# SOBOLEV EXTENSION DOMAINS ON METRIC SPACES OF HOMOGENEOUS TYPE

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Abstract. Let $(X, d, \mu)$ be a metric space of homogeneous type with a finite measure. Assume that $\Omega \subset X$ is a bounded domain, which satisfies an $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega) = 0$. We show that there exists a bounded linear extension operator $E$ from a Hajłasz space $M^{1,p}(\Omega, d, \mu)$ to $M^{1,p}(X, d, \mu)$, such that $Eu|_\Omega = u$.

## 1. Introduction

P. Hajłasz defined Sobolev spaces on a metric space, [Haj96]. Let $(X, d, \mu)$ be a metric space with a measure $\mu$. A function $u \in L^p(X, \mu)$ belongs to a Sobolev space $M^{1,p}(X, d, \mu)$, $1 < p \leq \infty$, if there exists a non-negative function $g \in L^p(X, \mu)$ such that

$$|u(x) - u(y)| \leq d(x, y)\big(g(x) + g(y)\big),$$

for almost every $x, y \in X$. When $d$ is an Euclidean metric $|\cdot|$, $\mu$ is an $n$-dimensional Lebesgue measure $m$ and $\Omega \subset \mathbb{R}^n$ is a Sobolev extension domain for some $1 < p \leq \infty$, then the function space $M^{1,p}(\Omega, |\cdot|, m)$ coincides with the classical Sobolev space $W^{1,p}(\Omega)$. In particular $M^{1,p}(\mathbb{R}^n, |\cdot|, m)$ coincides with $W^{1,p}(\mathbb{R}^n)$ for every $1 < p \leq \infty$.

Hajłasz and O. Martio proved that if a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies an $A(c)$-condition, then there exists a bounded linear extension operator

$$E^* : M^{1,p}(\Omega, |\cdot|, m) \to W^{1,p}(\mathbb{R}^n),$$

for every $1 < p \leq \infty$, such that $E^*u|_\Omega = u$, [HM97, Theorem 9, p.240]. The $A(c)$-condition tells that the domain can not be too thin near the boundary: for every $x \in \partial\Omega$ there exists $y \in \Omega$ such that $B^n(y, cr) \subset B^n(x, r) \cap \Omega$ with $0 < r \leq \operatorname{diam}(\Omega)$. For example uniform and John domains, [MS79b], satisfy the $A(c)$-condition. We generalize their result to metric spaces of homogeneous type.

Assume that $\mu$ is a non-negative Borel regular outer measure, which is finite on every bounded set. We also assume that $\mu$ satisfies the doubling condition and $\mu(X) < \infty$. We define a new plumpness condition

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which is weaker than the $A(c)$-condition. We write

$$\Omega(\varepsilon r) = \{z \in \Omega |\, \operatorname{dist}(z, X \setminus \bar{\Omega}) > \varepsilon r\}.$$

A domain $\Omega \subset X$ satisfies an $A^*(\varepsilon, \delta)$-condition, $0 < \varepsilon < 1$ and $0 < \delta \leq 1$, if

$$\mu(B(x, r) \cap \Omega(\varepsilon r)) \geq \delta \mu(B(x, r)),$$

for every $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$.

Assume that $\Omega \subset X$ is a bounded domain, which satisfies the $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega) = 0$. Then for every $1 < p \leq \infty$, there exists a bounded linear extension operator

$$E : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu),$$

such that $Eu|\Omega = u$ for every $u \in M^{1,p}(X, d, \mu)$.

In the Euclidean case we do not need the assumptation $\mu(X) < \infty$. We give the following characterization for a $W^{1,p}$-extension domain. Let $1 < p \leq \infty$. Assume that $\Omega \subset \mathbb{R}^n$ be a bounded domain, which satisfies the $A^*(\varepsilon, \delta)$-condition. Then the following two conditions are equivalent:

(1) $\Omega$ is a $W^{1,p}$-extension domain.
(2) $M^{1,p}(\Omega, |\cdot|, m) = W^{1,p}(\Omega)$.

Our results are new even in the Euclidean case and they extend the results of Hajłasz and Martio, [HM97, Theorems 9 and 10, pp.240 and 242]. The proof is based on the proof of Hajłasz and Martio.

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## 2. NOTATION

Throughout this paper $C$ will denote a general constant which may change even in a single string of estimate.

We let $(X, d, \mu)$ denote a metric space of homogenous type. This means that $\mu$ is a non-negative, non-trivial Borel regular outer measure on $X$, finite on every bounded set and there exists a constant $C > 0$ such that for every $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{2.1}$$

Here $B(x, r)$ is an open ball with center $x$ and radius $r$. Condition (2.1) is the doubling condition of the measure $\mu$ and the constant $C$ the doubling constant of the measure $\mu$. Note that by the doubling condition, if $B(x, R)$ is a ball in $X$, $z \in B(x, R)$ and $0 < r \leq R < \infty$ then

$$\frac{\mu(B(z, r))}{\mu(B(x, R))} \geq q\Big(\frac{r}{R}\Big)^Q. \tag{2.2}$$

The constants $q$ and $Q$ depend only on the doubling constant.

The Euclidean metric on $\mathbb{R}^n$ is denoted by $|\cdot|$ and the Lebesgue $n$-measure by $m$.

We let $\bar{A}$ be the closure of a set $A$ and $\partial A$ the boundary of a set $A$. By $A \setminus F$ we mean a set $\{x \in A : x \notin F\}$. By $\chi_A$ we donote the characteristic function of a set $A$.

We say that a condition holds almost every $x \in A$ if there exists a set $F \subset A$ with $\mu(F) = 0$ such that the condition holds for every $x \in A \setminus F$. We write $u \approx v$ if there exists a constant $C > 0$ so that $C^{-1}u \leq v \leq Cu$. By $⨍_A u d\mu = \mu(A)^{-1} \int_A u d\mu$ we denote the integral avarege of the function $u$ over the set $A$.

The set of $p$-integrable functions, $1 < p \leq \infty$, by measure $\mu$ in a set $A$ is denote by $L^p(A, \mu)$. In the Euclidean case we just write $L^p(A, m) = L^p(A)$. We say that a function $u$ belogs to a Sobolev space $M^{1,p}(A, d, \mu)$, $1 < p \leq \infty$, if $u \in L^p(A, \mu)$ and there exists a non-negative $g \in L^p(A, \mu)$ so that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{2.3}$$

for almost every $x, y \in A$. The function $g$ is a weak gradient of the function $u$. The Sobolev space $M^{1,p}(A, d, \mu)$ is equipped with a norm

$$\|u\|_{M^{1,p}(A,d,\mu)} = \|u\|_{L^p(A,\mu)} + \inf_g \|g\|_{L^p(A,\mu)},$$

where the infimum is taken over all weak gradients of the function $u$. These Sobolev spaces were defined by Hajłasz, [Haj96].

In the Euclidean case the classical Sobolev space on an open set $A \subset \mathbb{R}^n$ is denoted by $W^{1,p}(A)$, $1 \leq p \leq \infty$. It is equiped with a norm

$$\|u\|_{W^{1,p}(A)} = \|u\|_{L^p(A)} + \|\nabla u\|_{L^p(A)}.$$

Here $\nabla u$ is the weak gradient of the function $u$.

We say that a domain $\Omega \subset X$ is an $M^{1,p}$-extension domain, $1 < p \leq \infty$, if there exists a bounded extension operator

$$E : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu)$$

such that $Eu|_\Omega = u$ for every $u \in M^{1,p}(X, d, \mu)$. The extension operator is bounded if there exists a constant $C > 0$ such that

$$\|Eu\|_{M^{1,p}(X,d,\mu)} \leq C\|u\|_{M^{1,p}(\Omega,d,\mu)}$$

for every $u \in M^{1,p}(\Omega, d, \mu)$. In a similar way we define a $W^{1,p}$-extension domain in the Euclidean $n$-space $\mathbb{R}^n$.

Note that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, $1 < p \leq \infty$, then $M^{1,p}(\Omega, |\cdot|, m) = W^{1,p}(\Omega)$. This means that function spaces $M^{1,p}(\Omega, |\cdot|, m)$ and $W^{1,p}(\Omega)$ are the same as a set and the norms are equivalent. In particular it holds that $M^{1,p}(\mathbb{R}^n, |\cdot|, m) = W^{1,p}(\mathbb{R}^n)$. On the other hand, the reverse does not hold: A.S. Romanov has constructed a bounded domain $\Omega \subset \mathbb{R}^n$, which is not a $W^{1,p}$-extension domain, but $W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, m)$, [Rom98].

## 3. A plumpness condition

A bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the $A(c)$-condition, $0 < c < 1$, if for every $x \in \partial\Omega$ and every $0 < r < \operatorname{diam}(\Omega)$ there exists $y \in \Omega$ such that $B(y, cr) \subset \Omega \cap B(x, r)$, [HM97, p.225].

The $A(c)$-condition says that the domain $\Omega$ cannot be too "thin" near the boundary. The $A(c)$-condition for a bounded domain is equivalent with an $\alpha$-plump condition up to a constant. The $\alpha$-plump condition was defined by Martio and J. Väisälä, [MV88, p.310]. Another equivalent condition is the corkcrew condition, which is by D. Jerison and C. Kenig [JK82, 3.1, p.93].

Next we define a new plumpness condition, which in the Euclidean case is weaker than the $A(c)$-condition. Let

$$\Omega(h) = \{x \in \Omega |\operatorname{dist}(x, X \setminus \bar{\Omega}) > h\},$$

for every $h > 0$. We say that a bounded domain $\Omega \subset X$ satisfies the $A^*(\varepsilon, \delta)$-condition, $0 < \varepsilon < 1$ and $0 < \delta \leq 1$, if

$$\mu(B(z, r) \cap \Omega(\varepsilon r)) \geq \delta \mu(B(z, r))$$

for every $z \in \Omega$ and every $0 < r < \operatorname{diam}(\Omega)$.

3.1. **Lemma.** *Let $(\mathbb{R}^n, |\cdot|, m)$ be an Euclidean $n$-space. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, which satisfies the $A(c)$-condition. Then the domain $\Omega$ satisfies the $A^*\Big(\frac{c}{4}, (\frac{c}{4})^n\Big)$-condition.*

*Proof.* Fix $r \in (0, \operatorname{diam}(\Omega))$. Let $z \in \Omega$ be arbitrary. If $B^n(z, \frac{r}{2}) \subset \Omega$ then

$$\begin{aligned} m\big(B^n(z, r) \cap \Omega(\frac{c}{4}r)\big) &\geq m\big(B^n(z, (\frac{1}{2} - \frac{c}{4})r)\big) \\ &\geq \big(\frac{c}{4}\big)^n m\big(B^n(z, r)\big). \end{aligned}$$

If $B^n(z, \frac{r}{2}) \not\subset \Omega$ then there exists $x \in \partial\Omega \cap B^n(z, \frac{r}{2})$. By the $A(c)$-condition there exists a point $y \in \Omega$ such that $B^n(y, \frac{c}{2}r) \subset B^n(z, r)$. Hence we obtain

$$\begin{aligned} m\big(B^n(z, r) \cap \Omega(\frac{c}{4}r)\big) &\geq m\big(B^n(y, \frac{c}{4}r)\big) \\ &\geq \big(\frac{c}{4}\big)^n m\big(B^n(z, r)\big). \end{aligned}$$

This completes the proof of Lemma 3.1. □

It is clear that if a bounded domain in $\mathbb{R}^n$ satisfies the $A(c)$- or $A^*(\varepsilon, \delta)$-condition then it satisfies the following condition for some constant $C > 0$:

$$m(B(z, r) \cap \Omega) \geq C m(B(z, r)), \tag{3.2}$$

for every $z \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$.

P. Koskela has proved that if $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain with $n - 1 < p < \infty$, then it satisfies the condition (3.2), [Kos90, Theorem 6.5, p.28]. The condition (3.2) implies $m(\partial\Omega) = 0$, since $\partial\Omega$ can not contain points of $n$-density.

In Example 3.3 we show that there exists a bounded domain $\Omega \subset \mathbb{R}^2$ which satisfies the $A^*(\varepsilon,\delta)$-condition, but does satisfy neither the $A(c)$-condition nor the Whitney cube #-condition. This example is by R. Hurri, [Hur88, Remark 7.18, p.31]. More details about the Whitney cube #-condition can be found from [MV87, p.19]. In Example 3.4 we construct a bounded domain in $\mathbb{R}^2$ which satisfies the condition (3.2) but does not satisfy the $A^*(\varepsilon,\delta)$-condition.

3.3. **Example.** Let $d$ be the Euclidean metric $|\cdot|$ and $\mu$ the Lebesgue 2-measure $m$. Let $\Omega = Q \setminus C \subset \mathbb{R}^2$, where $Q$ is an open cube with a diameter $3\sqrt{2}$ and $C$ is a Cantor dust type cube construction in the plane, see [Hur88, Remark 7.18, p.31] for more details. Note that the Hausdorff dimension of the boundary $\partial\Omega$ is 2, hence by [HM97, Lemma 3, p.225] the domain $\Omega$ does not satisfy the $A(c)$-condition. Hurri has also proved that the domain $\Omega$ does not satisfy the Whitney cube #-condition with $2^{mk}$ for any $0 < m < 2$. Since $m(C) = 0$ it is easy to see that $\Omega$ satisfies the $A^*\big(\varepsilon, \frac{1-2\varepsilon}{\pi}\big)$-condition for every $0 < \varepsilon < \frac{1}{5}$.

3.4. **Example.** Let $d$ be the Euclidean metric $|\cdot|$ and $\mu$ the Lebesgue 2-measure $m$. Let

$$\Omega = (0,1)\times(0,1) \setminus \bigcup_{j=1}^{\infty} \big[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j\big] \times \big[0, \frac{1}{2}\big] \subset \mathbb{R}^2,$$

where $\varepsilon_j = \frac{1}{4^j}$. We denote by $\lfloor x \rfloor$ the largest natural number which is smaller that $x$. Let $r > 0$. The ball $B^2((1,0),r)$ intersects sets $[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j] \times [0, \frac{1}{2}]$ at least when $\frac{j-1}{j} \geq 1 - r$. Thus we assume that $j \geq \lfloor \frac{1}{r} \rfloor$. We obtain

$$\begin{aligned} m(\Omega \cap B^2((1,0),r)) &\geq \frac{1}{4}\pi r^2 - \sum_{j=\lfloor \frac{1}{r}\rfloor}^{\infty} r\frac{1}{4^j} \\ &= \frac{1}{4}\pi r^2 - r\frac{1}{4^{\lfloor \frac{1}{r}\rfloor}} \sum_{i=1}^{\infty}\frac{1}{4^i} \\ &= \frac{1}{4}\pi r^2 - r\frac{1}{4^{\lfloor \frac{1}{r}\rfloor}}\frac{1}{3} \geq \frac{1}{8}\pi r^2. \end{aligned}$$

Hence the domain $\Omega$ satisfies the condition (3.2).

The distance between sets $[\frac{j-1}{j}, \frac{j-1}{j} + \varepsilon_j] \times [0, \frac{1}{2}]$, when $j = i, i+1$, is less than $\frac{1}{i^2+i}$. The ball $B^2((1,0),r)$ intersects these sets at least when $\frac{i-1}{i} \geq 1 - r$ which implies that $i \geq \lfloor \frac{1}{r} \rfloor$. Assume that $i = \lfloor \frac{1}{r} \rfloor$. Let $0 < \varepsilon < 1$. Then $m(\Omega(\varepsilon r) \cap B^2((1,0),r)) = 0$ when $\frac{1}{i^2+i} \leq \varepsilon\frac{1}{i}$. This happens as soon as we assume that $i \geq \frac{1}{2\varepsilon} - 1$, which implies $r \leq \frac{2\varepsilon}{1-2\varepsilon}$. To see that the domain $\Omega$ does not satisfy the $A^*(\varepsilon,\delta)$-condition for any $0 < \varepsilon < 1$ we need only take a sequence on points $(x_i, y_i) \in \Omega$ with $(x_i, y_i) \to (0, 1)$ as $i \to \infty$.

## 4. Main theorem

We recall the following well known Whitney type covering lemma, [MS79a, Lemma 2.9].

4.1. **Lemma.** *Suppose that $E$ is an open set of finite measure in $(X, d, \mu)$, $E \neq X$. Let $r(x) = \frac{1}{4}d(x, X \setminus E)$. Then there exist $N \geq 1$ and a sequence $\{x_i\}_{i=1}^{\infty}$ such that by denoting $r(x_i) = r_i$, the following properties hold:*

(1) *$E = \bigcup_{i=1}^{\infty} B(x_i, r_i)$.*
(2) *$B(x_i, 2r_i) \subset E$ for every $i = 1, 2, \ldots$.*
(3) *The balls $B(x_i, \frac{1}{4}r_i)$ are pairwise disjoint.*
(4) *For every $i = 1, 2\ldots$ holds that if $x \in B(x_i, r_i)$ then*
$$2r_i \leq d(x, X \setminus E) \leq 6r_i.$$
(5) *For every $i = 1, 2, \ldots$ there exists $y_i \in X \setminus E$ such that $d(x_i, y_i) \leq 6r_i$*
(6) *$\sum_{i=1}^{\infty} \chi_{B(x_i, r_i)}(x) \leq N$ for every $x \in X$.*

It is easy to prove that balls in the above Whitney decomposition have the following properties:

(7) *There exists a constant $C > 0$ such that if $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset$ then $C^{-1}r_i \leq r_j \leq Cr_i$.*
(8) *Suppose that $B$ is a Whitney ball, that is a ball from the Whitney decomposition of $E$, then there is at most $C$ Whitney balls which intersect the ball $B$. The constant $C$ depends only on the doubling constant.*

By inequality (2.2) the following condition holds:

(9) *There exists a constant $C > 0$ such that if $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset$ then $C^{-1}\mu(B(x_i, r_i)) \leq \mu(B(x_j, r_j)) \leq C\mu(B(x_i, r_i))$. The constant $C$ depends only on the doubling constant.*

The following lemma is a simple consequence of the inequality (2.2).

4.2. **Lemma.** *Let $\lambda > 0$ be fixed. Let $B(x_1, r)$ and $B(x_2, r)$ be two balls in $(X, d, \mu)$, with $d(x_1, x_2) = \lambda r$. There exists a constant $C > 0$ which depends only on $\lambda$ and the doubling constant, such that*
$$C^{-1}\mu(B(x_2, r)) \leq \mu(B(x_1, r)) \leq C\mu(B(x_2, r)).$$

*The construction of the extension operator.* Assume that $\Omega$ is a bounded domain in a homogeneous type metric space $(X, d, \mu)$ with $\mu(X) < \infty$ and $\Omega \neq X$. Assume that $\Omega$ satisfies the $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega)=0$. Let $\mathcal{W}_0$ be the Whitney decomposition of $X\setminus\bar\Omega$. Let $\mathcal{W}$ be a collection of balls $B_i\in\mathcal{W}_0$ with $\operatorname{diam}(B_i)\le\min\{1,\operatorname{diam}(\Omega)\}$.

For every Whitney ball $B_i=B(x_i)=B(x_i,r_i)\in\mathcal{W}$ we pick up $x_i^*\in\Omega$ such that $\frac{6}{4}\operatorname{dist}(x_i,\Omega)\ge\operatorname{dist}(x_i,x_i^*)$. We set

$$U(x_i^*)=B(x_i^*,r_i)\cap\Omega(\varepsilon r_i),$$

here $\Omega(\varepsilon r_i)=\{x\in\Omega|\operatorname{dist}(x,X\setminus\bar\Omega)>\varepsilon r_i\}$. By the $A^*(\varepsilon,\delta)$-condition and Lemma 4.2 there exists a constant $C>0$ so that

$$C^{-1}\mu(B(x_i))\le\mu(U(x_i^*))\le C\mu(B(x_i))$$

for every $i=1,2,3\ldots$. We set

$$\tilde u(x_i)=\fint_{U(x_i^*)}u(z)d\mu(z),$$

and hence $\tilde u$ is defined on all center points of Whitney balls from the collection $\mathcal{W}$. Next we extend $\tilde u$ in a piecewise linear way onto $X\setminus\bar\Omega$. Let $\phi_i$, $i=1,2,3,\ldots$, be a partition of unity such that $\operatorname{spt}\phi_i=\overline{\{x\in X:\phi_i(x)\neq 0\}}\subset B(x_i,r_i)$, $\phi_i(x_i)=1$, $0\le\phi_i\le 1$, every $\phi_i$ is Lipschitz continuous with a Lipschitz constant $\frac{C}{r_i}$ here the constant $C$ depends only on the doubling constant and

$$\sum_{i=1}^{\infty}\phi_i(x)=1$$

for all $x\in X\setminus\bar\Omega$. Now we set

$$\tilde u(x)=\sum_{i=1}^{\infty}\phi_i(x)\tilde u(x_i).$$

It is easy to see that $\tilde u$ is defined and continuous in every point in $X\setminus\bar\Omega$. We set

$$Eu(x)=\begin{cases}u(x), & \text{if } x\in\Omega;\\ \tilde u(x), & \text{if } x\in X\setminus\bar\Omega.\end{cases}$$

Since $\mu(\partial\Omega)=0$, $Eu$ is defined almost everywhere in $X$.

We write for every $B_i\in\mathcal{W}$

$$\fint_{U^*(B_i)}|u(z)|d\mu(z)=\max_{B_i\cap B_j\neq\emptyset}\fint_{U(x_j^*)}|u(z)|d\mu(z),$$

where $U(x_j^*)=B(x_j^*,r_j)\cap\Omega(\varepsilon r_j)$. The maximum exists, since for every $B_i\in\mathcal{W}$ there are only uniformly bounded number of balls $B_j\in\mathcal{W}$ with $B_i\cap B_j\neq\emptyset$. For every $x\in X\setminus\bar\Omega$ we write

$$\fint_{U^*(x)}g(z)d\mu(z)=\max_{x\in B_i,B_i\cap B_j\neq\emptyset}\fint_{U(x_j^*)}g(z)d\mu(z).$$

4.3**. Lemma.** *There exists natural numbers $N_1$ and $N_2$ such that for every $z \in \Omega$*

$$\sum_{i=1}^{\infty} \chi_{U(x_i^*)}(z) \leq N_1 \tag{4.4}$$

*and*

$$\sum_{i=1}^{\infty} \chi_{U^*(B_i)}(z) \leq N_2. \tag{4.5}$$

*Proof.* Fix $z \in \Omega$ and assume that $z$ belongs to the sets $U(x_j^*)$, $j = 1, 2, \ldots$. By the $A^*(\varepsilon, \delta)$-condition and the construction of the extension operator there exists a constant $C > 0$ such that for every $j = 1, 2, \ldots$

$$C^{-1} d(z, X \setminus \bar{\Omega}) \leq r_j \leq C d(z, X \setminus \bar{\Omega}).$$

On the other hand $d(x_j, z) < 7r_j$. The balls $\frac{1}{4}B_j$ are disjoint, hence we obtain by the inequality (2.2) that the point $z$ can not belong to more than

$$N_1 = \frac{\mu(B(z, 8Cd(z, X \setminus \bar{\Omega})))}{\mu(B(y, \frac{1}{4}C^{-1}d(z, X \setminus \bar{\Omega})))} \leq q^{-1}(32C^2)^Q$$

sets $U(x_j^*)$, here $y \in B(z, 7Cd(z, X \setminus \bar{\Omega}))$. Inequality (4.4) is proved.

In a Whitney decomposition every ball $B_i$ intersects only uniformly bounded number of balls $B_j$, $j = 1, \ldots, N$. We obtain $N_2 = N \cdot N_1$. This completes the proof of Lemma 4.3. □

4.6**. Theorem.** *Let $(X, d, \mu)$ be a metric space of homogeneous type and $\mu(X) < \infty$. Assume that $\Omega \subset X$ is a bounded domain, $\Omega \neq X$, which satisfies the $A^*(\varepsilon, \delta)$-condition and $\mu(\partial\Omega) = 0$. Then for every $1 < p \leq \infty$, the domain $\Omega$ is an $M^{1,p}$-extension domain: there exists a bounded linear extension operator*

$$E : M^{1,p}(\Omega, d, \mu) \to M^{1,p}(X, d, \mu)$$

*such that $Eu|_{\Omega} = u$ for every $u \in M^{1,p}(\Omega, d, \mu)$.*

4.7. *Remark.* Note that we do not need the assumption $\mu(X) < \infty$ in the proof of Theorem 4.6. We need this assumptation only in the construction of the Whitney type covering lemma. Our proof works also if we replace balls by cubes. Thus in the Euclidean case we can use the standard Whitney decomposition, see [Ste70, Theorem 1, p.167], and assume that $X = \mathbb{R}^n$. The proof of Theorem 4.6 is based on the proof of Hajłasz and Martio, [HM97, Theorem 9, p.240].

*Proof of Theorem 4.6.* We prove Theorem 4.6 in three steps. First we prove that $\tilde{u}$ is $p$-integrable, then we prove that it has a weak gradient and finally we prove that the weak gradient is p-integrable.

*Step 1.* We show that $\tilde{u} \in L^p(X,\mu)$ and $\|\tilde{u}\|_{L^p(X\setminus\bar{\Omega})} \le C\|u\|_{L^p(\Omega)}$. Assume first that $1 < p < \infty$. Let $B \in \mathcal{W}$. By Hölder's inequality we obtain for every $x \in B$

$$
\begin{aligned}
|\tilde{u}(x)| &\le \fint_{U^*(B)} |u(z)|d\mu(z) \\
&\le \mu(U^*(B))^{-1}\Big(\int_{U^*(B))} d\mu\Big)^{\frac{p-1}{p}}\Big(\int_{U^*(B)} |u|^p d\mu\Big)^{\frac{1}{p}}.
\end{aligned}
\tag{4.8}
$$

By the $A^*(\varepsilon,\delta)$-condition and Lemma 4.2 $\mu(B) \approx \mu(U^*(B))$, hence we obtain

$$
\begin{aligned}
\int_B |\tilde{u}|^p d\mu &\le \int_B \mu(U^*(B))^{-1}\int_{U^*(B)} |u(z)|^p d\mu(z) d\mu \\
&\le C\int_{U^*(B)} |u|^p d\mu.
\end{aligned}
$$

The condition (4.5) implies

$$
\begin{aligned}
\int_{X\setminus\bar{\Omega}} |\tilde{u}|^p d\mu &\le \sum_{i=1}^{\infty}\int_{B_i} |\tilde{u}|^p d\mu \le \sum_{i=1}^{\infty} C\int_{U^*(B_i)} |u|^p d\mu \\
&\le CN_2\int_{\Omega} |u|^p d\mu.
\end{aligned}
$$

If $p = \infty$, as in the estimate of (4.8) we obtain

$$
\sup_{x\in B} |\tilde{u}(x)| \le \fint_{U^*(B)} |u(z)|d\mu(z) \le \operatorname*{ess\,sup}_{z\in\Omega} |u(z)|,
$$

for every $B \in \mathcal{W}$, and hence $\|\tilde{u}\|_{L^\infty(\mathbb{R}^n\setminus\bar{\Omega})} \le \|u\|_{L^\infty(\Omega)}$. This completes the step 1.

*Step 2.* We show that the function $\tilde{u}$ satisfies the inequality

$$
|Eu(x_1) - Eu(x_2)| \le Cd(x_1,x_2)(G(x_1) + G(x_2))
\tag{4.9}
$$

for almost every $x_1, x_2 \in X$, here

$$
G(x) = \begin{cases} \fint_{U^*(x)} g d\mu, & \text{if } x \in X\setminus\bar{\Omega} \\ g(x), & \text{if } x \in \Omega \end{cases}
\tag{4.10}
$$

and $g$ is a weak gradient of $u$.

Assume that $y_1 \in B_1 \in \mathcal{W}$, $y_2 \in B_2 \in \mathcal{W}$ and $\operatorname{dist}(y_1,y_2) \le \frac{1}{2}\max\{\operatorname{diam}(B_1), \operatorname{diam}(B_2)\}$. This implies $r_1 \approx r_2$. Let $B_1', B_2' \in \mathcal{W}$ be such that

$$
|\tilde{u}(x_1') - \tilde{u}(x_2')| = \max_{(j,i)} |\tilde{u}(x_j) - \tilde{u}(x_i)|.
$$

Here the maximum is taken over those $(j,i)$ for which $B_j, B_i \in \mathcal{W}$, $B_j \cap B_1 \neq \emptyset$ and $B_i \cap B_2 \neq \emptyset$. We obtain

$$\begin{aligned}
|\tilde{u}(y_1) - \tilde{u}(y_2)| &\leq |\sum_i \phi_i(y_1)\tilde{u}(x_i) - \sum_i \phi_i(y_2)\tilde{u}(x_i)| \\
&= |\sum_i \phi_i(y_1)\tilde{u}(x_i) - \sum_i \phi_i(y_1)\tilde{u}(x_1) \\
&- \sum_i \phi_i(y_2)\tilde{u}(x_i) + \sum_i \phi_i(y_2)\tilde{u}(x_1)| \\
&\leq \sum_i |\tilde{u}(x_i) - \tilde{u}(x_1)||\phi_i(y_1) - \phi_i(y_2)| \\
&\leq |\tilde{u}(x_1') - \tilde{u}(x_2')| \sum_i |\phi_i(y_1) - \phi_i(y_2)|
\end{aligned}$$

Each function $\phi_i$ is a Lipschitz continuous function with a constant $\frac{c}{r_i}$. By Lemma 4.1 (6) each $z \in X$ belongs at most $N$ cubes $B_i \in \mathcal{W}$. We write $r = \min\{r_i : B_i \in \mathcal{W}, B_1 \cap B_i \neq \emptyset \text{ or } B_2 \cap B_i \neq \emptyset\}$. There exists a constant $C > 0$ such that $r \approx r_1 \geq C \operatorname{dist}(x_1', x_2')$. This yields

$$\begin{aligned}
|\tilde{u}(y_1) - \tilde{u}(y_2)| &\leq 2N|\tilde{u}(x_1') - \tilde{u}(x_2')| \frac{C}{r} \operatorname{dist}(y_i, y_2) \\
&\leq Cd(y_1, y_2) \frac{\operatorname{dist}(x_1', x_2')}{r} \Big( \fint_{U((x_1')^*)} g d\mu + \fint_{U((x_2')^*)} g d\mu \Big) \\
&\leq Cd(y_1, y_2) \Big( \fint_{U^*(y_1)} g d\mu + \fint_{U^*(y_2)} g d\mu \Big).
\end{aligned}$$

Assume that $y_1 \in B_1 \in \mathcal{W}$, $y_2 \in B_2 \in \mathcal{W}$ and $\operatorname{dist}(y_1, y_2) > \frac{1}{2} \max\{\operatorname{diam}(B_1), \operatorname{diam}(B_2)\}$. We obtain

$$|\tilde{u}(y_1) - \tilde{u}(y_2)| \leq |\tilde{u}(y_1) - \tilde{u}(x_1)| + |\tilde{u}(x_1) - \tilde{u}(x_2)| + |\tilde{u}(x_2) - \tilde{u}(y_2)|$$

and this case follows by the previous case.

Assume that $x_1 \in X \setminus \bar{\Omega}$, $x_1$ is a center of a ball $B \in \mathcal{W}$, and $x_2 \in \Omega$. We obtain

$$\begin{aligned}
|\tilde{u}(x_1) - u(x_2)| &= |\fint_{U(x_1^*)} u d\mu - u(x_2)| \leq \fint_{U(x_1^*)} |u(z) - u(x_2)| d\mu(z) \\
&\leq \fint_{U(x_1^*)} d(z, x_2)(g(z) + g(x_2)) d\mu(z) \\
&\leq Cd(x_1, x_2) \Big( \fint_{U(x_1^*)} g(z) d\mu(z) + g(x_2) \Big).
\end{aligned}$$

The general case $x_1 \in X \setminus \bar{\Omega}$ and $x_2 \in \Omega$ follows from the previous cases. This completes the step 2.

*Step 3.* We show that $\|G\|_{L^p(X\setminus\bar\Omega)} \le C\|g\|_{L^p(\Omega)}$. Here $g$ is a weak gradient of the function u and $G$ is the weak gradient of the function $Eu$, see the inequality (4.9) and the definition (4.10).

Let $1 < p < \infty$. We use Hölder's inequality to obtain

$$\begin{aligned}
&\int_{X\setminus\bar\Omega} |G(z)|^p d\mu(z) \le \sum_{i=1}^{\infty} \int_{B_i} |G(z)|^p d\mu(z) \\
&= \sum_{i=1}^{\infty} \int_{B_i} \Big( \fint_{U^*(z)} g(x) d\mu(x)\Big)^p d\mu(z) \\
&\le \sum_{i=1}^{\infty} \int_{B_i} \Big(\mu(U^*(z))^{-1}\Big( \int_{U^*(z)} d\mu\Big)^{\frac{p-1}{p}} \Big( \int_{U^*(z)} g(x)^p d\mu(x)\Big)^{\frac{1}{p}}\Big)^p d\mu(z) \\
&= \sum_{i=1}^{\infty} \int_{B_i} \mu(U^*(z))^{-1} \int_{U^*(z)} g(x)^p d\mu(x) d\mu(z).
\end{aligned}$$

We write

$$\begin{aligned}
I(i) = \{ j \in \mathbb{N} : B_j \in \mathcal{W} &\text{ and there exists } B \in \mathcal{W} \text{ such that} \\
&B_i \cap B \neq \emptyset \text{ and } B \cap B_j \neq \emptyset \}.
\end{aligned}$$

The fact that $\mu(B_i) \approx \mu(U(x_j^*))$ for every $j \in I(i)$ yields

$$\begin{aligned}
\int_{X\setminus\bar\Omega} |G(z)|^p d\mu(z) &\le \sum_{i=1}^{\infty} \int_{B_i} \sum_{j\in I(i)} \mu(U(x_j^*))^{-1} \int_{U(x_j^*)} g(x)^p d\mu(x)\ d\mu(z) \\
&\le C \sum_{i=1}^{\infty} \sum_{j\in I(i)} \int_{U(x_j^*)} g(x)^p d\mu(x).
\end{aligned}$$

Every Whitney ball $B(x_i)$ intersects at most $N$ Whitney balls. By the condition (4.4) every point $x \in \Omega$ belongs at most than $N_1$ sets $U(x_i^*)$, $i = 1, 2, \ldots$. This implies

$$\sum_{i=1}^{\infty} \sum_{j\in I(i)} \chi_{U(x_j^*)}(z) \le N^2 \cdot N_1$$

for every $z \in \Omega$. We obtain

$$\int_{X\setminus\bar\Omega} G^p d\mu \le C \int_\Omega g^p d\mu.$$

If $p = \infty$, we obtain

$$\sup_{x\in X\setminus\bar\Omega} G(x) = \sup_{x\in X\setminus\bar\Omega} \fint_{U^*(x)} g d\mu \le \operatorname*{ess\,sup}_{x\in\Omega} g.$$

This completes the step 3.

By the steps 1 and 3 we obtain

$$\begin{aligned}\|Eu\|_{M^{1,p}(X,d,\mu)} &= \|\tilde{u}\|_{M^{1,p}(X\setminus\bar{\Omega},d,\mu)} + \|u\|_{M^{1,p}(\Omega,d,\mu)}\\ &\leq C\|u\|_{M^{1,p}(\Omega,d,\mu)}.\end{aligned}$$

This together with the step 2 completes the proof of Theorem 4.6. $\square$

We assume that $X$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$, $d$ is the Euclidean metric $|\cdot|$ and $\mu$ is the Lebesgue $n$-measure $m$.

If $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,p}$-extension domain, $1 < p \leq \infty$, then $M^{1,p}(\Omega, |\cdot|, m) = W^{1,p}(\Omega)$, [Haj96, Theorem 1, p.405]. By Theorem 4.6 and Remark 4.7 we obtain the following characterization for $W^{1,p}$-extension domains.

4.11. **Theorem.** *Let $1 < p \leq \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain which satisfies the $A^*(\varepsilon, \delta)$-condition. Then the following three conditions are equivalent:*

(1) *$\Omega$ is a $W^{1,p}$-extension domain.*
(2) *$M^{1,p}(\Omega, |\cdot|, m) = W^{1,p}(\Omega)$.*
(3) *The operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ is bounded.*

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