FACTORS OF EUCLIDEAN SPACES AND GENERALIZED MANIFOLDS

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Abstract

Applying the theory of generalized manifolds we give a characterization of factors for Euclidean spaces and in all dimensions, including $n = 3$, prove the equivalence of the generalized Poincare conjecture to the characterization of Siebenmann for $\mathbb{R}^n$ among n–manifolds as proper homotopy equivalent to $\mathbb{R}^n$.

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1 Introduction

For $n \geq 5$, J. Stallings [St$_2$] in the smooth and PL categories and then L.C. Siebenmann [Si$_2$] in the topological category have established criteria for detecting $\mathbb{R}^n$ among manifolds.

Recall that a space X is said to be 1–LC at ∞, 1*–locally connected at infinity*, if X is not compact and, for every compact set $C \subset X$, there is a compact set K such that $C \subset K \subset X$ and every loop in $X \setminus K$ is contractible in $X \setminus C$. Further, a topological space X is said to *have one end* if X is not compact and if, for every compact set $C \subset X$, there is a compact set K such that $C \subset K \subset X$ and $X \setminus K$ is 0–connected, i.e., path connected. Finally, the space X is said to be 1*–connected at infinity* if, for every compact set $C \subset X$, there is a compact set K with $C \subset K \subset X$ and $X \setminus K$ is 1–connected. In general, a space Y is said to be k–connected, $k = 0, 1, 2, \ldots$ if $\pi_i(X) = 0$ for all $i \leq k$.

A separable metric space M is said to be an n*–manifold* if every point $x \in M$ has a neighborhood $U \subset M$ which is homeomorphic to $\mathbb{R}^n$. All manifolds are assumed to be connected. A locally compact separable metric space X is said to be a *generalized* n*–manifold* if X is a finite–dimensional ANR, absolute neighborhood retract, which is a *homology* n*–manifold*, i.e., for each $x \in X$, $H_k(X, X \setminus \{x\})$ is isomorphic to $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ for all k (local singular homology with integer coefficients). For the introduction in the theory of generalized manifolds, see Section 2 in [MR$_1$], the surveys [Ca], [Da$_3$], [La$_1$], [La$_2$], [MiR], [Re$_1$], [Re$_3$] and the monography [Da$_2$].

A (generalized) compact n–manifold G^n with the same homologies as S^n, i.e., $H_k(G^n) \cong H_k(S^n)$, is said to be a *(generalized) homology n–sphere*. A (generalized) compact n–manifold G^n of the same homotopy type as S^n is said to be a *(generalized) homotopy n–sphere*. Similarly, a (generalized) open n–manifold G^n of the same homotopy type as $\mathbb{R}^n$ (in the other words, contractible) is called a *(generalized) homotopy n–space*. Finally, a (generalized) open n–manifold G^n which is proper homotopy equivalent to $\mathbb{R}^n$ is called a *(generalized) proper homotopy n–space*. Here open means noncompact. For compact spaces, the concept of proper homotopy coincides with the ordinary concept of homotopy, see [Mas], p. 38.

Denote by $S(n)$, $S^*(n)$, $L(n)$ and $P(n)$ the following statements:

1) $S(n)$: Every homotopy n–space, 1–LC at ∞, is homeomorphic to $\mathbb{R}^n$;

2) $S^*(n)$: Every proper homotopy n–space is homeomorphic to $\mathbb{R}^n$;

3) $L(n)$: Every homotopy n–space, 1–connected at ∞, is homeomorphic to $\mathbb{R}^n$;

4) $P(n)$: Every homotopy n–sphere is homeomorphic to S^n.

The generalized Poincare conjecture $P(n)$ has been proved for $n \geq 4$ in the series of the works [Fr], [St$_1$], [Sm$_1$], [Sm$_2$], [Ne] and [Co], and $S(n)$ and $S^*(n)$ for $n \geq 5$ in [Si$_2$]. By definition it follows that $S(n) \Rightarrow L(n)$ for all n. Note also that the property 1–LC at ∞ is invariant under proper homotopy equivalence, see [Si$_2$], p. 245 and [ADMQ], p. 206 and 213. Thus, $S(n) \Rightarrow S^*(n)$, too. E. Luft in [Lu], p. 200, has shown that $L(n) \Rightarrow P(n)$ for all n. He also asked whether $L(n)$ and $P(n)$ are equivalent; it is known that $L(n)$ and $P(n)$ are equivalent for $n = 3$, see Corollary 2 in [Wa].

We prove that the statements $P(n)$ and $S^*(n)$ are equivalent for all n. In particular, this makes possible to obtain the generalized Poincare conjecture $P(n), n \geq 5$, from the criterion of Siebenmann $S^*(n), n \geq 5$, and, conversely, to extend the characterization of $\mathbb{R}^n$, $n \geq 5$, by Siebenmann $S^*(n)$ to $n = 4$ as a consequence of the affirmative answer to the generalized Poincare conjecture for $n \geq 4$.

More precisely, we prove that the one–point compactification of a generalized proper homotopy n–space is a generalized homotopy n–sphere. In particular, in the case of a real proper homotopy n–space, the one–point compactification is a generalized homotopy n–sphere with ∞ as the only potential singular (non–manifold) point. Then, using the theory of removable singularities, we conclude that the one–point compactification is a homotopy n–sphere for $n \geq 4$. If the Poincare conjecture is true, then the same holds for $n = 3$.

A space X is called a *factor* of another space Y if there is a third space Z such that $X \times Z$ is homeomorphic to Y, wirtten $Y \simeq X \times Z$. It is known that the factors of ANR's are ANR's, see [Bor$_2$], p. 295. The factors of generalized manifolds are also generalized manifolds and, moreover, orientability (and nonorientability) is inherited by the factors, see [Bo], p. 15. In particular, 1– and 2–dimensional topological factors of generalized manifolds are always usual manifolds because by [Wi] 1– and 2–dimensional generalized manifolds are usual manifolds. Furthermore, 1– and 2–dimensional topological factors of Euclidean spaces $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2$, respectively see [Bor$_2$], p. 275 and 286.

In [MZ], it has been shown that

$$M^n \times \mathbb{R}^2 \simeq \mathbb{R}^{n+2} \tag{1.1}$$

for contractible *combinatorial* n–manifolds M^n. We prove that

$$G^n \times \mathbb{R}^2 \simeq \mathbb{R}^{n+2} \tag{1.2}$$

for resolvable contractible generalized n–manifolds G^n and, in particular, for all contractible n–manifolds. Since every triangulable generalized manifold G^n is resolvable, i.e., it is a cell–like image of a manifold (and well resolvable, i.e., $M^n \times \mathbb{R}$ is a manifold), see Corollary 2.8 (2.7) in [MR$_2$], (1.2) holds for all triangulable contractible generalized n–manifolds.

In fact, we show that among general topological spaces X the relation

$$X \times \mathbb{R}^k \simeq \mathbb{R}^{n+k}, \;\; k \neq 1, \;\; n \neq 3, \tag{1.3}$$

holds if and only if X is a resolvable contractible generalized n–manifold.

Moreover, McMillan and Zeeman asked whether (1.1) can be improved. Stallings has shown that

$$M^n \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \;\; n \geq 4, \tag{1.4}$$

holds for every contractible *combinatorial* n–manifold M^n, see [St$_2$]. We show that, for a manifold M^n, (1.4) holds if and only if M^n is an arbitrary contractible n–manifold. Furthermore, we show that, for an arbitrary topological space X,

$$X \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \;\; n \neq 3, \tag{1.5}$$

holds if and only if X is a generalized homotopy n–space which is well resolvable. In particular, for a countable (locally finite) polyhedron P,

$$P \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \;\; n \neq 3, \tag{1.6}$$

holds if and only if P is a generalized homotopy n–space, cf. [AQ].

The case $n = 3$ is special. We prove that

$$G^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{1.7}$$

if G^3 is a generalized *proper* homotopy 3–sphere which is well resolvable. In particular,

$$P^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{1.8}$$

if P^3 is a polyhedral generalized proper homotopy 3–space. For a manifold M^3,

$$M^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{1.9}$$

if M^3 is a proper homotopy 3–space.

It is easy to show that, for an arbitrary topological space X, the realation

$$X \times \mathbb{R} \simeq \mathbb{R}^4 \tag{1.10}$$

implies that X is a generalized homotopy 3–space.

However, the conclusions in (1.7)–(1.9) cannot be reversed. Indeed, by a result of Shapiro, see [Gli], the well–known Whitehead manifold W^3 in [Wh] satisfies the relation

$$W^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{1.11}$$

although W^3 is not homeomorphic to $\mathbb{R}^3$. The same is true for uncountable collection of Whitehead manifolds generalized by McMillan, see [Mc$_1$] and [Mc$_2$]. These manifolds are not proper homotopy equivalent to $\mathbb{R}^3$, see [Bro$_2$], p. 509.

2 One–point compactification

By the well–known theorem of Alexandroff, see, e.g., [Bou], p. 92, every locally compact topological space X has the one–point compactification $\overline{X} = X \cup \{\infty\}$ where ∞ is an ideal element with open neighborhoods of the form $(X \setminus K) \cup \{\infty\}$ where K is a compact subset of X. Since a natural extension $\overline{f} : \overline{X} \to \overline{Y}$ with $\overline{f}(\infty) = \infty$ for every proper continuous mapping $f : X \to Y$ is continuous, see [Bou], p. 104, we have the following general statement.

2.1. Proposition. *Let X and Y be proper homotopy equivalent locally compact topological spaces. Then their one–point compactifications $\overline{X}$ and $\overline{Y}$ are (proper) homotopy equivalent.*

Among finite–dimensional metric spaces, X is ANR if and only if X is locally contractible, i.e., for every $x_0 \in X$ and neighborhood U of x_0 there is a neighborhood V of x_0 such that the inclusion $V \to U$ is homotopic in U to the constant map x_0, see, e.g., [MaS], p. 40. It is also known that the local contractibilty is invariant under homotopy equivalence, see [Bor_1], p. 29. Hence by the metrizability criterion of the Urysohn, see, e.g., [Ku], p. 241–242, we obtain the following consequence of Proposition 2.1.

2.2. Corollary. *Let X and Y be locally compact separable metric spaces of a finite dimension and let X and Y be proper homotopy equivalent. Then $\overline{X}$ is ANR if and only if $\overline{Y}$ is ANR.*

Now, using the Poincare duality and the invariance of singular cohomology with compact support under proper homotopy equivalence, see [Mas], p. 39, and hence of orientability, see [Bo], p. 12, we come by [Ra_1], p. 958, to the following.

2.3. Corollary. *Let X be a generalized proper homotopy n–space. Then the one–point compactification of X is a generalized homotopy n–sphere.*

2.4. Remark. The *proper* condition in Corollary 2.3 is essential because, for every $n \geq 3$, there exist examples, see [MS], p. 121, of homotopy n–spaces X whose one–point compactifications $\overline{X}$ are not simply connected. These spaces $\overline{X}$ are not generalized n–manifolds. Indeed, assume $\overline{X}$ is a generalized manifold. Then by Lemma 2.1 in [MR_1], for every loop $\varphi_0 : C \to \overline{X}$, $\varphi_0(c_0) = x_0 \in X$, $c_0 \in C$ where C is the unit circle, there is a loop $\varphi_1 : C \to \overline{X}$, $\varphi_1(c_0) = x_0$, which is homotopic to φ_0 and such that $\infty \notin \varphi_1(C)$. And then $\overline{X}$ would be simply connected because X is a contractible space.

The next lemma makes possible to conclude, in particular, that all possible counterexamples to the Poincare conjecture are just the one–point compactifications of proper homotopy 3–spaces and that the generalized Poincare conjecture for $n \geq 5$ is a consequence of the criterion $S^*(n)$ by Siebenmann for detecting $\mathbb{R}^n, n \geq 5$.

2.5. Lemma. *Let N be a homotopy n–sphere. Then $N \setminus \{y_0\}$ is a proper homotopy n–space for every $y_0 \in N$.*

Proof. As a compactum N is proper homotopy equivalent to $S^n \simeq \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. By the Hopf theorem there is a proper homotopy equivalence $h : N \to \overline{\mathbb{R}^n}$ such that $B^* = h^{-1}(B)$ is a topological ball in N for a topological ball B in $\overline{\mathbb{R}^n}$ and the restriction of h to B^* is a homeomorphism, see [Ho] and [Ep]. Applying isotopies of N and $\overline{\mathbb{R}^n}$ we may assume that $h^{-1}(\infty) = y_0$, see, e.g., [Ep], p. 376. Then the restriction of h to $N \setminus \{y_0\}$ is a proper homotopy equivalence onto $\mathbb{R}^n$, see [FTW], p. 21.

A point x of a generalized n–manifold X is said to be *singular*, if x has no neighborhood homeomorphic to $\mathbb{R}^n$; otherwise x is called a *regular point* of X. The set of all singular points of X is denoted by $S(X)$. For $n \geq 5$, if $S(X) \subset Z$ where Z is a closed set and 1–LCC embedded in X with $2k + 2 \leq n$, $k = \dim Z$, then X is a real manifold, see [CBL], [La$_1$], p. 281, and [Da$_2$], p. 286. For $n = 4$, much less is known and, for $n = 3$, the corresponding statement on removability of singularities is equivalent to the Poincare conjecture. More precisely, denote by $D(3)$ the following statement: every generalized 3–manifold X with a singular set $S(X)$ 1–LCC embedded in X such that $\dim S(X) \leq 0$ is a real 3–manifold.

2.6. Proposition. *The statement $D(3)$ is equivalent to the Poincare conjecture.*

Proof. If the Poincare conjecture is not true, then there is the so–called Wilder generalized 3–manifold X with the only singular point which is 1–LCC embedded in X, see Example 7.1, p. 87, and Proposition 7.7, p. 90, in [Re$_1$]. Conversely, if the Poincare conjecture is true, then there are no fake cubes, i.e., homotopy 3–cells which are not homeomorphic to $\overline{\mathbb{B}^3}$, see [He], and, thus, the so–called Kneser finiteness authomatically holds. Hence the Poincare conjecture implies $D(3)$ by [BL] and [Re$_2$], see also Theorem 7.5*, p. 89, and Proposition 7.7, p. 90, in [Re$_1$].

2.7. Proposition. *Let N be a generalized 4–manifold with only one potential singular point $y_0 \in N$. If y_0 has an open neighborhood U such that $U \setminus \{y_0\}$ is proper homotopy equivalent to $\mathbb{R}^4 \setminus \{0\}$, then N is a real 4–manifold.*

Proof. Indeed, $\mathbb{R}^4 \setminus \{0\} \simeq S^3 \times \mathbb{R}$ and hence $U \setminus \{x_0\} \simeq S^3 \times \mathbb{R}$ by the Freedman–Siebenmann lemma, see [Fr], p. 367, cf. [Si$_1$], p. 288. Consequently, x_0 is a removable singularity by the uniqueness of the end point (Freudenthal) compactification, see, e.g., [Fre] and [Ra$_1$].

2.8. Lemma. *Let M be a proper homotopy n–space, $n \neq 3$. Then its one–point compactification is a homotopy n–sphere. The same is true for $n = 3$ up to the Poincare conjecture.*

Proof. By Corollary 2.3 $\overline{M} = M \cup \{\infty\}$ is a generalized homotopy n–sphere with ∞ as the only potential singular point. For $n = 1$ and 2, generalized manifolds coincide with manifolds, see [Wi]. For $n = 3$, the statement follows from

Proposition 2.6 and, for $n \geq 5$, by [CBL], [La$_1$], p. 281, or [Da$_2$], p. 286, because M is 1–LC at ∞. For $n = 4$, it follows from Proposition 2.7 applied to $N = \overline{M} = M \cup \{\infty\}$ and $y_0 = \infty$. Indeed, by the Hopf theorem there is a proper homotopy equivalence $h : M \to \mathbb{R}^4$ such that $B^* = h^{-1}(B)$ is a topological ball in M for a topological ball B in $\mathbb{R}^4$ and the restriction of h to B is a homeomorphism, see [Ho] and [Ep]. Without loss of generality we may assume that $0 \in B$. Let $x_0 = h^{-1}(0)$. Then the restriction h to $M \setminus \{0\}$ is a proper homotopy equivalence onto $\mathbb{R}^4 \setminus \{0\}$, see, e.g., [FTW], p. 16.

By Lemmas 2.5 and 2.8 we come to the following conclusion.

2.9. Theorem. *The statements $S^*(n)$ and $P(n)$ are equivalent for all n.*

Thus, the following Siebenmann theorem extended to $n = 4$ can be obtained as a consequence of the affirmative answer to the generalized Poincare conjecture for $n \geq 4$.

2.10. Theorem. *Let M^n be a proper homotopy n–space, $n \geq 4$. Then M^n is homeomorphic to $\mathbb{R}^n$.*

Since a resolution is a proper homotopy equivalence, see [La$_3$], p. 719, and [Da$_2$], p. 130, we have also the following consequence of Theorem 2.10.

2.11. Proposition. *Let X be a topological space. Then X is a resolvable generalized proper homotopy n–space, $n \geq 4$, if and only if*

$$X \simeq \mathbb{R}^n/D \tag{2.12}$$

where D is a cell–like decomposition of $\mathbb{R}^n$.

2.13. Remark. If X is a resolvable generalized homotopy n–space which is not proper, then

$$X \simeq M^n/D \tag{2.14}$$

where the n–manifold M^n is not $\mathbb{R}^n$ and only a homotopy n–space. It is well known that there exist uncountably many topologically distinct homotopy n–spaces for every dimension $n \geq 3$, see [Wh], [Mc$_2$], [CK] and [Gl]. There is no one of them which is a proper homotopy 3–space, see, e.g., [Bro$_2$], p. 509.

3 Factors of Euclidean spaces

We start from the following simple statement.

3.1. Proposition. *Let X be a topological space such that*

$$X \times \mathbb{R}^k \simeq \mathbb{R}^{n+k} \tag{3.2}$$

for some $k = 1, 2, \ldots$. Then X is a contractible generalized n–manifold which, for $n \neq 3$, is resolvable. For $k = 1$, X is a generalized homotopy n–space.

Proof. X is a generalized n–manifold by [Bo], see also [Ra$_2$], p. 17, [KR], p 113, [Bre], Section 2D in [MR$_1$]. Moreover, X as well as $\mathbb{R}^n$ are deformation retracts of $\mathbb{R}^{n+k}$ and, consequently, X has the same homotopy type as $\mathbb{R}^n$, see, e.g., [Ko], p. 115. X is resolvable for $n \geq 4$ by [Qu$_1$], p. 287, and [Re$_1$], p. 84. For $k = 1$, X cannot be compact because $X \times \mathbb{R} \simeq \mathbb{R}^{n+1}$ must have one end.

3.3. Theorem. *Let X be an arbitrary topological space. Then*

$$X \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \quad n \neq 3, \tag{3.4}$$

if and only if X is a well resolvable generalized homotopy n–space.

Proof. If X is a well resolvable generalized homotopy n–space, then $X \times \mathbb{R}$ is a space which is 1–connected at infinity, see [St$_2$], p. 482. Without loss of generality we may assume that $n \geq 4$. Then the criterion of Siebenmann $S(n)$ for detecting $\mathbb{R}^n$, see [Si$_2$], p. 245, (3.4) holds because $X \times \mathbb{R}$ is a contractible $(n+1)$–manifold. The converse conclusion is obtained from Proposition 3.1.

By [Da$_2$], p. 289, we have the following consequences of Theorem 3.3, cf. [AQ].

3.5. Corollary. *The relation (3.4) holds for every generalized homotopy n–space G^n, $n \neq 3$, whose singular set has dimension $\leq n-2$.*

3.6. Corollary. *For a polyhedron P,*

$$P \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \quad n \neq 3, \tag{3.7}$$

if and only if P is a generalized homotopy n–space.

The next consequence includes the affirmative answer to the McMillan–Zeeman question, see [MZ], p. 221.

3.8. Corollary. *For a manifold M^n,*

$$M^n \times \mathbb{R} \simeq \mathbb{R}^{n+1}, \quad n \neq 3, \tag{3.9}$$

if and only if M^n is a homotopy n–space.

As an immediate consequence of Theorem 2.10 we also obtain the following statement.

3.10. Theorem. *Let G^3 be a well resolvable generalized proper homotopy 3–space. Then*

$$G^3 \times \mathbb{R} \simeq \mathbb{R}^4. \tag{3.11}$$

3.12. Corollary. *For a 3–manifold M^3,*

$$M^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{3.13}$$

if M is proper homotopy equivalent to $\mathbb{R}^3$.

3.14. Remark. For $k = 1$ and $n = 3$ in Proposition 3.1, the relation (3.2) implies that X is a generalized homotopy 3–space. But X is, generally speaking, not proper homotopy eqivalent to $\mathbb{R}^3$. Shapiro, see [Gli], has shown that

$$W^3 \times \mathbb{R} \simeq \mathbb{R}^4 \tag{3.15}$$

where W^3 is the Whitehead manifold in [Wh]. McMillan, see [Mc_1], p. 513, and [Mc_2], has constructed an abundance of topologically distinct Whitehead manifolds W^3 with the factor property. However, these are not proper homotopy 3–spaces, see [Bro_2], p. 509.

3.16. Theorem. *Let G^n be a contractible resolvable generalized n–manifold. Then*

$$G^n \times \mathbb{R}^2 \simeq \mathbb{R}^{n+2}. \tag{3.17}$$

Proof. For $n \geq 3$, $H^{n+1} = G^n \times \mathbb{R}$ is well resolvable by the Daverman theorem, see [Da_1], p. 283, and, thus, the statement follows from Theorem 3.3.

3.18. Corollary. *For every contractible n–manifold M^n,*

$$M^n \times \mathbb{R}^2 \simeq \mathbb{R}^{n+2}. \tag{3.19}$$

3.20. Corollary. *All triangulable contractible generalized n–manifolds G^n satisfy (3.17).*

Finally, combining Theorem 3.16 and Proposition 3.1 we obtain the following conclusion.

3.21. Theorem. *Let X be an arbitrary topological space and $k \neq 1$ and $n \neq 3$. Then*

$$X \times \mathbb{R}^k \simeq \mathbb{R}^{n+k} \tag{3.22}$$

if and only if X is a contractible resolvable generalized n–manifold.

Since a resolution is a proper homotopy equivalence we have the following general construction for well resolvable generalized proper homotopy spaces by [DR] and [Da_2], p. 289.

3.23. Proposition. *Let D be a cell–like decomposition of $\mathbb{R}^n$ and let the quotient space $G^n = \mathbb{R}^n/D$ have a singular set $S(G^n)$ such that*

$$\dim S(G^n) \leq n - 2 \tag{3.24}$$

for $n \geq 4$ and

$$\dim S(G^n) = 0 \tag{3.25}$$

for $n = 3$. Then G^n is a generalized proper homotopy n–space which is well resolvable.

By Proposition 3.23 we obtain the following consequence of Theorems 3.10 and 3.21.

3.26. Corollary. *Under conditions of Proposition 3.23,*

$$(\mathbb{R}^n/D) \times \mathbb{R} \simeq \mathbb{R}^{n+1}. \tag{3.27}$$

In particular, the following consequence can be obtained from the well–known result of Bryant, see [Bry$_1$] and [Bry$_2$], cf. also [AC].

3.28. Corollary. *Let C be a k–cell in $\mathbb{R}^n$, $n \geq 3$. Then $\mathbb{R}^n/C$ is a generalized proper homotopy n–space and*

$$(\mathbb{R}^n/C) \times \mathbb{R} \simeq \mathbb{R}^{n+1}. \tag{3.29}$$

If C is tamely embedded in $\mathbb{R}^n$, then the only potential singularity of $\mathbb{R}^n/C$ can be removed and $\mathbb{R}^n/C \simeq \mathbb{R}^n$. It is well known the so–called horned sphere in $\mathbb{R}^3$ by Alexander whose interior is just a topological ball and exterior is not even simply connected and the examples by Artin and Fox [AF] of wild arcs in $\mathbb{R}^3$ whose complements are not simple connected and also similar examples for all $1 \leq k \leq n$ and all $n \geq 3$, see [Bro$_1$], p. 219, cf. also [Kw], [DS], [Sin] and [FPV]. These generate a number of non–trivial examples of generalized proper homotopy n–spaces with the only singular point.

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