# TRACES AND SOBOLEV EXTENSION DOMAINS

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ABSTRACT. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. We show that if there exists a bounded trace operator $T : W^{1,p}(\Omega) \to B^p_{1-\alpha}(\partial\Omega)$, $1 < p < \infty$ and $0 < \alpha \le \frac{n-m}{p}$, so that $Tu|_{\partial\Omega} = u$ for every $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, and $(1-\lambda)$-Hölder continuous functions are dense in $W^{1,p}(\Omega)$, $0 \le \lambda < n-m$, then the domain $\Omega$ is a $W^{1,p}$-extension domain.

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## 1. Introduction

We say that a domain $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain if there exists a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$, so that $Eu|_{\Omega} = u$ for every $u \in W^{1,p}(\Omega)$. In a filosofical point of view when we extend function it does not matter how irregular the function is inside the domain as soon as it is smooth enough on the boundary. Thus it is natural to try characterize the extension property by traces.

A. Jonsson showed that a trace of a Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ in an $m$-set $F \subset \mathbb{R}^n$, $0 < m < n$, belongs to a Besov space $B^p_{1-\frac{n-m}{p}}(F)$, [Jon79, Theorem 3, p.4]. A bounded set $A \subset \mathbb{R}^n$ is an $m$-set if

$$C^{-1} r^m \le H^m(B^n(x,r) \cap A) \le C r^m$$

for every $x \in A$ and every $r$, $0 < r \le \operatorname{diam}(A)$. There also exists a bounded extension operator $B^p_{1-\frac{n-m}{p}}(F) \to W^{1,p}(\mathbb{R}^n)$, [Jon79, Theorem 3, p.4]. However, the problem is that the extension of $u \in W^{1,p}(\Omega)$ should have weak derivatives across the boundary.

Let $1 < p < \infty$ and $0 < \alpha \le \frac{n-m}{p}$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. Assume that $(1-\lambda)$-Hölder continuous functions are dense in $W^{1,p}(\Omega)$, $0 \le \lambda < n-m$. Assume that there exists a bounded linear trace operator

$$T : W^{1,p}(\Omega) \to B^p_{1-\alpha}(\partial\Omega),$$

so that $Tu|_{\partial\Omega} = u$ for every $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then we show that the domain $\Omega$ is a $W^{1,p}$-extension domain.

Jonsson's theorem implies the following result. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,p}$-extension domain, $p > n-1$, and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. Then there exists a bounded trace operator

$$T : W^{1,p}(\Omega) \to B^p_{1-\frac{n-m}{p}}(\partial\Omega),$$

so that $Tu(x) = \lim_{r\to 0} \fint_{B^n(x,r)\cap\Omega} u(z)\,dH^n(z)$.

We obtain a characterization for Sobolev extension domains. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. Then the following two conditions are equivalent for every $n-1 < p < \infty$:

(1) The domain $\Omega$ is a $W^{1,p}$-extension domain.
(2) There exists a bounded linear trace operator

$$T : W^{1,p}(\Omega) \to B^p_{1-\frac{n-m}{p}}(\partial\Omega),$$

so that $Tu(x) = \lim_{r\to 0} \fint_{B^n(x,r)\cap\Omega} u(z)\,dH^n(z)$ and and Lipschitz continuous functions are dense in $W^{1,p}(\Omega)$.

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## 2. Notations and preliminaries

Throughout this paper we let $\Omega$ be a domain of Euclidean n-space $\mathbb{R}^n$, $n \ge 2$. The norm of $x \in \mathbb{R}^n$ is $|x| = \sqrt{x_1^n + ... + x_n^2}$. By an open ball centered at $x$ and with radius $r > 0$ we mean the set $B^n(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. By an open cube we mean the set $Q(x,r) = \{y \in \mathbb{R}^n : \max_{j=1,..,n} |y_j - x_j| < r\}$ and we write $kQ$ for the cube with the same center as $Q$ and dilated by a factor $k > 0$. We let $\operatorname{diam}(A) = \sup\{|x-y| : x, y \in A\}$ denote the diameter of a set $A$. The boundary of a domain $\Omega$ is denoted by $\partial\Omega$. Let $|A|$ be a Lebesgue $n$-measure of a set $A$. Let $H^m$ be the $m$-dimensional Hausdorff measure. We write $u \approx v$ if there exists a constant $C > 0$ so that $C^{-1}u \le v \le Cu$. By $\fint_A u(x)\,dx$ we denote the integrate average of the function $u$ over the set $A$.

We say that a bounded set $F \subset \mathbb{R}^n$ is an $m$-set, $0 < m \le n$, if there exists a constant $C > 0$ so that

$$C^{-1} r^m \le H^m(B^n(x,r) \cap F) \le C r^m$$

for every $x \in F$ and every $r$, $0 < r \le \operatorname{diam}(F)$. A typical example of $m$-sets is the boundary of Koch's snowflake domain with $m = \frac{\ln 4}{\ln 3}$. For more details about m-sets we refer to [JW84, Chapter II].

Let $F \subset \mathbb{R}^n$ be a closed set. Let $\mathcal{W}_0$ be a Whitney decomposition of $\mathbb{R}^n \setminus F$, [Ste70, Theorem 1, p.167]. We say that a set $F$ satisfies a Whitney cube #-condition for $\lambda$, $0 < \lambda \le n$, if there exists a constant $M > 0$ and an integer $k_0$ so that $N_k \le M 2^{\lambda k}$ for every $k \ge k_0$. Here $N_k$ is the number of cubes from the $\mathrm{k}^{\text{th}}$ generation in $\mathcal{W}_0$. Note that if $Q \in \mathcal{W}_0$ is a Whitney cube from k$^{\text{th}}$ generation, then $\operatorname{diam}(Q) \approx 2^{-k}$. Every bounded domain satisfies the Whitney cube #-condition for $\lambda = n$. For more details about Whitney cube #-condition we refer to [MV87, 2.1, p.18].

By $C(A)$ we denote the set of continuous functions in a set $A$. By $C^{0,\lambda}(A)$, $0 < \lambda \leq 1$, we denote the set of $\lambda$-Hölder continuous functions: $u \in C^{0,\lambda}(A)$ if there exists a constant $C > 0$ so that

$$|u(x) - u(y)| \leq C|x - y|^{\lambda}$$

for every $x, y \in A$. If $\lambda = 1$ we say that function $u$ is Lipschitz continuous.

By $L^p(A)$, $1 \leq p \leq \infty$, we denote the class of all $p$-integrable functions in a set $A \subset \mathbb{R}^n$. By $L^1_p(A)$, $1 \leq p \leq \infty$, we denote the class of all functions whose first distrubutional derivatives are in $L^p(A)$. The classical Sobolev space in a set $A$ is denoted by $W^{1,p}(A) = L^p(A) \cap L^1_p(A)$, $1 \leq p \leq \infty$. We equip $W^{1,p}(A)$ with a norm $\|u\|_{W^{1,p}(A)} = \|u\|_{L^p(A)} + \|\nabla u\|_{L^p(A)}$, where $\nabla u$ is the distributional gradient.

By $M^{1,p}(\Omega)$, $1 < p \leq \infty$, we denote a Hajłasz space: $u \in M^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and there exists a non-negative $g \in L^p(\Omega)$ so that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y))$$

for almost every $x, y \in \Omega$. For an arbitrary domain $\Omega$ holds $M^{1,p}(\Omega) \subset W^{1,p}(\Omega)$. If a domain $\Omega$ is a $W^{1,p}$-extension domain then holds $M^{1,p}(\Omega) = W^{1,p}(\Omega)$, [Haj96, Theorem 1, p.405]. These spaces were defined by P. Hajłasz, [Haj96].

Let $F \subset \mathbb{R}^n$ be an $m$-set, $0 < m \leq n$. The Besov space $B^p_{1-\alpha}(F)$ on an $m$-set $F$ , $1 < p < \infty$ and $0 < \alpha < 1$, consists of all functions $u \in L^p(F)$ such that for every $j = 0, 1, 2, \ldots$ holds

$$\Big(2^{jm} \iint_{|x-y|<2^{-j}} |u(x) - u(y)|^p dH^m(x) dH^m(y)\Big)^{\frac{1}{p}} \leq 2^{-j(1-\alpha)} a_j, \tag{2.1}$$

with some sequence $(a_j)$ of positive numbers with $\sum_{j=0}^{\infty} a_j^p < \infty$. The Besov space is equipped with a norm

$$\|u\|_{B^p_{1-\alpha}(F)} = \|u\|_{L^p(F)} + \inf \Big(\sum_{i=0}^{\infty} a_i^p\Big)^{\frac{1}{p}},$$

where the infimum is taken over all possible sequences $(a_j)$. An equivalent norm is obtained by taking the double integral in (2.1) over $|x - y| < C2^{-j}$, where $C > 0$ is a constant, [JW84, Remark 1, p.123].

By [JW84, Definition 1, p.122; Proposition 3, p.125] the Besov norm is equivalent to the following norm in an $m$-set

$$\|u\|_{L^p(F)} + \Big(\iint_{|x-y|<1} \frac{|u(x) - u(y)|^p}{|x - y|^{m+(1-\alpha)p}} dH^m(x) dH^m(y)\Big)^{\frac{1}{p}}.$$

The Besov space $B^p_{1-\alpha}(\mathbb{R}^n)$ coincides with the Slobodeckiĭ space $W^{1-\alpha,p}(\mathbb{R}^n)$. More details about classical Besov spaces and Besov spaces on an $m$-set can be found for example from [KJF77, Sections 8.3 and 8.4] and [JW84, Chapter V].

We say that a domain $\Omega \subset \mathbb{R}^n$ is a $W^{1,p}$-extension domain, if there exists a bounded linear extension operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),$$

that $Eu|_\Omega = u$, and a constant $C$, $0 < C < \infty$, that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$.

## 3. Extension operator

First we prove that $m$-sets satisfy the Whitney cube #-condition for $m$. Then we construct an extension operator using the Whitney decomposition. When we show that this extension operator is bounded, we need the Whitney cube #-condition.

3.1. **Lemma.** *Let $F \subset \mathbb{R}^n$ be a bounded $m$-set, $0 < m \le n$. Then $F$ satisfies the Whitney cube #-condition for $m$.*

*Proof.* By O. Martio and M. Vuorinen [MV87, Theorem 3.11, p.25] it is enough to prove that $\lim_{r\to 0}\sup H_m(F,r) < \infty$; where

$$H_m(F,r) = \inf\Big\{kr^m : F \subset \bigcup_{i=1}^{k} B^n(x_i,r)\Big\}$$

is an $m$-dimensional Hausdorff type premeasure.

Let

$$\begin{aligned} P(F,r) = \max\{k|\ &\text{there are disjoint balls } B^n(x_i,r), i=1,\dots,k, \\ &\text{with } x_i \in F\}. \end{aligned}$$

We obtain

$$\begin{aligned} H_m(F,r) &\le P\Big(F,\frac{r}{2}\Big)r^m = 2^m P\Big(F,\frac{r}{2}\Big)\Big(\frac{r}{2}\Big)^m \\ &\le C2^m \sum_{i=1}^{P(F,\frac{r}{2})} H^m\Big(B^n\big(x_i,\frac{r}{2}\big)\cap F\Big) = C2^m H^m(F) \end{aligned}$$

and hence

$$\lim_{r\to 0}\sup H_m(F,r) \le C2^m H^m(F) < \infty.$$

This completes the proof of Lemma 3.1. □

*The construction of an extension operator.* Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. Assume that there exists a bounded trace operator $T : W^{1,p}(\Omega) \to B^p_{1-\alpha}(\partial\Omega)$, $1 < p < \infty$ and $0 < \alpha < 1$. Let $\mathcal{W}_0$ be a Whitney decomposition of $\mathbb{R}^n \setminus \bar\Omega$. Let $\mathcal{W}$ be a family of those cubes $Q_i \in \mathcal{W}_0$, for which

$\operatorname{diam}(Q_i) < \min\{1, \operatorname{diam}(\Omega)\}$ for every $i$. Let $(\phi_i)$ be a partition of the unity corresponding to the collection $\mathcal{W}$ so that each $\phi_i \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\phi_i) \subset \frac{9}{8}Q_i$ and $|\nabla\phi_i| \le C \operatorname{diam}(Q_i)^{-1}$. For each cube $Q_i \in \mathcal{W}$ we pick up $x_i \in \partial\Omega$ so that $\operatorname{dist}(Q_i, \partial\Omega) = \operatorname{dist}(Q_i, x_i)$. Let

$$a_i = \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu(y)\,dH^m(y) \tag{3.2}$$

where $Tu \in B^p_{1-\alpha}(\partial\Omega)$ and $r_i = \operatorname{diam}(Q_i)$. We define an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$,

$$Eu(x) = \begin{cases} u(x), & \text{for} \quad x \in \Omega; \\ Tu(x), & \text{for} \quad x \in \partial\Omega; \\ \sum_{i=1}^\infty a_i\phi_i(x), & \text{for} \quad x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Note that the extension operator is the same for every $p$ and $\alpha$. It is also easy to see that $Eu \in C^\infty(\mathbb{R}^n \setminus \bar{\Omega})$.

For technical reasons we have chosen slightly different extension operator than Jonsson [Jon79, Theorem 3, p.4]. For every cube we first pick up the nearest point $x_i$ from the boundary and then set the value of $a_i$, see (3.2). Jonsson just set

$$a_i^* = \fint_{B^n(z_i,Cr_i)\cap\partial\Omega} Tu(y)\,dH^m(y),$$

where $z_i$ is the center of the cube $Q_i \in \mathcal{W}_0$, $r_i = \operatorname{diam}(Q_i)$ and $C > 0$ is a constant.

3.3. **Lemma.** *Let $1 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. If there exists a bounded trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, then there exists a constant $0 < C < \infty$ such that*

$$\|Eu\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$$

*for every $u \in W^{1,p}(\Omega)$.*

*Proof.* Let $x \in \frac{9}{8}Q_0$ for some $Q_0 \in \mathcal{W}$. Hence we obtain

$$|Eu(x)| = |\sum_{i=1}^\infty a_i\phi_i(x)| \le \sup_i |a_i|,$$

where the supremum is taken over all those cubes $Q_i$ for which $Q_i \cap Q_0 \neq \emptyset$. Since $\partial\Omega$ is an $m$-set and $r_i = \operatorname{diam}(Q_i)$, we obtain by Hölder's inequality

$$\begin{aligned}
\sup_i |a_i| &\le \sup_i \fint_{B^n(x_i,r_i)\cap\partial\Omega} |Tu|\,dH^m \\
&= \sup_i \frac{1}{H^m(B^n(x_i,r_i)\cap\partial\Omega)} \int_{B^n(x_i,r_i)\cap\partial\Omega} |Tu|\,dH^m \\
&\le \sup_i C \operatorname{diam}(Q_i)^{-m} H^m(B^n(x_i,r_i)\cap\partial\Omega)^{1-\frac{1}{p}} \\
&\cdot \Big( \int_{B^n(x_i,r_i)\cap\partial\Omega} |Tu|^p dH^m \Big)^{\frac{1}{p}} \\
&\le C \operatorname{diam}(Q_0)^{-\frac{m}{p}} \Big( \int_{B^n(x_0,Cr_0)\cap\partial\Omega} |Tu|^p dH^m \Big)^{\frac{1}{p}}.
\end{aligned}$$

Here $C > 0$ is a constant so that $B^n(x_i,r_i) \subset B^n(x_0,Cr_0)$ for every $i$, $Q_i \cap Q_0 \neq \emptyset$. It is possible to choose this constant, because for every two touching cubes the distance of corresponding points in boundary is comparable with the diameter of cubes. Note that $\mathcal{W}$ is a collection of Whitney cubes with $\operatorname{diam}(Q_i) < \min\{1, \operatorname{diam}(\Omega)\}$ for every $i = 1,2,\ldots$. Let $N_k$ be a number of cubes from the $\mathrm{k}^{\mathrm{th}}$ generation in the Whitney decomposition. We assume that the Whitney cubes belong to $\mathcal{W}$, when $k \ge k_0$. This implies

$$\begin{aligned}
\int_{R^n\setminus\bar\Omega} |Eu|^p dH^n &\le \sum_{k=k_0}^{\infty}\sum_{i=1}^{N_k} \int_{Q_i} |Eu|^p dH^n \\
&\le \sum_{k=k_0}^{\infty}\sum_{i=1}^{N_k} C 2^{-k(-m)} \int_{B^n(x_i,Cr_i)\cap\partial\Omega} |Tu|^p dH^m \cdot |Q_i| \\
&\le \sum_{k=k_0}^{\infty}\sum_{i=1}^{N_k} C 2^{-k(n-m)} \int_{B^n(x_i,Cr_i)\cap\partial\Omega} |Tu|^p dH^m.
\end{aligned}$$

Let $z \in \partial\Omega$ be an arbitrary point. Fix $k \in [k_0,\infty)$. If $z \in B^n(x_i,Cr_i)$ then $\operatorname{dist}(z,Q_i) \le C\operatorname{diam}(Q_i) \approx 2^{-k}$. Thus we see that $z$ cannot belongs more than

$$\frac{(C2^{-k})^n}{2^{-kn}} = C^n$$

cubes $Q_i^k$. We obtain

$$\begin{aligned}
\int_{R^n\setminus\bar\Omega} |Eu|^p dH^n &\le C \sum_{k=k_0}^{\infty} 2^{-k(n-m)} \sum_{i=1}^{N_k} \int_{B^n(x_i,Cr_i)\cap\partial\Omega} |Tu|^p dH^m \\
&\le \|Tu\|^p_{L^p(\partial\Omega)} \sum_{k=k_0}^{\infty} 2^{-k(n-m)}.
\end{aligned}$$

This sum is convergent for every $1 < p < \infty$. This implies

$$\|Eu\|_{L^p(\mathbb{R}^n \setminus \bar{\Omega})} \leq C \|Tu\|_{L^p(\partial\Omega)}.$$

There exists a bounded trace operator, and hence we obtain

$$\|Eu\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

This completes the proof of Lemma 3.3. □

3.4. **Lemma.** *Let $1 < p < \infty$ and $0 < \alpha \leq \frac{n-m}{p}$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \leq m < n$. If there exists a bounded trace operator $T : W^{1,p}(\Omega) \to B^p_{1-\alpha}(\partial\Omega)$ then there exists a constant $0 < C < \infty$ such that*

$$\|\nabla Eu\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

*for every $u \in W^{1,p}(\Omega)$.*

*Proof.* If $Q_i$ and $Q_0$ are two Whitney cubes from $\mathcal{W}$, with $Q_i \cap Q_0 \neq \emptyset$, then $\operatorname{diam}(Q_0) \approx \operatorname{diam}(Q_i)$ and furthermore $|x_i - x_0| \leq C \operatorname{diam}(Q_0)$, where $\operatorname{dist}(Q_0, \partial\Omega) = \operatorname{dist}(Q_0, x_0)$, $x_0 \in \partial\Omega$, and $\operatorname{dist}(Q_i, \partial\Omega) = \operatorname{dist}(Q_i, x_i)$, $x_i \in \partial\Omega$. We obtain by Fubini's theorem for cubes $Q_i$ and $Q_0$, with $Q_i \cap Q_0 \neq \emptyset$, that

$$\begin{aligned}
|a_i - a_0| &= \Big| \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu(z)\,dH^m(z) - \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu(w)\,dH^m(w) \Big| \\
&\leq \Big| \fint_{B^n(x_i,r_i)\cap\partial\Omega} \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu(z)\,dH^m(w)\,dH^m(z) \\
&\quad - \fint_{B^n(x_i,r_i)\cap\partial\Omega} \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu(w)\,dH^m(w)\,dH^m(z) \Big| \\
&\leq \fint_{B^n(x_i,r_i)\cap\partial\Omega} \fint_{B^n(x_0,r_0)\cap\partial\Omega} |Tu(z) - Tu(w)|\,dH^m(w)\,dH^m(z) \\
&\leq \frac{C}{r_0^{2m}} \int_{B^n(x_i,r_i)\cap\partial\Omega} \int_{B^n(x_0,r_0)\cap\partial\Omega} |Tu(z) - Tu(w)|\,dH^m(w)\,dH^m(z).
\end{aligned}$$

By Hölder's inequality we obtain

$$
\begin{aligned}
&|a_i - a_0| \leq \\
&\leq \frac{CH^m(B^n(x_i,r_i)\cap\partial\Omega)^{1-\frac{1}{p}} H^m(B^n(x_0,r_0)\cap\partial\Omega)^{1-\frac{1}{p}}}{r_0^{2m}} \\
&\cdot \Big(\int_{B^n(x_i,r_i)\cap\partial\Omega}\int_{B^n(x_0,r_0)\cap\partial\Omega} |Tu(z)-Tu(w)|^p dH^m(w)dH^m(z)\Big)^{\frac{1}{p}} \\
&\leq r_0^{\frac{-2m}{p}}\Big(\int_{B^n(x_0,Cr_0)\cap\partial\Omega}\int_{B^n(x_0,r_0)\cap\partial\Omega} |Tu(z)-Tu(w)|^p dH^m(w)dH^m(z)\Big)^{\frac{1}{p}} \\
&\leq \Big(r_0^{-2m}\int_{B^n(x_0,Cr_0)\cap\partial\Omega}\int_{B^n(x_0,r_0)\cap\partial\Omega} |Tu(z)-Tu(w)|^p dH^m(w)dH^m(z)\Big)^{\frac{1}{p}} \\
&\leq \Big(r_0^{-2m}\iint_{|z-w|<Cr_0} |Tu(z)-Tu(w)|^p dH^m(w)dH^m(z)\Big)^{\frac{1}{p}}.
\end{aligned}
$$

The constant $C$ does not depend on the cube $Q_0$. Assume that $Q_0$ is a Whitney cube from the k$^{\text{th}}$ generation, then $r_0 \approx 2^{-k}$. We obtain by the definition of the Besov space (2.1) that

$$|a_i - a_0|^p \leq C2^{-k(1-\alpha)p} a_k^p.$$

Let $x \in \frac{9}{8}Q_0$ for $Q_0 \in \mathcal{W}$. This implies

$$|\nabla Eu(x)| = |\nabla \sum_{i=1}^{\infty} a_i\phi_i(x)| = |\sum_{i=1}^{\infty}(a_i - a_0)\nabla\phi_i(x)|.$$

We obtain

$$
\begin{aligned}
&\int_{Q_j} |\nabla Eu|^p dH^n \leq C \sup_{Q_i\cap Q_j\neq\emptyset} |a_i - a_j|^p C \operatorname{diam}(Q_j)^{-p}|Q_j| \\
&\leq C2^{-k(1-\alpha)p} a_k^p 2^{-k(n-p)} \leq C2^{-k(n-\alpha p)} a_k^p.
\end{aligned}
$$

By Lemma 3.1 we obtain $N_k \leq M2^{mk}$ for some $M > 0$. This implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus\bar{\Omega}} |\nabla Eu|^p dH^m \leq \sum_{k=k_0}^{\infty}\sum_{j=1}^{N_k}\int_{Q_j}|\nabla Eu|^p dH^n \\
&\leq \sum_{k=k_0}^{\infty}\sum_{j=1}^{N_k} C2^{-k(n-\alpha p)} a_k^p \\
&\leq C\sum_{k=k_0}^{\infty} 2^{mk}2^{-k(n-\alpha p)} a_k^p \\
&\leq C\sum_{k=k_0}^{\infty} 2^{-k(n-m-\alpha p)} a_k^p.
\end{aligned}
$$

When $n-m-\alpha p \ge 0$ then $2^{-k(n-m-\alpha p)} < 2^{(n-m-\alpha p)}$ for every $k = 1, 2, \ldots$. Hence the sum is convergent and

$$\|\nabla Eu\|_{L^p(\mathbb{R}^n\setminus\bar\Omega)} \le C\Big(\sum_{k=0}^{\infty} a_k^p\Big)^{\frac{1}{p}}.$$

Since there exists a bounded trace operator, we obtain

$$\|\nabla Eu\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

This completes the proof of Lemma 3.4. □

The extension of $u \in W^{1,p}(\Omega)$ is a $C^\infty$-function in $\mathbb{R}^n \setminus \bar\Omega$. We are left to prove that the weak gradient of $Eu$ exists across the boundary. First we prove that the extension $Eu$ is Hölder continuous in $\mathbb{R}^n$ if the function $u$ is Hölder continuous in $\Omega$.

3.5. **Lemma.** *Let $1 < p < \infty$ and $0 < \lambda \le 1$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and there exists a trace operator $T$ to the boundary $\partial\Omega$ so that $Tu = u|_{\partial\Omega}$ for every $u \in W^{1,p}(\Omega) \cap C(\bar\Omega)$. If $u \in W^{1,p}(\Omega) \cap C^{0,\lambda}(\Omega)$, then $Eu \in C^{0,\lambda}(\mathbb{R}^n)$.*

*Proof.* We prove Lemma 3.5 in several steps. Let $x \in \mathbb{R}^n \setminus \bar\Omega$ and $y \in \partial\Omega$. Assume first that $|x-y| = \operatorname{dist}(x,\partial\Omega)$. Let $Q_0$ be a Whitney cube with $x \in Q_0$. Let $Q_i$ be a Whitney cube with $Q_0 \cap Q_i \ne \emptyset$ and $x_i \in \partial\Omega$ so that $\operatorname{dist}(Q_i,\partial\Omega) = \operatorname{dist}(Q_i, x_i)$. We obtain

$$\begin{aligned}
&\Big| \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu(z)\,dH^m(z) - Tu(y)\Big| \\
&= \Big|\frac{1}{H^m(B^n(x_i,r_i)\cap\partial\Omega)}\int_{B^n(x_i,r_i)\cap\partial\Omega} Tu(z)\,dH^m(z) \\
&- \frac{H^m(B^n(x_i,r_i)\cap\partial\Omega)\cdot Tu(y)}{H^m(B^n(x_i,r_i)\cap\partial\Omega)}\Big| \\
&\le \frac{1}{H^m(B^n(x_i,r_i)\cap\partial\Omega)}\int_{B^n(x_i,r_i)\cap\partial\Omega} |Tu(z)-Tu(y)|\,dH^m(z) \\
&\le \frac{|x-y|^\lambda}{H^m(B^n(x_i,r_i)\cap\partial\Omega)}\int_{B^n(x_i,r_i)\cap\partial\Omega} \frac{|Tu(z)-Tu(y)|}{|x-y|^\lambda}\,dH^m(z) \\
&\le \frac{C|x-y|^\lambda}{H^m(B^n(x_i,r_i)\cap\partial\Omega)}\int_{B^n(x_i,r_i)\cap\partial\Omega} \frac{|Tu(z)-Tu(y)|}{|z-y|^\lambda}\,dH^m(z).
\end{aligned}$$

Since the function $Tu$ is $\lambda$-Hölder continuous, we obtain

$$\Big| \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu(z)\,dH^m(z) - Tu(y)\Big| \le C|x-y|^\lambda.$$

This implies that

$$\begin{aligned}
|Eu(x) - Tu(y)| &\le |\sum_{i=1}^{\infty} a_i\phi_i(x) - Tu(y)| \\
&\le |\sum_{i=1}^{\infty} \big(a_i\phi_i(x) - Tu(y)\phi_i(x)\big)| \\
&\le \sum_{i=1}^{\infty} \phi_i(x)|a_i - Tu(y)| \\
&\le C|x-y|^{\lambda}.
\end{aligned}$$

Assume then that $x \in \mathbb{R}^n \setminus \bar{\Omega}$, $y \in \partial\Omega$ and $|x-y| > \operatorname{dist}(x, \partial\Omega)$. Let $x_0 \in \partial\Omega$ so that $\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, x_0)$. We obtain

$$\begin{aligned}
|Eu(x) - Tu(y)| &\le |Eu(x) - Tu(x_0)| + |Tu(x_0) - Tu(y)| \\
&\le C|x-x_0|^{\lambda} + C|x_0 - y|^{\lambda} \\
&\le C|x-y|^{\lambda} + C(2|x-y|)^{\lambda} \\
&\le C|x-y|^{\lambda}.
\end{aligned}$$

Let $x, y \in \mathbb{R}^n \setminus \bar{\Omega}$. Assume first that

$$|x-y| \ge \frac{1}{2}\max\{\operatorname{dist}(x,\partial\Omega), \operatorname{dist}(y,\partial\Omega)\}. \tag{3.6}$$

Let $x_0, y_0 \in \partial\Omega$ so that $\operatorname{dist}(x,\partial\Omega) = \operatorname{dist}(x, x_0)$ and $\operatorname{dist}(y,\partial\Omega) = \operatorname{dist}(y, y_0)$. We obtain

$$\begin{aligned}
|Eu(x) - Eu(y)| &\le |Eu(x) - Tu(x_0)| + |Tu(x_0) - Tu(y_0)| \\
&\quad + |Eu(y) - Tu(y_0)| \\
&\le C|x-x_0|^{\lambda} + C|x_0 - y_0|^{\lambda} + C|y-y_0|^{\lambda} \\
&\le C|x-y|^{\lambda}.
\end{aligned}$$

Suppose then that the inequality (3.6) does not hold. Let $x \in \frac{9}{8}Q_0$ and $y \in \frac{9}{8}Q'$ then $\operatorname{diam}(Q_0) \approx \operatorname{diam}(Q')$. We obtain

$$\begin{aligned}
|Eu(x) - Eu(y)| &= |\sum_{i=1}^{\infty} a_i\phi_i(x) - \sum_{i=1}^{\infty} a_i\phi_i(y)| \\
&\le |\sum_{i=1}^{\infty}(a_i - M)\phi_i(x) - \sum_{i=1}^{\infty}(a_i - M)\phi_i(y)| \\
&\le |\sum_{i=1}^{\infty}(a_i - M)(\phi_i(x) - \phi_i(y))| \\
&\le \sup_i |a_i - M||x-y|\frac{C}{\operatorname{diam}(Q_0)},
\end{aligned}$$

for any constant $M$, with $M > 0$. We choose

$$M = a_0 = \fint_{B^n(x_0,r)\cap\partial\Omega} Tu dH^m.$$

Next we approximate the first part. Since (3.6) does not hold, $r \approx \operatorname{diam}(Q_0)$ and $r_i \approx \operatorname{diam}(Q_0)$ we obtain

$$\begin{aligned}
&\Big| \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu dH^m - \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu dH^m \Big| \\
&\leq \Big| \frac{H^m(B^n(x_i,r_i)\cap\partial\Omega)}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \int_{B^n(x_0,r_0)\cap\partial\Omega} Tu dH^m \\
&- \frac{H^m(B^n(x_0,r_0)\cap\partial\Omega)}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \int_{B^n(x_i,r_i)\cap\partial\Omega} Tu dH^m \Big| \\
&\leq \frac{1}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \\
&\cdot \Big| \int_{B^n(x_0,r_0)\cap\partial\Omega} \int_{B^n(x_i,r_i)\cap\partial\Omega} Tu(z) - Tu(w) dH^m(z) dH^m(w) \Big|
\end{aligned}$$

and furthermore

$$\begin{aligned}
&\Big| \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu dH^m - \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu dH^m \Big| \\
&\leq \frac{r_0^\lambda}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \\
&\cdot \int_{B^n(x_0,r_0)\cap\partial\Omega} \int_{B^n(x_i,r_i)\cap\partial\Omega} \frac{|Tu(z) - Tu(w)|}{r_0^\lambda} dH^m(z) dH^m(w) \\
&\leq \frac{C r_0^\lambda}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \\
&\cdot \int_{B^n(x_0,r_0)\cap\partial\Omega} \int_{B^n(x_i,r_i)\cap\partial\Omega} \frac{|Tu(z) - Tu(w)|}{|z-w|^\lambda} dH^m(z) dH^m(w).
\end{aligned}$$

The function $Tu$ is $\lambda$-Hölder continuous on the boundary. This implies that

$$\begin{aligned}
&\Big| \fint_{B^n(x_0,r_0)\cap\partial\Omega} Tu dH^m - \fint_{B^n(x_i,r_i)\cap\partial\Omega} Tu dH^m \Big| \\
&\leq C \frac{r_0^\lambda}{H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega)} \\
&\cdot H^m(B^n(x_0,r_0)\cap\partial\Omega)H^m(B^n(x_i,r_i)\cap\partial\Omega) \\
&\leq C r_0^\lambda.
\end{aligned}$$

We obtain

$$\begin{aligned}|Eu(x)-Eu(y)| &\le Cr_0^\lambda|x-y|\frac{C}{\operatorname{diam}(Q_0)}\\ &\le C\operatorname{diam}(Q_0)^\lambda|x-y|\frac{1}{\operatorname{diam}(Q_0)}\\ &\le C|x-y|^\lambda\Big(\frac{|x-y|}{\operatorname{diam}(Q_0)}\Big)^{1-\lambda}\\ &\le C|x-y|^\lambda.\end{aligned}$$

Let $x \in \mathbb{R}^n \setminus \bar{\Omega}$ and $y \in \Omega$. Let $x \in \frac{9}{8}Q_0$ and $x_0 \in \partial\Omega$ so that $\operatorname{dist}(Q_0, \partial\Omega) = \operatorname{dist}(Q_0, x_0)$. We obtain

$$\begin{aligned}|Eu(x)-u(y)| &\le |Eu(x)-Tu(x_0)| + |Tu(x_0)-u(y)|\\ &\le C|x-x_0|^\lambda + C|x_0-y|^\lambda\\ &\le C|x-y|^\lambda + C\big(2|x-y|\big)^\lambda\\ &\le C|x-y|^\lambda.\end{aligned}$$

The cases $x, y \in \Omega$; $x \in \Omega$, $y \in \partial\Omega$ and $x, y \in \partial\Omega$ are clear. This completes the proof of Lemma 3.5. □

The following result is due to Hajłasz and Martio, [HM97, Lemma 11, p.237]: If a compact set $K \subset \mathbb{R}^n$ satisfies $H^{n-\lambda}(K) = 0$, where $0 \le \lambda < 1$, then for every p, $1 \le p \le \infty$,

$$W^{1,p}(\mathbb{R}^n \setminus K) \cap C^{0,1-\lambda}(\mathbb{R}^n) \subset W^{1,p}(\mathbb{R}^n).$$

Since $\partial\Omega$ is an $m$-set, $H^{n-\lambda}(\partial\Omega) = 0$ for every $\lambda$, $0 \le \lambda < n-m$, the general case follows by Lemmata 3.3, 3.4 and 3.5 when we assume that $W^{1,p}(\Omega) \cap C^{0,1-\lambda}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for some $\lambda$, $0 \le \lambda < n-m$.

3.7. **Theorem.** *Let $1 < p < \infty$ and $0 < \alpha \le \frac{n-m}{p}$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \le m < n$. Assume that $W^{1,p}(\Omega) \cap C^{0,1-\lambda}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for some $0 \le \lambda < n-m$. Assume that there exists a bounded linear trace operator*

$$T : W^{1,p}(\Omega) \to B^p_{1-\alpha}(\partial\Omega),$$

*so that $Tu = u|_{\partial\Omega}$ for every $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Then the domain $\Omega$ is a $W^{1,p}$-extension domain: there exists a bounded linear extension operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),$$

*so that $Eu|_\Omega = u$ for every $u \in W^{1,p}(\Omega)$.*

3.8. *Remark.* If $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,p}$-extension domain, $p > n-1$, then $H^n(\partial\Omega) = 0$, [Kos90, Theorem 6.5, p.28]. Thus it is natural to asume that $m$ is strictly less than $n$. Lipschitz continuous functions are dense in $W^{1,p}(\Omega)$ if $M^{1,p}(\Omega) = W^{1,p}(\Omega)$, [Haj96, Theorem 5, p.408]. In particulary $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ if $\Omega$ is a $W^{1,p}$-extension domain, [Haj96, Theorem 1, p.405]. The reverse does not hold, there exists a bounded domain $\Omega \subset \mathbb{R}^n$ with $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ but $\Omega$ is not a $W^{1,p}$-extension domain, [Rom98].

By Lemmata 3.3 and 3.4 we obtain the following corollary.

3.9. **Corollary.** *Let $1 < p < \infty$ and $0 < \alpha \leq \frac{n-m}{p}$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \leq m < n$. Then there exists a bounded linear extension operator*

$$E^* : B^p_{1-\alpha}(\partial\Omega) \to W^{1,p}(\Omega).$$

## 4. Characterization for Sobolev extension domains

Using Theorem 3.7 we give a characterization for Sobolev extension domains. We need the following trace theorem, which is due to Jonsson.

4.1. **Theorem.** [Jon79, Theorem 3, p.4],[JW84, Theorem 1, p.182] *Let $1 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \leq m < n$. Then the operator*

$$T : W^{1,p}(\mathbb{R}^n) \to B^p_{1-\frac{n-m}{p}}(\partial\Omega),$$

*$Tu(x) = \lim_{r\to 0} \fint_{B^n(x,r)} u(z)dH^n(z)$, is bounded and linear.*

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \leq m < n$. We have to show that if $\Omega$ is a $W^{1,p}$-extension domain then

$$\lim_{r\to 0} \fint_{B^n(x,r)\cap\Omega} u(z)dH^n(z) = \lim_{r\to 0} \fint_{B^n(x,r)} Eu(z)dH^n(z),$$

for every $x \in \partial\Omega \setminus F$, with $H^m(F) = 0$.

The proof is similar to the proof in [Wal91, Theorem 1, p. 121] as soon as we know that $\Omega$ is an $n$-set. The case $m = n-1$ has been proved in [JW84, Proposition 2, p.206]. A bounded $W^{1,p}$-extension domain, $n-1 < p < \infty$, is an $n$-set by [Kos90, Theorem 6.5, p.28] and [HK91, Corollary 4.9, p.184].

By Theorem 4.1 we obtain the following result.

4.2. **Corollary.** *Let $n-1 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,p}$-extension domain and its boundary $\partial\Omega$ is an $m$-set, $n-1 \leq m < n$. Then there exists a bounded linear trace operator*

$$T : W^{1,p}(\Omega) \to B^p_{1-\frac{n-m}{p}}(\partial\Omega),$$

*so that $Tu(x) = \lim_{r\to 0} \fint_{B^n(x,r)\cap\Omega} u(z)dH^n(z)$.*

By Theorem 3.7, Corollary 4.2 and Remark 3.8 we obtain the following characterization for Sobolev extension domain.

4.3. **Theorem.** *Let $n-1 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and its boundary $\partial\Omega$ is an $m-set$, $n-1 \leq m < n$. Then the following two conditions are equivalent:*

(1) *The domain* $\Omega$ *is a* $W^{1,p}$*-extension domain: there exists a bounded linear extension operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),$$

*that* $Eu|_\Omega = u$ *for every* $u \in W^{1,p}(\Omega)$.

(2) *There exists a bounded linear trace operator*

$$T : W^{1,p}(\Omega) \to B^p_{1-\frac{n-m}{p}}(\partial\Omega),$$

*that* $Tu(x) = \lim_{r\to 0} \fint_{B^n(x,r)\cap\Omega} u(z)dH^n(z)$, *and Lipschitz continuous functions are dense in* $W^{1,p}(\Omega)$.

4.4. *Remark.* By Theorem 3.7 the set of Lipschitz continuous functions in Theorem 4.3 (2) can be relace by the set $W^{1,p}(\Omega)\cap C^{0,1-\lambda}(\Omega)$, where $0 \le \lambda < n - m$.

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