Stability and higher integrability of derivatives of solutions in double obstacle problems

by

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Abstract. Higher integrability of the derivatives of solutions to double obstacle problems associated with the second order quasilinear elliptic differential equation $\nabla \cdot A(x, \nabla u) = 0$ is obtained under natural assumptions on obstacles. This result is used to prove a stability result for solutions to double obstacle problems for varying equations.

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1. Introduction

Let Ω be a domain in $\mathbb{R}^n$ and let $W^{1,p}(\Omega),\ 1 < p < \infty$, be the first order Sobolev space of functions $u \in L^p(\Omega)$ whose distributional gradient ∇u belongs to $L^p(\Omega)$. If $\theta \in W^{1,p}(\Omega)$ and $\varphi, \psi : \Omega \to [-\infty, \infty]$, then we write

$$\mathcal{K}^{\theta,p}_{\varphi,\psi} = \{u \in W^{1,p}(\Omega) : u - \theta \in W^{1,p}_0(\Omega), \quad \varphi \le u \le \psi \ \text{ a.e. in } \ \Omega\}.$$

Given a quasilinear second order elliptic equation

$$\nabla \cdot A_p(x, \nabla u) = 0 \tag{1.1}$$

with $A_p(x, h) \cdot h \approx |h|^p$ a function $u \in \mathcal{K}^{\theta,p}_{\varphi,\psi}$ is called a solution to the $\mathcal{K}^{\theta,p}_{\varphi,\psi}(A_p)$–obstacle problem if

$$\int_\Omega A_p(x, \nabla u) \cdot \nabla(v - u)\, dx \ge 0$$

for all $v \in \mathcal{K}^{\theta,p}_{\varphi,\psi}$. Obstacle problems naturally appear in the nonlinear potential theory, see [HKM].

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In this paper we address two questions associated with the $\mathcal{K}^{\theta,p}_{\varphi,\psi}(A_p)$–obstacle problem. The first is a higher integrability problem for solutions. We show that under some natural assumptions a solution u to the $\mathcal{K}^{\theta,p}_{\varphi,\psi}(A_p)$–obstacle problem belongs to $W^{1,s}(\Omega)$ where $s > p$. This result can then be used to study the following stability problem. If we have a sequence of mappings A_{p_i} such that $A_{p_i} \to A_0$ (this convergence will be specified later), then for the solutions u_i of the $\mathcal{K}^{\theta,p_i}_{\varphi,\psi}(A_{p_i})$–obstacle problem it holds that $u_i \to u_0$ in $W^{1,s}(\Omega)$ for some $s > p_0$ and u_0 is a solution to the $\mathcal{K}^{\theta,p_0}_{\varphi,\psi}(A_{p_0})$–obstacle problem. This kind of stability has been previously studied in [LM2] for single obstacle problems. Solutions to single obstacle problems are either super– or subsolutions to (1.1), but this is not true for double obstacle and hence a different approach must be employed. The higher integrability of solutions to single obstacle problems was consideread in [LM1]. Hölder continuity of solutions to double obstacle problems has been studied in [KZ].

Our assumptions and main results are presented in Section 2. Sections 3 and 4 are devoted to the proofs of the main results. Our notation is standard and generally as in [HKM].

2. Main results

Let Ω be a bounded domain. We assume that our mappings $A_p : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $1 < p < \infty$, satisfy the following assumptions for fixed $0 < \alpha \leq \beta < \infty$:
(a) A_p satisfies the Caratheodory measurability condition [HKM, p. 56],
(b) for a.e. $x \in \Omega$

$$A_p(x,\xi)\cdot\xi \geq \alpha|\xi|^p,$$

$$|A_p(x,\xi)| \leq \beta|\xi|^{p-1}$$

for all $\xi \in \mathbb{R}^n$, and
(c) for a.e. $x \in \Omega$

$$(A_p(x,\xi_1) - A_p(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$$

whenever $\xi_1 \neq \xi_2$.
Given mappings $A_{p_0}, A_{p_1}, A_{p_2}, \ldots$ as in (a)–(c) we say that $A_{p_i} \to A_{p_0}$ if for a.e. $x \in \Omega$

$$A_{p_i}(x,\xi) \to A_{p_0}(x,\xi) \tag{2.1}$$

locally uniformly in $\mathbb{R}^n$. Note that we assume that all the mappings A_{p_i}, $i = 0, 1, \ldots$, satisfy (b) with the same fixed constants α and β and that (2.1) implies $p_i \to p_0$.

We also need a regularity condition for $\partial\Omega$. We assume that $\mathbb{R}^n \setminus \Omega$ is thick in the following sense: For each cube $Q(r)$ with side length r and $Q(3r/2) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$

$$|(\mathbb{R}^n \setminus \Omega) \cap Q(2r)| \geq \mu|Q(2r)| \tag{2.2}$$

where $\mu > 0$ is independent of the cube $Q(r)$. Here $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^n$. Here and in the sequel we denote by $Q(\lambda r)$ the cube parallel to the cube $Q(r)$ with the same center and side length λr. This condition is slightly weaker than the corkscrew condition for $\mathbb{R}^n \setminus \Omega$, see [HKM, p. 123], which generalizes the well known uniform outside cone condition for Ω.

To formulate our main results let $1 < p_0 < \infty$ and $s > p_0$. Suppose that $\varphi, \psi \in W^{1,s}(\Omega)$ with $\varphi \leq \theta \leq \psi$ a.e. in Ω. Given a sequence of mappings A_{p_i} such that $A_{p_i} \to A_{p_0}$ there exists a unique solution u_i to the $\mathcal{K}^{\theta,p_i}_{\varphi,\psi}(A_{p_i})$–obstacle problem for $i = 0$ and for large i, in fact for $p_i \leq s$, see [HKM, Appendix I, p. 332]. We also assume that Ω satisfies the thickness condition (2.2).

2.3. **Theorem.** *There is $t > p_0$ such that $u_i \to u_0$ in $W^{1,t}(\Omega)$ and t depends only on n, p_0, s, α, β and μ.*

For the proof of Theorem 2.3 we need the following uniform higher integrability result. We assume that u is a solution to the $\mathcal{K}^{\theta,p}_{\varphi,\psi}(A_p)$–obstacle problem in a bounded domain satisfying (2.2).

2.4. **Theorem.** *There exists a constant $\varepsilon_0 = \varepsilon_0(n, p, \alpha, \beta, \mu) > 0$ uniformly for $p > 1$ such that for each $\varepsilon \in [0, \varepsilon_0]$*

$$\begin{aligned}(2.5)\qquad &\int_\Omega |\nabla u|^{p+\varepsilon} dx \\ &\leq C\Big[\Big(\int_\Omega |\nabla u|^p dx\Big)^{1+\varepsilon/p} + \int_\Omega |\nabla \varphi|^{p+\varepsilon} dx + \int_\Omega |\nabla \psi|^{p+\varepsilon} dx + \int_\Omega |\nabla \theta|^{p+\varepsilon} dx\Big]\end{aligned}$$

where $C = C(n, p.\alpha, \beta, \mu, d(\Omega)) < \infty$ uniformly for $p > 1$.

The condition $\varepsilon_0 > 0$ uniformly for $p > 1$ means that for each compact set $F \subset (1, \infty)$ and for each n, α, β, μ there is $\varepsilon' > 0$ such that we can use ε' instead of ε_0 for all $p \in F$. The condition for $C < \infty$ uniformly for $p > 1$ is similar.

2.6. **Remark.** The proof for Theorem 2.4 shows, in fact, that there are continuous functions ε_0 and C such that $\varepsilon_0 > 0$ and $C < \infty$ in the domain of their arguments n, p, α, β, μ and $d(\Omega)$. Hence the uniformity can be extended to all argument values.

3. Proof for Theorem 2.4

The proof for Theorem 2.4 is similar to the higher integrability result for a single obstacle problem, see [LM1]. However, there is an important difference in choosing a suitable test function. Unlike in the single obstacle problem the choice is less apparent.

We prove Theorem 2.4 for $d(\Omega) \leq 1$ with the constant C in (2.5) independent of $d(\Omega)$. For $d(\Omega) > 1$ a simple scaling argument completes the proof.

For the proof pick a cube $Q_0 = Q_0(2r_0)$ such that $\Omega \subset Q(r_0)$ and $r_0 = d(\Omega) \leq 1$. Next take an arbitrary cube $Q(2r)$ in Q_0 and consider two possibilities

(i) $Q(3r/2) \subset \Omega$,

(ii) $Q(3r/2) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$.

In the case (i) we let $\eta \in C_0^\infty(Q(3r/2))$ be a cut off function such that $0 \le \eta \le 1$, $|\nabla\eta| \le c/r$ and $\eta = 1$ on $Q(r)$. We denote by c a constant whose value may change even on the same line. Write

$$c_u = \fint_{Q(3r/2)} u dx$$

for the mean value of u in $Q(3r/2)$.

Consider the function

$$v = (1-\eta^p)(u-c_u) + \eta^p w$$

where

$$w = (\psi - c_u)^- + \min\left((\varphi - c_u)^+, (\psi - c_u)^+\right).$$

For the proof of Theorem 2.4 we can assume that $\nabla\varphi, \nabla\psi \in L^{p+\varepsilon}(\Omega)$ where $\varepsilon > 0$ and hence $w \in W^{1,p}(\Omega)$. Moreover, $v \in \mathcal{K}^{\theta - c_u, p}_{\varphi - c_u, \psi - c_u}$ because

$$w = \begin{cases} (\varphi - c_u)^+, & \psi \ge c_u, \\ \psi - c_u, & \psi < c_u, \end{cases}$$

and this yields $\varphi - c_u \le v \le \psi - c_u$ a.e. in Ω. Since $u - c_u$ is a solution to the $\mathcal{K}^{\theta - c_u}_{\varphi - c_u, \psi - c_u}(A_p)$–obstacle problem we have

$$\int_\Omega A(x, \nabla u) \cdot (\nabla v - \nabla u) du \ge 0$$

and the structure assumption (b) in Section 2 yields

$$\alpha \int_\Omega \eta^p |\nabla u|^p dx \le \int_\Omega \eta^p A(x, \nabla u) \cdot \nabla u dx$$

$$\le \beta \int_\Omega \eta^p |\nabla u|^{p-1} |\nabla w| dx + p\beta \int_\Omega |\nabla u|^{p-1} \eta^{p-1} |\nabla \eta| (|w| + |u - c_n|) dx.$$

Now Young's inequality together with

$$|w| \le \begin{cases} |\varphi - c_\varphi|, & \text{if } \psi \ge c_u, \\ |\psi - c_\psi|, & \text{if } \psi < c_u, \end{cases}$$

implies

$$\alpha \int_\alpha \eta^p |\nabla n|^p dx \le c \int_\Omega \eta^p (|\nabla\varphi|^p + |\nabla\psi|^p) dx$$

(3.1)

$$+ cr^{-p}\int_\Omega \eta^{p-1}(|\varphi - c_\varphi|^p + |\psi - c_\psi|^p + |u - c_u|^p)dx$$

where c_φ and c_ψ denote the mean values of φ and ψ in $Q(3r/2)$, respectively. Note also that $c_\varphi \le c_u \le c_\psi$.

Next the Poincaré inequality, see e.g. [GT, Lemma 7.6] or [LM1], and the Sobolev–Poincaré inequality [GT, p. 176] together with (3.1) yield

$$\begin{aligned}\fint_{Q(r)} |\nabla u|^p dx \le c\Big(\fint_{Q(3n/2)} |\nabla u|^{np/(p+n)} dx\Big)^{(p+n)/n} \\ + c\Big(\fint_{Q(3r/2)} |\nabla\psi|^p dx + \fint_{Q(3r/2)} |\nabla\varphi|^p dx\Big)\end{aligned} \tag{3.2}$$

where $c = c(n,p,\alpha,\beta) < \infty$ uniformly for $p > 1$.

In the case (ii) let $v = u - \eta^p(u-\theta)$ where $\eta \in C_0^\infty(Q(2r))$ is a similar cut off function as in the first case. Since $\varphi \le \theta \le \psi$ it is easy to see that $v \in \mathcal{K}^{\theta,p}_{\varphi,\psi}$ and as in [LM1,LM2] we obtain

$$\int_\Omega \eta^p|\nabla u|^p dx \le c\int_\Omega (\eta^p|\nabla\theta|^p + |\nabla\eta|^p|\theta - u|^p)dx \tag{3.3}$$

where $c = c(n,p,\alpha,\beta) < \infty$ uniformly for $p > 1$. Next we extend $\theta - u$ to be zero in $\mathbb{R}^n \setminus \Omega$ and use the inequality

$$\Big(\int_{Q(2r)} |w|^p dx\Big)^{1/p} \le c\Big(\int_{Q(2r)} |\nabla w|^q dx\Big)^{1/q} \tag{3.4}$$

where $q = np/(n+p)$; (3.4) is valid for functions $w \in W^{1,p}(Q(2r))$ provided that $w = 0$ in $Q(2r) \cap (\mathbb{R}^n \setminus \Omega)$ and that the condition (2.2) holds, see [LM1, Lemma 3.11]. Then, as in [LM1,LM2], we obtain

$$\int_\Omega \eta^p|\nabla u|^p dx \le c\Big[r^{-p}\Big(\int_{Q(2r)\cap\Omega} |\nabla u|^q dx\Big)^{(p+n)/n} + \int_{Q(2r)\cap\Omega} |\nabla\theta|^p dx\Big] \tag{3.5}$$

where $c = c(n,p,\alpha,\beta,\mu) < \infty$ uniformly for $p > 1$.

To complete the proof we write

$$g = |\nabla u|^q, \quad f = (|\nabla\theta| + |\nabla\varphi| + |\nabla\psi|)^q$$

in Ω and $g = 0 = f$ in $\mathbb{R}^n \setminus \Omega$. ¿From (3.2) and (3.5) we obtain

$$\tag{3.6} \fint_{Q(r)} g^\kappa dx \le c\Big[\Big(\fint_{Q(2r)} g dx\Big)^\kappa + \fint_{Q(2r)} f^\kappa dx\Big]$$

where $\kappa = (p+n)/n$ and $c = c(n, p, \alpha, \beta, \mu) < \infty$ uniformly for $p > 1$. This is the basic reverse Hölder inequality needed for the higher integrability for $|\nabla u|$. As in [LM1] we can now employ [I, Proposition 6.1], see also [LM1. Lemma 3.15], to conclude that the inequality (2.5) of Theorem 2.4 holds. This completes the proof.

4. Proof of Theorem 2.3

Since $p_i \to p_0$ and $s > p_0$ we may assume that $p_i < s$ for all $i = 1, 2, \ldots$. Write $A_i = A_{p_i}$. The proof is carried out in several steps.

Step I. We show that

$$\tag{4.1} \int_\Omega |\nabla u_i|^{p_i} dx \le c$$

where $c < \infty$ is independent of i. Since $\varphi \le \theta \le \psi$ and $\theta \in W^{1,s}(\Omega)$, $\theta \in \mathcal{K}^{\theta, p_i}_{\varphi,\psi}$ and hence

$$\int_\Omega A_i(x, \nabla u_i) \cdot (\nabla\theta - \nabla u_i) dx \ge 0.$$

This together with the Hölder inequality and the structure assumptions (b) of Section 2 yields (4.1).

Step II. There exists $t > p_0$ such that

$$\tag{4.2} \int_\Omega |\nabla u_i|^t dx \le c$$

where $c < \infty$ is independent of i. This follows from (4.1) and from Theorem 2.4.

Step III. Since $\theta \in W^{1,s}(\Omega)$ and $u_i - \theta \in W^{1,p_i}_0(\Omega)$ the Sobolev imbedding theorem and (4.2) give

$$\tag{4.3} \|u_i\|_{1,t} = \|u_i\|_{L^t(\Omega)} + \|\nabla u_i\|_{L^t(\Omega)} \le c$$

where $c < \infty$ is again independent of i.

Step IV. From (4.3) it follows that there is a subsequence, denoted again by u_i, of u_i and $\overline{u} \in W^{1,t}(\Omega)$ such that $u_i \to \overline{u}$ weakly in $W^{1,t}(\Omega)$ and $u_i \to \overline{u}$ in $L^t(\Omega)$. Now the function $\overline{u}$ belongs to $\mathcal{K}^{\theta,p_0}_{\varphi,\psi}$. Indeed, since $\varphi \le u_i \le \psi$ we have $\varphi \le \overline{u} \le \psi$. To show that $\overline{u}-\theta \in W_0^{1,p_0}(\Omega)$ Lemma 3.25 in [LM2] can be employed; note that this result is not immediate if $p_i < p_0$.

Step V. We show that

$$\nabla u_i \to \nabla \overline{u} \text{ a.e.} \tag{4.4}$$

This immediately implies that

$$u_i \to \overline{u} \text{ in } W^{1,t'}(\Omega) \tag{4.5}$$

for some $t' > p_0$, see Step IV.

In order to prove (4.4) a different approach than in [LM2] is used. This is due to the fact that a solution to the double obstacle problem is no longer a supersolution. To this end, fix domains $G \subset\subset G' \subset\subset \Omega$ and $\eta \in C_0^\infty$ such that $0 \le \eta \le 1,$ spt $\eta \subset G'$ and $\eta = 1$ on G. Let

$$v_i = u_i + \eta(\overline{u} - u_i).$$

Since $\overline{u} \in \mathcal{K}_{\varphi,\psi}^{\theta,p_i}$ we have

$$v_i \ge (1-\eta)\varphi + \eta\varphi = \varphi,$$

$$v_i \le (1-\eta)\psi + \eta\psi = \psi$$

and since $v_i - \theta \in W_0^{1,p_i}(\Omega),\ v_i \in \mathcal{K}_{\varphi,\psi}^{\theta,p_i}$ and hence

$$\int_\Omega A_i(x,\nabla u_i)\cdot(\nabla v_i - \nabla u_i)dx \ge 0.$$

This leads to

$$\int_\Omega A_i(x,\nabla u_i)\cdot(\nabla u_i - \nabla\overline{u})\eta dx \le \int_\Omega A_i(x,\nabla u_i)\cdot\nabla\eta(\overline{u}-u_i)dx$$

$$\le \beta c \int_\Omega |\nabla u_i|^{p_i-1}|\overline{u}-u_i|dx \le c\|u_i - \overline{u}\|_t \to 0$$

as $i \to \infty$ because of (4.2) and Step IV. Now

$$\int_\Omega (A_i(x,\nabla u_i) - A_i(x,\nabla\overline{u}))\cdot(\nabla u_i - \nabla\overline{u})\eta dx$$

$$\le \int_\Omega A_i(x,\nabla u_i)\cdot\nabla\eta(\overline{u}-u_i)dx - \int_\Omega \eta A_i(x,\nabla\overline{u})\cdot(\nabla u_i - \nabla\overline{u})dx$$

$$= I_i^1 + I_i^2.$$

We already know that $I_i^1 \to 0$ as $i \to \infty$ and the second integral is estimated as follows:

$$|I_i^2| = \left| \int_\Omega [\eta(A_i(x,\nabla\overline{u}) - A_0(x,\nabla\overline{u})) \cdot (\nabla u_i - \nabla\overline{u}) + \eta A_0(x,\nabla\overline{u})(\nabla u_i - \nabla\overline{u})]dx \right|$$

$$\leq J_i^1 + J_i^2.$$

As $\nabla u_i \to \nabla\overline{u}$ weakly in $L^t(\Omega)$ we have that

$$J_i^2 = \left| \int_\Omega \eta A_0(x,\nabla\overline{u}) \cdot (\nabla u_i - \nabla\overline{u})dx \right| \to 0$$

and to show that

$$J_i^1 = \left| \int \eta(A_i(x,\nabla\overline{u}) - A_0(x,\nabla\overline{u})) \cdot (\nabla u_i - \nabla\overline{u})dx \right| \to 0$$

as $i \to \infty$ we observe that (4.1) and $\nabla\overline{u} \in L^t(\Omega),\ t > p_i$, yield

$$J_i^1 \leq c\|A_i(x,\nabla\overline{u}) - A_0(x,\nabla\overline{u})\|_{p_i'}$$

where p_i' is the conjugate exponent of p_i. Hence it suffices to show that

$$\int_\Omega |A_i(x,\nabla\overline{u}) - A_0(x,\nabla\overline{u})|^{p_i'}dx \to 0, \quad i \to \infty. \tag{4.6}$$

Now the assumption $A_i(x,\xi) \to A_0(x,\xi)$, see (2.1), implies that

$$|A_i(x,\nabla\overline{u}(x)) - A_0(x,\nabla\overline{u}(x))|^{p_i'} \to 0$$

a.e. in Ω and for any measurable set $E \subset \Omega$

$$\begin{aligned} \int_E |A_i(x,\nabla\overline{u}) - A_0(x,\nabla\overline{u})|^{p_i'}dx &\leq c\left[\int_E |\nabla\overline{u}|^{p_i}dx + \int_E |\nabla\overline{u}|^{(p_0-1)p_i'}dx\right] \\ &\leq c\left[|E|^{1-p_i/t}\left(\int_E |\nabla\overline{u}|^t dx\right)^{p_i/t} + |E|^{1-(p_0-1)p_i'/t}\left(\int_E |\nabla\overline{u}|^t dx\right)^{(p_0-1)p_i^2/t}\right] \end{aligned} \tag{4.7}$$

where we have used the fact that $\overline{u} \in W^{1,t}(\Omega)$. The constant c is independent of i. Since the right hand side of (4.7) can be made arbitrary small by making $|E|$ small, the Vitali convergence theorem implies (4.6) and hence $I_i^2 \to 0$ as $i \to \infty$. Thus we have shown that

$$\int_G (A_i(x,\nabla u_i) - A_i(x,\nabla\overline{u})) \cdot (\nabla u_i - \nabla\overline{u})dx$$

$$\leq \int_\Omega \eta(A_i(x,\nabla u_i) - A_i(x,\nabla\overline{u})) \cdot (\nabla u_i - \nabla\overline{u})dx \to 0$$

as $i \to \infty$. The condition (c) of Section 2 now yields

$$\lim_{i\to\infty} I_i(x) = \lim_{i\to\infty} (A_i(x, \nabla u_i(x)) - A_i(x, \nabla \overline{u}(x)) \cdot (\nabla u_i - \nabla \overline{u}) = 0$$

for a.e. $x \in G$ and thus for a.e. $x \in \Omega$.

In order to complete the proof for (4.4) fix $x_0 \in \Omega$ such that

(i) $I_i(x_0) \to 0,\ i \to \infty$,

(ii) $|\nabla \overline{u}(x_0)| < \infty$,

(iii) the assumptions (b) and (c) of Section 2 hold for $A_p = A_i,\ i = 1, 2, \ldots$, at x_0,

(iv) $A_i(x_0, \xi) \to A_0(x_0, \xi)$ locally uniformly in $\mathbb{R}^n$.

If $|\nabla u_i(x_0)| \to \infty$, then

$$\begin{aligned} &\alpha |\nabla u_i(x_0)|^{p_i} \le A_i(x_0, \nabla u_i(x_0)) \cdot \nabla u_i(x_0) \\ &\le I_i(x_0) + \beta |\nabla \overline{u}(x_i)|^{p_i - 1} (|\nabla u_i(x_0)| + |\nabla \overline{u}(x_0)|) + \beta |\nabla u_i(x_0)|^{p_i - 1} |\nabla \overline{u}(x_0)| \end{aligned}$$

and this is a contradiction if $|\nabla u_i(x_0)| \to \infty$. Passing to a subsequence, if necessary, we may thus assume that $\nabla u_i(x) \to \xi \in \mathbb{R}^n$. If now $\xi \neq \nabla \overline{u}(x_0)$, then

$$0 = \lim_{i\to\infty} I_i(x_0) = (A_0(x_0, \xi) - A_0(x_0, \nabla \overline{u})) \cdot (\xi - \nabla \overline{u}(x_0)) > 0$$

provides a contradiction. Thus (4.4) follows.

Step VI. We show that $u_0 = \overline{u}$. Since $\overline{u} \in \mathcal{K}^{\theta, p_0}_{\varphi, \psi}$, we have

$$\int_\Omega A_0(x, \nabla u_0) \cdot (\nabla \overline{u} - \nabla u_0) dx \ge 0. \tag{4.8}$$

On the other hand, $u_0 - \theta \in W^{1,p_0}_0(\Omega)$ and hence there is a sequence $\varphi_i \in C^\infty_0(\Omega)$ such that $\varphi_i \to u_0 - \theta$ in $W^{1,p_0}_0(\Omega)$, From the properties of the functions φ, ψ and θ it follows that

$$w_i = \min\,[\max\,(\varphi_i, \varphi - \theta), \psi - \theta] + \theta \in \mathcal{K}^{\theta, t}_{\varphi, \psi} \subset \mathcal{K}^{\theta, p_i}_{\varphi, \psi}$$

for large i. Moreover $w_i \to u_0$ in $W^{1,p_0}(\Omega)$. Let $v \in \mathcal{K}^{\theta, t'}_{\varphi, \psi},\ t' > p_0$. Then

$$\int_\Omega A_i(x, \nabla u_i) \cdot (\nabla v - \nabla u_i) dx \ge 0 \tag{4.9}$$

and as $u_i \to \overline{u}$ in $W^{1,t'}(\Omega)$, see (4.5), the inequality (4.9) yields

$$\int_\Omega A_0(x, \nabla \overline{u}) \cdot (\nabla v - \nabla \overline{u}) dx \ge 0. \tag{4.10}$$

Substituting $v = w_i$ in (4.10) we obtain

$$\int_\Omega A_0(x, \nabla\overline{u}) \cdot (\nabla w_i - \nabla\overline{u}) dx \geq 0.$$

Now $\nabla w_i \to \nabla u_0$ in $L^{p_0}(\Omega)$ and the above inequality implies

$$\int_\Omega A_0(x, \nabla\overline{u}) \cdot (\nabla u_0 - \nabla\overline{u}) dx \geq 0$$

and this together with (4.8) yields $u_0 = \overline{u}$ as required.

The proof for Theorem 2.3 is complete since we have shown that each subsequence of u_i has a subsequence which converges to u_0 in $W^{1,t}(\Omega)$ for some fixed $t > p_0$ and hence the whole sequence u_i converges to u_0 in $W^{1,t}(\Omega)$.

References

[GT] Gilbarg, D, and Trudinger, N.S., *Elliptic Partial Equations of Second Order*, 2nd Edition, Springer–Verlag, 1983.

[I] Iwaniec, T., *The Gehring Lemma*, Quasiconformal Mappings and Analysis: A collection of papers honoring Frederick W. Gehring on his 70th birthday. (P. L. Duren et al., eds.), Proceedings of the International Symposium, Ann Arbor, August 1995, Springer–Verlag, 1998, 181–204.

[HKM] Heinonen, J,. Kilpeläinen T. and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, 1993.

[KZ] Kilpeläinen, T. and Ziemer, W.P., *Pointwise regularity of solutions to nonlinear double obstacle problems*, Ark. Math. 29 (1991), 83–106.

[LM1] Li, Gongbao and Martio, O., *Local and global integrability of gradients in obstacle problems*, Ann. Acad. Sci. Fenn. Ser. A I Math. 19 (1994), 25–34.

[LM2] Li Gongbao and Martio, O., *Stability of solutions of varying degenerate elliptic equations.* Indiana Math. J. 47 (1998), 873–891.

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