varepsilon)^2\right]^{3/2}}{M^2}-M(1+\beta^2)^{3/2}=p(\varepsilon) \tag{3.42}$$

we will study it as a function of the parameter ε. Since $p(0)=0$, $p''(\varepsilon)\geq 0$ we obtain that $p(\varepsilon)\geq p'(0)\varepsilon=-6(1+\beta^2)^{1/2}\varepsilon$. The latter together with (3.40) and (3.41) yield $g(x)\geq w(\nu)x-6(1+\beta^2)^{1/2}\varepsilon$. Thus, the verification of the inequality (3.19) is completed.

4. Proofs for Theorems

By our assumption $I(r) \to 0$ as $r \to 0$. Hence

$$(4.1) \qquad \frac{1}{\omega_{n-1} r^n} \int\limits_{|x|<r} (L_f(x) - 1)dx \le \frac{1}{\omega_{n-1}} \int\limits_{|x|<r} \frac{L_f(x) - 1}{|x|^n} dx = I(r) \to 0$$

is immediate.

Proof of Theorem 2.3. 1. Taking into account Remark 2.7, it remains to prove only the last two assertions of the theorem.

2. The assumption (2.4) implies (4.1) and by Lemma 3.1 we obtain that the infinitesimal space $T(0, f)$ consists of linear isometric mappings only. Now we focus on the proving that the inverse linear isometric mappings form the space $T(0, f^{-1})$ and each pair $F \in T(0, f)$ and $F^{-1} \in T(0, f^{-1})$ is generated by the same sequence $t_n \to 0$ as $n \to \infty$. Since

$$(4.2) \qquad \lim_{x \to 0} |f(x)|/|x| = \lambda$$

for some λ, $0 < \lambda < \infty$, we can write the approximating families for $T(0, f)$ and $T(0, f^{-1})$ as

$$(4.3) \qquad F_t(x; f) = f(tx)/\lambda t, \quad F_t(x; f^{-1}) = f^{-1}(tx)/\tau t, \quad \tau = 1/\lambda,$$

respectively. For simplicity we will later assume that $\lambda = 1$. Otherwise we can perform a renormalization of the mapping f which does not effect on the final result. Suppose that $F \in T(0, f)$. The latter means that there is a sequence $t_n \to 0$ as $n \to \infty$ such that

$$(4.4) \qquad \lim_{n \to \infty} F_{t_n}(x; f) = F(x)$$

locally uniformly in $x \in \mathbb{R}^n$. We will prove that (4.4) implies

$$(4.5) \qquad \lim_{n \to \infty} F_{t_n}(y; f^{-1}) = F^{-1}(y)$$

locally uniformly in $y \in \mathbb{R}^n$. Let us assume the converse. Then there exist a compact set $\mathcal{C} \subset \mathbb{R}^n$ and sequences $y_j \in \mathcal{C}$, $t_{n_j} \to 0$ as $j \to \infty$, such that

$$(4.6) \qquad |F_{t_{n_j}}(y_j; f^{-1}) - F^{-1}(y_j)| \ge \varepsilon > 0$$

for every $j = 1, 2, \ldots$. Set $x_j = f^{-1}(t_{n_j} y_j)/t_{n_j}$. By (4.2) with $\lambda = 1$ it follows that $|x_j|/|y_j| \to 1$ as $j \to \infty$. Therefore the sequence x_j belongs to a compact set of $\mathbb{R}^n$. We may also assume that $y_j \to y_0 \in \mathcal{C}$ and $x_j \to x_0$, $|x_0| = |y_0|$, as $j \to \infty$. Otherwise one can pass to some appropriate subsequences. Since $F_{t_{n_j}}(x_j;f) = y_j \to F(x_0)$ as $j \to \infty$ and F is a linear isometry, it follows that

$$0 = F(x_0) - y_0 = x_0 - F^{-1}(y_0) = \lim_{j\to\infty}(x_j - F^{-1}(y_j)) = \lim_{j\to\infty}(F_{t_{n_j}}(y_j; f^{-1}) - F^{-1}(y_j)). \tag{4.7}$$

Formula (4.7) provides a contradiction to the inequality (4.6). Since the above arguments are invertible, we complete the proof of the second assertion of Theorem 2.3.

3. Assume now that $T(0, f)$ is simple. This means, by the definition, that $T(0, f)$ consists of a single element which in our case is linear isometric mapping $\mathcal{O}$ being generated by the formula

$$\frac{f(tx)}{\lambda t} \to \mathcal{O}x, \quad \text{as } t \to 0, \tag{4.8}$$

locally uniformly in $x \in \mathbb{R}^n$. The latter is equivalent to the conformal differentiation of f at 0. Thus the proof of Theorem 2.3 is complete.

Proof of Theorem 2.9. By Theorem 2.3 with $n = 3$ we see that

$$\lim_{x\to 0}\frac{|\varphi(x)|}{|x|} = \tau, \quad \tau = 1/\lambda, \tag{4.9}$$

and the infinitesimal space $T(0, \varphi)$, $\varphi = f^{-1}$, consists of linear isometric mappings only.

Let us assume that $T(0, \varphi)$ is not simple. In this case there exist two sequences $\{a_n\}$, $\{b_n\}$, $0 < a_n < b_n$, $b_n \to 0$ as $n \to \infty$, and two distinct linear isometric mappings $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$F_{a_n}(x;\varphi) = \frac{\varphi(a_n x)}{\tau a_n} \to \mathcal{O}_1(x) \tag{4.10}$$

and

$$F_{b_n}(x;\varphi) = \frac{\varphi(b_n x)}{\tau b_n} \to \mathcal{O}_2(x), \tag{4.11}$$

as $n \to \infty$ uniformly in $x \in \mathbb{R}^3 \setminus \{0\}$. Since f is a K – quasiconformal rotation about the e_3 - axis, the mappings φ, $\mathcal{O}_1(x)$ and $\mathcal{O}_2(x)$ are also quasiconformal rotations about e_3 - axis.

It is well - known that every linear isometric mapping $\mathcal{O}$ of $\mathbb{R}^3$ is a rotation of $\mathbb{R}^3$ by an angle θ about some vector $\mu \in \mathbb{R}^3$ and has the explicit representation, see, e.g., [2], p. 230,

$$\mathcal{O}x = x\cos\theta + \mu\langle\mu, x\rangle(1 - \cos\theta) + (\mu \times x)\sin\theta \tag{4.12}$$

where $\langle \mu, x\rangle$ and $(\mu \times x)$ stand for the inner and the vector product, respectively. In our case $\mu = e_3$ and thus

$$\mathcal{O}_j(x) = \Big((x_1 + ix_2)e^{i\theta_j}\ ,\ x_3\Big)\,,\quad j = 1, 2, \tag{4.13}$$

with some angles θ_1, θ_2.

Since the angular displacement $\mathcal{A}_\varphi(x)$ of the point x under the mapping φ is a continuous single–valued function, the asymptotic formulae (4.10) and (4.11) imply that

$$\mathcal{A}_\varphi(a_n\nu) = \arg\frac{\Phi_\varphi(a_n\nu)}{e^{i\omega}} \to \theta_1 \tag{4.14}$$

and

$$\mathcal{A}_\varphi(b_n\nu) = \arg\frac{\Phi_\varphi(b_n\nu)}{e^{i\omega}} \to \theta_2 \tag{4.15}$$

as $n \to \infty$ uniformly in $\nu \in S^2$. The assumption that $T(0,\varphi)$ is not simple now is equivalent to the condition

$$\theta_1 - \theta_2 = \theta \neq 0. \tag{4.16}$$

In order to obtain a contradiction to the assumption (4.16) we make use of Lemma 3.10. Since (4.9) holds, we see that for each small positive number ε there is N_1 such that the space ring $\mathcal{R} = \varphi(R(a_n, b_n))$ will contain the spherical annulus $R(\tau a_n(1+\varepsilon), \tau b_n/(1+\varepsilon))$ and be contained in $R(\tau a_n/(1+\varepsilon),\ \tau b_n(1+\varepsilon))$ for all $n > N_1$. Hence for $n > N_1$ the mapping φ is admissible for Lemma 3.10.

Let us fix a radial segment $\gamma^\nu_{a_nb_n} = \{t\nu : a_n \le t \le b_n\}$, $\nu = (e^{i\omega}\sin\alpha,\ \cos\alpha) \in S^2$, in $R(a_n, b_n)$ and estimate the total rotation of the curve $\varphi\circ h(\gamma^\nu_{a_nb_n})$ with a test function $h(x)$ as in (3.11). We shall choose the constant b and the sign for $h(x)$ later. From now on we will assume that the test function h is extended to the space $\mathbb{R}^3$ by $h(x) = x$ for $|x| \ge b$. We may also assume that $t \mapsto \varphi \circ h(t\nu)$ provides an absolutely continuous parametrization $\eta(t)$, $a_n \le t \le b_n$. Since $\varphi(x)$ and $h(x)$ are rotations about the e_3 - axis, we have

$$\frac{\langle \varphi(h(t\nu)), e_3\rangle}{|\varphi(h(t\nu))|} = \frac{\langle h(t\nu), e_3\rangle}{|h(t\nu)|} = \langle \nu, e_3\rangle = \cos\alpha \tag{4.17}$$

for all $a_n \le t \le b_n$. Hence $\nu(t)$ has the following representation in the spherical coordinates

$$\eta(t) = \Big(\rho(t)e^{i\sigma(t)}\sin\alpha,\ \rho(t)\cos\alpha\Big)\,,\quad a_n \le t \le b_n, \tag{4.18}$$

where by (3.8) and (3.9)

$$\sigma(t) = \Omega_\varphi(h(t\nu)) = \mathcal{A}_\varphi(h(t\nu)) \pm \log \frac{b_n}{t} + \omega. \tag{4.19}$$

Then

$$|\eta'(t)|^2 = {\rho'}^2 + \rho^2 {\sigma'}^2 \sin^2 \alpha, \quad |\eta(t)|'^2 = {\rho'}^2, \tag{4.20}$$

and (3.6) yields

$$\delta_{\varphi\circ h(\gamma^\nu_{a_n b_n})} = \int\limits_{a_n}^{b_n} |\sigma'(t)| \sin \alpha dt \geq |\sigma(a_n) - \sigma(b_n)| \sin \alpha. \tag{4.21}$$

Now by (4.19) and (4.21) we otain that

$$\delta_{\varphi\circ h(\gamma^\nu_{a_n b_n})} \geq |\mathcal{A}_\varphi(h(a_n\nu)) - \mathcal{A}_\varphi(h(b_n\nu)) \pm \log(b_n/a_n)| \sin \alpha, \tag{4.22}$$

where the sign at the logarithmic function agrees with the sign in (3.11).

To proceed with the proof we first assume that θ, defined by (4.16), is positive and apply the inequality (3.12) with the test function $h(x)$ of the form

$$h(x) = \left((x_1 + ix_2)e^{i \log \frac{b_n}{|x|}},\ x_3\right). \tag{4.23}$$

Since $h(b_n\nu) = b_n\nu$ for every unit vector $\nu \in S^2$ we see, by (4.15), that

$$\mathcal{A}_\varphi(h(b_n\nu)) = \mathcal{A}_\varphi(b_n\nu) \to \theta_2 \quad \text{as } n \to \infty \tag{4.24}$$

uniformly in $\nu \in S^2$. Hence there exists N_2 such that

$$\mathcal{A}_\varphi(h(b_n\nu)) \leq \theta_2 + \varepsilon \tag{4.25}$$

for all $n > N_2$ uniformly in $\nu \in S^2$. Next, taking into account that $h(a_n\nu) = a_n\mu_n$, where

$$\mu_n = \left(e^{i(\omega + \log(b_n/a_n))} \sin \alpha,\ \cos \alpha\right) \in S^2, \tag{4.26}$$

and assuming, without loss of generality, that $\mu_n \to \mu_0 \in S^2$ as $n \to \infty$, we obtain by (4.14) that

$$\mathcal{A}_\varphi(h(a_n\nu)) = \mathcal{A}_\varphi(b_n\mu_n) \to \theta_1 \quad \text{as } n \to \infty \tag{4.27}$$

uniformly in $\nu \in S^2$. Hence there exists N_3 such that for $n > N_3$

$$\mathcal{A}_\varphi(h(a_n\nu)) \geq \theta_1 - \varepsilon \tag{4.28}$$

uniformly in $\nu \in S^2$. The inequalities (4.25), (4.28) and (4.22) with the sign plus yield

$$\delta_{\varphi \circ h(\gamma^{\nu}_{a_n b_n})} \geq (\theta_1 - \theta_2 + \log(b_n/a_n) - 2\varepsilon)\sin\alpha \tag{4.29}$$

for all $\nu \in S^2$ if $n > \max\{N_2, N_3\}$. By formula (3.21), $\delta_{h(\gamma^{\nu}_{a_n b_n})} = (\sin\alpha)\log(b_n/a_n)$ and we arrive at the following estimate

$$\delta_{\varphi \circ h(\gamma^{\nu}_{a_n b_n})} - \delta_{h(\gamma^{\nu}_{a_n b_n})} \geq (\theta - 2\varepsilon)\sin\alpha, \tag{4.30}$$

which holds for every α, $0 < \alpha < \pi$.

Since the integral $I(r)$ converges, there exists N_4 such that

$$\int\limits_{a_n \leq |x| \leq b_n} \frac{L_f(x) - 1}{|x|^3} dx < \varepsilon \tag{4.31}$$

for all $n > N_4$. Now for every fixed $n > \max\{N_k\}$, $k = 1,2,3,4$, the inequality (3.12) yields

$$3\int\limits_{S^2} (\theta - 2\varepsilon)\sin^3\alpha dm(\nu) \leq \int\limits_{S^2} (\theta - 2\varepsilon)\sin\alpha w(\nu) dm(\nu) \leq (c_1 + c_2)\varepsilon, \tag{4.32}$$

since $w(\nu) \geq 3\sin^2\alpha$. For sufficiently small ε, (4.32) provides a contradiction to the assumption that $\theta > 0$.

The case when $\theta < 0$ can be handled in the same way with the test function $h(x)$ of the form

$$h(x) = \left((x_1 + ix_2)e^{-i\log\frac{b_n}{|x|}},\ x_3\right). \tag{4.33}$$

Thus, we have shown that the infinitesimal space $T(0,\varphi)$ is simple. Hence $T(0,f)$ is also simple and by Theorem 2.3 the mapping f is conformally differentiable at the origin. The proof for Theorem 2.9 is complete.

Proof of Theorem 2.13. Since f is conformally differentiable at the origin, we see that there exist a $\tau = 1/\lambda > 0$ and a linear isometric mapping $\mathcal{O}$ such that

$$\frac{\varphi(tx)}{\tau t} \to \mathcal{O}x \text{ as } t \to 0 \tag{4.34}$$

locally uniformly in $\mathbb{R}^3$. Let $\mathcal{O}$ be the rotation of $\mathbb{R}^3$ about e_3 - axis by an angle θ, i.e.

$$\mathcal{O}(x) = \left((x_1 + ix_2)e^{i\theta},\ x_3\right). \tag{4.35}$$

Given a unit vector $\nu = (e^{i\omega}\sin\alpha,\ \cos\alpha)$ we see that (4.34) and (4.35) imply that

$$\mathcal{A}_\varphi(r\nu) \to \theta \tag{4.36}$$

as $r \to 0$ uniformly in $\nu \in S^2$. Here $\mathcal{A}_\varphi(x)$ stands for the angular displacement of the point x under the mapping φ.

On the other hand, given $\varepsilon > 0$ we deduce from (4.34) that

$$\tau/(1+\varepsilon) < |\varphi(x)|/|x| < \tau(1+\varepsilon) \tag{4.37}$$

whenever $|x|$ is sufficiently small. Hence the restriction of the mapping $\varphi(x)$ on the spherical annulus $R(r,1)$ for all $r < \rho$ satisfies the assumptions of Lemma 3.10. In order to apply this lemma we have to estimate the total rotation of the curve $\varphi \circ h(\gamma^\nu_{r\,1})$ for fixed $r < \rho$ and $\nu \in S^2$. Let $\eta(t) = \varphi(h(t\nu))$, $r \le t \le 1$, be its absolutely continuous parametric representation. Then, by formula (4.22),

$$\delta_{\varphi\circ h(\gamma^\nu_{r\,1})} \ge |\mathcal{A}_\varphi(h(r\nu)) - \mathcal{A}_\varphi(h(\nu)) \pm \log(1/r)| \sin\alpha \tag{4.38}$$

where the sign at the logarithmic function agrees with the sign in (3.11). Since $f(x) = x$ for $|x| = 1$, we have $\mathcal{A}_\varphi(h(\nu)) = \mathcal{A}_\varphi(\nu) = 0$. Thus

$$\delta_{\varphi\circ h(\gamma^\nu_{r\,1})} \ge |\mathcal{A}_\varphi(h(r\nu)) \pm \log(1/r)| \sin\alpha. \tag{4.39}$$

Let us assume that the rotation angle θ is positive and apply the inequality (3.12) with the test function $h(x)$ of the form

$$h(x) = \left((x_1 + ix_2)e^{i\log\frac{1}{|x|}},\ x_3\right). \tag{4.40}$$

Since $h(r\nu) = r\mu(r)$, where

$$\mu(r) = \left(e^{i(\omega+\log(1/r))}\sin\alpha,\ \cos\alpha\right) \in S^2, \tag{4.41}$$

there exists a sequence $r_j \to 0$ as $j \to \infty$ such that $\mu(r_j) \to \mu_0 \in S^2$. Now (4.36) yields $\mathcal{A}_\varphi(h(r_j\nu)) \to \theta$ as $j \to \infty$ uniformly in $\nu \in S^2$. Hence the inequality

$$\mathcal{A}_\varphi(h(r_j\nu)) \ge \omega - \varepsilon \tag{4.42}$$

holds for all $j > N$ and every $\nu \in S^2$. The latter inequality together with the inequality (4.39) yield

$$\delta_{\varphi\circ h(\gamma^\nu_{r_j\,1})} \ge (\theta + \log(1/r_j) - \varepsilon)\sin\alpha \tag{4.43}$$

Since $\delta_{h(\gamma^\nu_{r_j\,1})} = \log(1/r_j)$, we have

$$\delta_{\varphi\circ h(\gamma^{\nu}_{r_j\,1})} - \delta_{h(\gamma^{\nu}_{r_j\,1})} \geq (\theta - \varepsilon)\sin\alpha. \tag{4.44}$$

By Lemma 3.1 with $w(\nu)$ replaced by $3\sin^2\alpha$ for all $\nu \in S^2$ we see that

$$\int\limits_{S^2} (\theta - \varepsilon)\sin^3\alpha dm(\nu) \leq \frac{2\sqrt{2}}{3} \int\limits_{r_j<|x|<1} \frac{L_f(x) - 1}{|x|^3} dx + c_2\varepsilon. \tag{4.45}$$

Passing to the limit first as $j \to \infty$ and then as $\varepsilon \to 0$ and making use of the formula

$$\int\limits_{S^2} \sin^3\alpha dm(\nu) = 2\pi \int\limits_0^{\pi} \sin^4\alpha d\alpha = \frac{3}{4}\pi^2, \tag{4.46}$$

we arrive at the estimate

$$\theta \leq \frac{8\sqrt{2}}{9\pi^2} \int\limits_{|x|<1} \frac{L_f(x) - 1}{|x|^3} dx = \frac{32\sqrt{2}}{9\pi} I(1). \tag{4.47}$$

The case when $\theta < 0$ can be handled in the same way with the test function $h(x)$ of the form

$$h(x) = \left((x_1 + ix_2)e^{-i\log\frac{1}{|x|}},\ x_3 \right). \tag{4.48}$$

The proof is complete.

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