Plane with A_∞–weighted metric not bilipschitz embeddable to R^n

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Abstract

We show that there exists a plane set G that is bilipschitz equivalent to (G, d^z) where (G, d) is not bilipschitz embeddable to any uniformly convex Banach space. Here $z \in (0, 1)$ and d^z denotes zth power of the metric d. This proves that there exists a strong A_∞ weight in the plane such that the corresponding deformed geometry admits no bilipschitz mappings to any uniformly convex Banach space. Such a weight cannot be comparable to the Jacobian of a quasiconformal self–mapping of the plane.

1. Introduction

The problem of characterising the Jacobian of a quasiconformal self–mapping of the plane is very difficult. About a decade ago in [DS1] David and Semmes introduced the class of strong A_∞–weights. See the Definition 1.1 below. It is easy to see that Jacobians of quasiconformal self–mappings of R^n belong to that class. In [DS1] they used slightly different definition, but equivalence of the definitions is proved in [S1].

David and Semmes asked in [DS1], whether all strong A_∞–weights in R^n are comparable to the Jacobian of a quasiconformal self–mapping of R^n. Semmes showed that answer is *no* in R^3. In this paper we show that also in the plane there are strong A_∞–weights that are not comparable to Jacobians of quasiconformal self–mappings. Our work is purely geometric and can be returned to the characterisation problem by well known results of Semmes.

There has been recent progress in the characterisation problem. In [BHR] they characterise the Jacobian of a quasiconformal self–mapping of the plane, when restricted to a sphere. The characterisation does not work in greater dimensions. To prove this they use as one ingredient the construction presented here. For this application we need to strengthen one of our claims in Appendix 2.

Terminology. We say that (X,d) is a quasimetric space if d it satisfies the conditions of a metric except that the usual triangle inequality is replaced by $d(x,z) \leq C(d(x,y)+d(y,z))$ for some C independent of the points. If (X,d) and (Y,d') are quasimetric spaces and $f: X \to Y$ satisfies $d(a,b)/C \leq d'(f(a),f(b)) \leq Cd(a,b)$ for some C independent of a and b, we say f is bilipschitz. We can also call f bilipschitz embedding. If $f(X)=Y$ we say (X,d) and (Y,d') are bilipschitz equivalent. If X is a normed vector space and $x \in X$, we denote the norm of x by $||x||$, except that the ordinary Euclidean norm is denoted by $|x|$. If we talk about norms themselves, we denote them by $||\ ||$ or $|\ |$ in the Euclidean case.

We say that a metric space (X,d) is *rounded ball space* if for every $\epsilon>0$ there exists a $\delta_\epsilon>0$ such that

$$\mathrm{diam}(\ B(z,(1+\delta_\epsilon)d(z,w)/2)\ \cap\ B(w,(1+\delta_\epsilon)d(z,w)/2)) < \epsilon d(z,w)$$

for all z and w. If X is a Banach space, it is rounded ball space if and only if it is uniformly convex. Recall that a Banach space is uniformly convex if for every $\epsilon>0$ there exists a $\delta>0$ such that if $||x||=||y||=1$ and $||(x+y)/2||>1-\delta$, then $||x-y||<\epsilon$.

To prove the equivalence of the two notions, note that x, y and $(x+y)/2$ as above are in the intersection of the closed unit ball and the closed ball of radius 1 centered at $x+y$. The diameter of this convex intersection goes uniformly to zero with δ by rounded ball property.

On the other hand, suppose

$$\{a,b\} \subset B(z,(1+\delta)||z-w||/2)\ \cap\ B(w,(1+\delta\ )||z-w||/2)$$

in a uniformly convex Banach space. Since this intersection is convex, it contains $(a+b)/2$, implying $||w-(a+b)/2|| \geq (1-\delta)||z-w||/2$. Let $\{a',b'\}$ be the central projection of $\{a,b\}$ to the boundary of $B(w,(1+\delta\ )||z-w||/2)$. Now $(a'+b')/2$ is not closer to w than $(a+b)/2$. By uniform convexity $||a'-b'||/||z-w||$ goes to zero uniformly with δ. The same is true for $||a-b||/||z-w||$ since the central projection cannot move a and b more than $\delta||z-w||$. Therefore we have a rounded ball space.

The class of rounded ball spaces contains hence all L^p spaces for $1<p<\infty$, in particular all Hilbert spaces.

Theorem 1.0 There exists a plane set G and $s>1$ such that $(G,|\ |^s)$ is bilipschitz equivalent to a metric space (G,d) that cannot be bilipschitz embedded to any rounded ball space.

Definition 1.1 A *Strong* A_∞ *weight* in R^n is a locally integrable nonnegative function in R^n such that the induced measure $d\mu = w dm_n$ is doubling and the distance function

$$D_w(x,y) = (\int_{B_{x,y}} w dm_n)^{1/n}, \qquad x,y \in R^n,$$

is bilipschitz equivalent to a metric in R^n. Here $B_{x,y}=B((x+y)/2,|x-y|/2)$.

See [DS1], [S1] and [S2] for the theory of strong A_∞ weights. We need the following Theorem 6.3 of [S2]. (In [S2] it was stated with using definition 6.1 that we do not use.)

Theorem. Suppose that for a metric space (Q,d) and some $z\in(0,1)$ we can bilipschitz embed (Q,d^z) to R^n. Then there exists a strong A_∞ weight w in the same R^n such that (Q,d) is bilipschitz embeddable to (R^n,D_w).

The following corollary answers question 1 of [HS], which originates from [DS1]. The corollary follows from Theorem 1.0 and the previous theorem of Semmes.

Corollary 1.2 There exists a strong A_∞ weight w in R^2 such that (R^2,D_w) is not bilipschitz embeddable to any rounded ball space.

Corollary 1.2 answers also partly question 2.3 of [S2]. Semmes proved already in [S2] that for some $n\geq 5$ that is in principle computable there exists a strong A_∞ weight w in R^n such that (R^n,D_w) is not bilipschitz embeddable to any R^m.

It is well known since [DS1] that 1.2 implies the following corollary that answers another formulation of question 1 of [HS]. For R^3 the corollary was already proved in [S2].

Corollary 1.3 There exists a strong A_∞ weight w in R^2 that is not comparable to the Jacobian of any quasiconformal map $f:R^2\to R^2$.

For convenience we will prove Corollary 1.3 from Corollary 1.2. The latter implies actually the following stronger form of Corollary 1.3 by the same well known reasons. Note that quasiconformal maps $f:R^2\to R^2$ are always quasisymmetric. Recall that Ahlfors Q–regular spaces are such that the Hausdorff Q–measure of any ball $B(x,r)$ is comparable to r^Q.

Corollary 1.3.2 There exists a strong A_∞ weight w in R^2 that is not comparable to the volume derivate of any quasisymmetric map $f:R^2\to(Y,d')$ such that $f(R^2)$ is Ahlfors 2-regular and Y is a rounded ball space. Here we take the volume derivate with respect to Hausdorff 2–measures.

Proof. Quasisymmetry implies that $f(B_{x,y})$ contains a ball with radius comparable to $d'(f(x),f(y))$ and is contained in another such ball. Thus $H^2(f(B_{x,y}))$ is comparable to $d'(f(x),f(y))^2$ by Ahlfors 2-regularity. By Theorem 7.11 of [HK] we know that f is absolutely continuous and usual change of variables rule holds. If w is the volume derivate of f, then $D_w(x,y)=H^2(f(B_{x,y}))^{1/2}$. Thence f is bilipschitz from (R^2,D_w) to $f(R^2)$ that is impossible by 1.2. If w_1 and w_2 are comparable, corresponding pseudometric space (R^2,D_{w_1}) is bilipschitz equivalent to (R^2,D_{w_2}).

Recall that linear local contractibility means that $B(x,r)$ can be contracted to a point in $B(x,Cr)$. As noted in [HS], no-answer to question 1 implies no-answer to question 4 of [HS]. This is so basically because deformation of geometry through a strong A_∞ weight preserves the linear local contractibility and Ahlfors regularity. Moreover, Theorem 6.3 of [S2] has a natural extension to S^n. Thus we got the following corollary.

Corollary 1.4 There exists a metric d in S^2 such that it is linearly locally contractible and Ahlfors 2-regular, but not bilipschitz embeddable to any rounded ball space.

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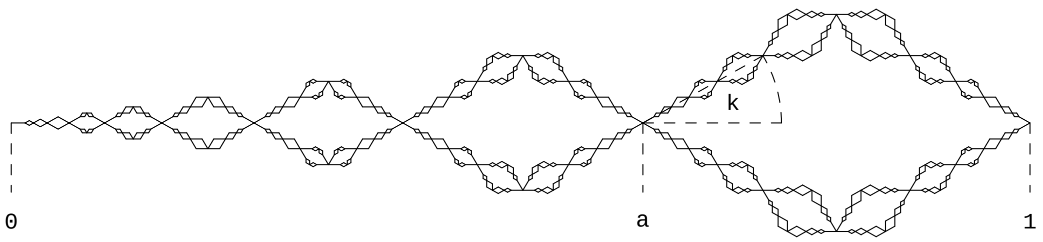
2. Proof of theorem 1.0.

Construction of the plane set G. We generate G from its self–similarities. We use complex notation. Pick $3/4 < a < 1$ and angle $0 < k < \pi/6$.



The similarity mappings of G are the following compositions of scalings, rotations and translations. Set

$$\begin{aligned} &A_1(z) = az \,, \\ &A_2(z) = a + be^{ik}z\,, \quad A_3(z) = a + be^{-ik}z\,, \\ &A_4(z) = 1 - be^{-ik}z\,, \quad A_5(z) = 1 - be^{ik}z\,, \end{aligned}$$

where positive b is chosen such that $A_2(1) = A_4(1)$. Thus also $A_3(1) = A_5(1)$. The set G is the closure of images of the origin under finite compositions of similarity mappings. Note that the origin is the fixed point of A_1. Thus $G = \cup_{i=1}^{5} A_i(G)$.

We call $A_i(G)$ for $i = 1...5$ first order similarity parts. Likewise $A_i(A_j(G))$ are called second order similarity parts and so on. The boundary of a similarity part consists of the images of 0 and 1 under corresponding composition of similarity mappings. (It is probably easier to see this elementary fact by Yourself, but read Appendix 1 if You like.) In the case one of the images is the actual origin, they are not necessarily boundary points and the similarity part may then have only one boundary point. In this case we adopt the obvious abuse of language that also the origin is a boundary point of the similarity part.

Metric d in G. Choose $s > 1$ such that $a^s + 2b^s = 1$. We construct a metric d that is bilipschitz equivalent to the quasimetric $|\ |^s$ in G.

Let G_n be the set of boundary points of nth order similarity parts. Define in G_n a metric as

$$d_n(x, y) = \inf_{\epsilon_1, \ldots, \epsilon_m} \sum_{i=1}^{m-1} |\epsilon_i - \epsilon_{i+1}|^s,$$

where infimum is taken over all chains of points in G_n such that $\epsilon_1 = x$, $\epsilon_m = y$ and ϵ_i and ϵ_{i+1} are boundary points of the same nth order similarity part for each i.

Now $d_1(0,1) = 1$ by the way s was chosen. Similarly the d_n–distance of boundary points of arbitrary similarity part of order $n-1$ is the same as $|\ |^s$–quasidistance. Likewise for all similarity parts of order less that n. To go through such a similarity part, a chain of points as in the definition of d_n therefore needs length that is equal of the $|\ |^s$–diameter of the similarity part. Such chain also exists. Thus d_{n+1} agrees with d_n where the latter is defined. We obtain hence a metric d in $\cup_n G_n$.

Clearly $(\cup_n G_n, d)$ has the same self–similarities as $(\cup_n G_n, |\ |^s)$, even with the same scale factors. Since $\cup_n G_n$ is dense in G in the Euclidean topology, we can extend d to G if it agrees with the Euclidean topology in $\cup_n G_n$. We show next that d and $|\ |^s$ are even bilipschitz equivalent there. The same holds naturally for extended d in the entire G.

Bilipschitz equivalence of $|\ |^s$ and d in $\cup_n G_n$. Suppose x does not belong to the first–order similarity part containing the origin. Then in both metrics the distance of x to the origin is more than a^s. It is also less than 1. Therefore $|\ |^s$ and d are comparable in this case. By self–similarity this extends to points x in the same similarity part as the origin. Likewise one estimates distances of x and 1. Using self–similarity again one extends this to distance of a boundary point of an arbitrary similarity part and its given interior point. In the same way one also sees that bilipschitz condition is true for points in different first–order similarity parts that do not have a common boundary point.

Let x and y be points in adjacent first order similarity parts with common boundary point z. Then $d(x,y) = d(x,z) + d(z,y)$. In the Euclidean metric we have $|x-y|/C \leq |x-z|+|z-y| \leq C|x-y|$, because there is a lower limit for the angle between lines from z to x and z to y. (Read Appendix 1 for the latter claim if You like, but probably it is easier to see it by Yourself.) Raising all distances to power s maintains the double inequality with increased constant. We know already that $|x-z|^s$ and $|y-z|^s$ are comparable to $d(x,z)$ and $d(z,y)$, respectively. We conclude that also $|x-y|^s$ and $d(x,y)$ have to be comparable. If x and y are in the same first order similarity part and not equal, we can use self–similarity to reduce it to cases already considered. Hence $|\ |^s$ and d are bilipschitz equivalent in $\cup_n G_n$.

Remark. If scale factor a in the construction is big enough, then also $a^s > 3/4$. We can then bilipschitz embed (G, d^z) to R^2 with any $z \in [1/s, 1)$. To see this let $a' = a^{zs}$ and $b' = b^{zs}$. Since s is chosen such that $a^s + 2b^s = 1$ and $sz < s$, we have $a' + 2b' > 1$. Then there exists $0 < k' \leq k$ such that $a' + 2b' \cos(k') = 1$. Since also $a' > a^s > 3/4$ one can construct G' with scale factors a' and b' and angle k' with parameters satisfying the bounds required in the beginning. The exponent s' used to construct d' in G' is then $1/z$. Now (G,d) and (G',d') have the same scale factors. Clearly (G_1, d_1) and (G'_1, d'_1) are isometric, with isometry i satisfying $i(A_j(x)) = A'_j(i(x))$ for $j = 1, ..., 5$. In other words, points of the similarity part corresponding to A_j go to similarity part corresponding to A'_j. Using self–similarity one obtains inductively that (G_n, d_n) and (G'_n, d'_n) are isometric for any n with isometry that preserves the self–similarity as above. Thus (G,d) is isometric to (G', d') and both are bilipschitz equivalent to $(G', |\ |^{1/z})$. Raising all metrics to power z preserves the conclusion.

Nonexistence of bilipschitz embeddings of (G,d) to a rounded ball space. Suppose (Y,d') is a rounded ball space with δ_ϵ as in the definition and $f : X \to Y$ is L-bilipschitz embedding. We denote by $L_{x,y}$ the Lipschitz constant of $f|_{\{x,y\}}$, i.e. $L_{x,y} = d'(f(x),f(y))/d(x,y)$. Set $c = A_2(1)$ and $d = A_3(1)$. They are both contained in $\bar{B}(a,d(a,1)/2) \cap \bar{B}(1,d(a,1)/2)$, but $d(c,d) = d(a,1)$. Hence $d'(f(c),f(d)) \geq L^{-2}d'(f(a),f(1))$. If r is big enough for

$$f(\{c,d\}) \ \subset \ \bar{B}(f(a),r) \ \cap \ \bar{B}(f(1),r)),$$

then $r > (1+\delta_{L^{-2}})d'(f(a),f(1))/2$ by rounded ball property. Thus

$$\max\{L_{a,c},L_{a,d},L_{1,c},L_{1,d}\} \ \geq \ (1+\delta_{L^{-2}})L_{a,1}.$$

Any pair in the left hand side are the boundary points of a similarity part.

For $n \in N$, a^n and a^{n+1} are images of 1 and a under A_1^n. Therefore the above reasoning works for them as well. Hence for any $n \in N$ there exist boundary points p and q of a similarity part such that $L_{p,q} \geq (1+\delta_{L^{-2}})L_{a^n,a^{n+1}}$.

Now the chain $\{a^n\}$, $n = 0,1,2,...$ begins from 1, converges to 0 and has length $d(0,1)$. By triangle inequality there is n such that $L_{a^n,a^{n+1}} \geq L_{0,1}$. Then there exist boundary points p and q of a similarity part S with $L_{p,q} \geq (1+\delta_{L^{-2}})L_{0,1}$. Because of self–similarity also S contains similarity part S' with boundary points p' and q' such that $L_{p',q'} \geq (1+\delta_{L^{-2}})L_{p,q}$. Continuing this way eventually contradicts the bilipschitz condition. This concludes the proof of Theorem 1.0.

Appendix 1.

Set $k' = \sup\{|\arg(z)|, z \in G, z \neq 0\}$. Let L be the line segment from 0 to 1. Denote by $D_H(U,V) = \sup_{u\in U}|u-V|$. In estimating $D_H(G,L)$ and k' one can omit $A_1(G)$ that is scaling of G. Thus

$$D_H(G,L) \ \leq \ D_H(\cup_{j=2}^5 A_j(L) \ , \ L) \ + \ \sup_{j=2,\ldots,5}\{D_H(A_j(G) \ , \ A_j(L))\}.$$

Now $D_H(\cup_{j=2}^5 A_j(L) \ , \ L) = b\sin(k)$ and $D_H(A_j(G),A_j(L)) = bD_H(G,L)$ for $j = 2,...,5$. Hence $D_H(G,L) < 2b\sin(k) < 1/8$. Because the estimates above hold for $D_H(G - A_1(G) \ , \ L - A_1(L))$, too, points of $G - A_1(G)$ cannot have real part less than $a - 1/8 > 1/2 > 2b\cos(k)$. Since the absolute value of their imaginary part is bounded by $D_H(G,L)$, we obtain $k' < k$.

Let $x \neq a$ and $y \neq a$ be in $A_2(G)$ and $A_3(G)$, respectively. The angle between lines from a to x and a to y has lower limit $2(k-k')$ by self–similarity. If $z \neq 1$, $|\arg(1-z)| \leq k+k'$. By self similarity this implies that for $w \neq A_4(1)$ in $A_4(G)$, the angle between lines from $A_2(1) = A_4(1)$ to x and w has lower limit $2(\pi/2-k-k-k') > 0$. Similarly we see that the other adjacent first order similarity parts have some definite angle between them in this sense, too. In particular, the boundary of a similarity parts consists solely of images of 1 and 0 under the corresponding similarity mapping.

Appendix 2: Nonexistence of DS–regular mappings from (G, d) to a rounded ball space.

For the application in [BHR] we need to strengthen the nonexistence of bilipschitz mappings from (G, d) to rounded ball spaces to cover also DS–regular mappings.

Definition A mapping $f : X \to Y$ between metric spaces is called *David–Semmes regular with data* (L, M, N) if f is L–Lipschitz and if for each closed ball B in Y the preimage $f^{-1}(B)$ can be covered by M closed balls in X each of diameter at most N times the diameter of B. We abbreviate the term to *DS–regular*. In [DS] they are called just regular.

We will use concepts *mapping package* and *convergence of mapping packages* from [DS, p. 59]. The latter is an extension of Gromov–Hausdorff -convergence to mappings between different spaces. Recall that doubling metric spaces are such that any ball can be covered by C balls with half radius, with C independent of the ball. From [DS, 8.22 and 8.17] we know the existence of convergent subsequence for any sequence of mapping packages with uniformly doubling spaces and uniformly Lipschitz mappings. Given metric spaces (Z, m) and (W, n) and $f : Z \to W$, denote by $L_{x,y}(f)$ the Lipschitz constant of $f|_{\{x,y\}}$, i.e. $L_{x,y}(f) = n(f(x), f(y))/m(x, y)$.

Theorem. There does not exist a DS-regular mapping from (G, d) to a rounded ball space.

Proof. Suppose $f : G \to X$ is DS–regular with data (L, M, N) and (X, d') is a rounded ball space. Given positive integer n pick $x, y \in G$ with $L_{x,y}(f) > (1 - 1/n)L$. We can suppose that x and y are boundary points of a similarity part S_n. Namely all points can be connected by ϵ–chain realising their distance such that each successive pair of points except the first and the last consists of boundary points of similarity parts.

For each such S_n consider mapping $f \circ s_n : G \to X$, where s_n is the natural self–similarity mapping from G to S_n. Let t be the scaling factor of s_n^{-1}. The sequence

$$\{(G, d, 0), (f \circ s_n(G), td', f \circ s_n(0)), f \circ s_n\}$$

of mapping packages has a convergent subsequence. In the limit package

$$\{(G, d, 0), (\bar{f}(G), \bar{d}, \bar{f}(0)), \bar{f}\}$$

the image $\bar{f}(G)$ is still a rounded ball space and $\bar{f}$ is Lipschitz with the same constant L and even DS–regular. (See [DS2, 8.20 and 12.7].) Since 1 is the only point at distance 1 from the base point 0 in G, it is also limit of 1:s of the sequence, even though identity can generally converge to some other isometry in this context. Hence, $L_{0,1}(\bar{f}) = L$. This implies by triangle inequality that for any points z and w in geodesics between 0 and 1 we have likewise $L_{z,w}(\bar{f}) = L$. Thus all geodesics from 0 to 1 will be mapped to geodesics between their images. But in rounded ball spaces geodesics are unique. In G they are dense, so the entire G will be mapped to a single geodesic. Hence the mapping cannot be DS–regular.

The use of limits of mapping packages is not necessary but can be used to avoid mechanical troubles. One can do essentially the same argument in approximative sense by working in arbitrary S_n as above.

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