GEOMETRIC BRANCHED COVERS BETWEEN GENERALIZED MANIFOLDS

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0. Introduction

It is a difficult problem to determine when a given metric space is locally bi-Lipschitzly equivalent to an open subset of Euclidean space. Recall that a map $f : X \to Y$ is L–*Lipschitz* if

$$|f(a) - f(b)| \leq L|a - b|$$

for each pair of points $a, b \in X$, and for some $L \geq 1$ independent of the points. A homeomorphism f is L–*bi–Lipschitz* if both f and f^{-1} are L–Lipschitz. (Generally, in this paper, we use the distance notation $|x - y|$ in every metric space.)

About twenty years ago, Siebenmann and Sullivan [SS] noted the curious fact that there are, for each $n \geq 5$, compact pure n-dimensional polyhedra that are topological manifolds but do not admit local bi-Lipschitz parametrizations. The double suspension of a homology sphere with nontrivial π_1 serves as an example. After some positive recent results due to Toro [T1], [T2], Semmes exhibited a family of geometrically nice metrics on S^3 for which no local bi-Lipschitz parametrizations exist [S2], [S3]. At this point, it is not clear if a simple geometric characterization can be found for the metric spaces which admit local bi-Lipschitz parametrizations. (Dimension $n = 2$ could be special here, cf. [S1], [DS2], [HK2].)

The situation is interestingly different if one allows for parametrizations by maps with branching. In this work, we shall consider maps that are natural multivalent analogs of bi-Lipschitz maps, called BLD–maps. We shall show, in particular, that none of the metrics d exhibited by Semmes on S^3 carries an obstruction for the existence of "branched Lipschitz", or "BLD–coordinates". In fact, global BLD–maps $f : (S^3, d) \to S^3$ exist. The same is true for the double suspension examples, but this is easier, for the polyhedra in question can be triangulated so that an Alexander type PL–branched cover is possible.

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Sullivan and the first author [HS] have recently outlined an approach to the parametrization problem which is based on differential (Whitney) forms. In particular, in [HS] a characterization of a "branched Euclidean metric gauge" is given. Among other things, the results of the present paper demonstrate the sharpness of the characterization in [HS] for dimensions $n \geq 3$.

In the first part of this work, we develop a basic theory of geometrically controlled branched covers between generalized metric manifolds. A generalized n–manifold is a finite dimensional locally compact Hausdorff space whose local cohomology groups over the integers are similar to those of an n–manifold in dimensions $n-1$ and higher. (For a precise definition, see 1.6).

We call a continuous map between topological spaces a *branched cover* if it is open and if the preimage of each point consists of isolated points. The latter property is also expressed by saying that the map is *discrete*. Thus, "branched cover" in this paper means "discrete and open".

The algebraic condition in the definition of a generalized manifold allows for a local degree theory for continuous maps between oriented generalized manifolds. In particular, it makes sense to speak about sense-preserving maps. (See Section 2.)

The main class of maps we shall study in this paper is the class of BLD–maps:

0.1. **Definition.** A sense-preserving branched cover $f : X \to Y$ between two oriented generalized metric n–manifolds is said to be an L–BLD, or an L*-bounded length distortion* map, $L \geq 1$, if

$$\frac{1}{L} \text{ length } \alpha \leq \text{ length } f \circ \alpha \leq L \text{ length } \alpha \tag{0.2}$$

for all (nonconstant) paths α in X.

Note that requirement (0.2) has interesting content only if the spaces X and Y are rich in rectifiable curves.

BLD–maps in Euclidean space were first studied by Martio and Väisälä [MVä] who gave the following analytic (quantitatively equivalent) characterization of BLD–maps in R^n: a continuous map $f : G \to R^n$, where $G \subset R^n$ is open and $n \geq 2$, is BLD if and only if f is locally uniformly Lipschitz and the Jacobian determinant $\det Df$ is positive and uniformly bounded away from zero almost everywhere in G. (Lipschitz maps are almost everywhere differentiable, so the assumption on the Jacobian makes sense.) This description of a BLD–map in Euclidean space does not include the assumption that the map be discrete and open, nor that it be sense-preserving. In fact, maps satisfying the above analytic condition form a (strict) subclass of *quasiregular maps*, or *maps of bounded distortion*, introduced by Reshetnyak in 1960's. It follows from a deep theorem of Reshetnyak [Re1] that each (locally nonconstant) quasiregular map is discrete, open, and sense-preserving. (See [Re2], [Ri2] for the basic theory of quasiregular maps.)

The above remarks about BLD and quasiregular maps hold on smooth Riemannian manifolds. In this work, we shall not attempt to extend the theory of general quasiregular maps from the smooth setting; it seems reasonable first to study the BLD case, where the tools of Lipschitz analysis are available. We emphasize that BLD–maps already can exhibit quite complicated (local) behavior. Indeed, each quasiregular map $f : R^n \to R^n$ factors $f = \varphi \circ g$, where g is a quasisymmetric homeomorphism from R^n to a metric space X_f, and $\varphi : X_f \to R^n$ is a BLD–map. (The metric space X_f, the "Riemann surface of f", is of type $A124$ as defined in Section 5 below.) On the other hand, it is true that the global behavior of BLD–maps is more limited than that of general quasiregular maps. (Compare Theorem 6.8 below.)

In the second part of this work, we prove the following general existence theorem for branched covers in dimension $n = 3$; this theorem is crucial in showing the existence of branched coordinates for metric spaces as discussed above.

0.3. **Theorem.** *Let W be a connected, compact, oriented PL 3–manifold with boundary ∂W consisting of $p \geq 2$ components $M_0, \ldots, M_{p-1}$ with the induced orientation. We assume that W is a polyhedron in some R^m and is equipped with the metric induced from R^m. There exists an integer $d_0 \geq 3$ such that the following holds: Let N be an oriented polyhedral 3–sphere in R^4 and let $B_0, \ldots, B_{p-1}$ be disjoint closed polyhedral 3–balls in N; the boundaries $\partial B_0, \ldots, \partial B_{p-1}$ are given the orientation induced from $W' = N \setminus \cup_{j=0}^{p-1} \operatorname{int} B_j$. For each $j = 0, 1, \ldots, p-1$, let $\varphi_j : M_j \to \partial B_j$ be a sense-preserving PL branched cover of degree $d = 3i \geq d_0$. Then there exists a sense-preserving PL branched cover $\varphi : W \to W'$ of degree d that extends the φ_j's.*

Theorem 0.3 was proved by Berstein and Edmonds [BE, (6.2)] for $p = 2$ and $d_0 = 3$ (without the restriction $d = 3i$). In our proof, we use the techniques of [Ri1] to reduce the general case to the case considered by Berstein and Edmonds. This reduction is presented here in a self-contained manner; no reference to [Ri1] is needed.

A special case of Theorem 0.3 was proved in [HR], where we showed the existence of a BLD–map $f : S^3 \to S^3$ whose branch set contains a wild Cantor set. The general construction of the present paper simplifies that of [HR], except the use of the Berstein-Edmonds theorem. (In [HR] we did not use [BE] but relied instead on the second author's paper [Ri1] in a more substantial way.)

By the aid of Theorem 0.3, we can construct BLD–maps from some interesting compact generalized 3–manifolds onto S^3. Semmes [S3] showed that the classical (compact) decomposition spaces arising from the Whitehead continuum, Bing's dogbone space, and Bing's double, for example, all admit metrics that are smooth Riemannian outside a totally disconnected closed singular set, and are indistinguishable from the standard metric on S^3 by means of classical analysis. We shall show that all these spaces, although bi-Lipschitzly inequivalent to the standard, or PL 3-sphere, can be mapped onto standard S^3 by a BLD–map. As alluded to above, the claims made by Semmes, as well as our constructions, work for each space arising in a self–similar manner from an "excellent package" as defined in [S3]. All this will be made more precise in Section 8.

At present, the method of constructing interesting branched covers in the spirit of [Ri1], [HR] is limited to dimension $n = 3$. When $n = 2$, every generalized manifold is actually a manifold, and the existence and nature of branched covers is well understood. In higher dimensions, every closed orientable n–pseudomanifold (without boundary) can be mapped onto S^n by a BLD branched cover. For smooth manifolds this was proved by Alexander [A] in 1920, and his method works for orientable pseudomanifolds, too. Recall that a closed *n–pseudomanifold* (cf. [ST, §24]) is a pure n–dimensional finite complex where every $(n-1)$–simplex is a face of precisely two n–simplexes and where every two n–simplexes can be joined by an arc that only meets n– or $(n-1)$–simplexes. An n–pseudomanifold X is *orientable* if $H_n(X) = \mathbb{Z}$. A particular example of a closed n–pseudomanifold is a finite complex X that satisfies

$$(0.4) \qquad H_*(X, X \setminus \{x\}) = H_*(R^n, R^n \setminus \{0\})$$

for all $x \in X$. Complexes X that satisfy (0.4) are generalized n–manifolds as defined in this paper and they can be characterized by the property that the link of each vertex has the homology of the $(n-1)$–sphere. The homology groups above, and everywhere in this paper, are integral.

Finally, let us compare the BLD–maps as defined above in 0.1 with another related class of maps. Remember that condition (0.2) becomes uninteresting if the spaces X and Y contain no rectifiable paths. In [D], David introduced "regular" maps between "regular" subsets of Euclidean space. Later David and Semmes [DS1], [DS2], [DS3], [S2] took up a more general definition valid in arbitrary metric spaces:

0.5. **Definition.** A map $f : X \to Y$ between metric spaces is called *regular* with data (L, M, N) if f is L–Lipschitz, and if for each closed ball B in Y the preimage $f^{-1}(B)$ can be covered by M closed balls in X, each of diameter at most N times the diameter of B.

Regular maps as defined in 0.5 are natural finite–to–one analogs of bi–Lipschitz maps. They serve as a natural generalization of BLD–maps to spaces where one cannot define sense-preserving maps, or where no rectifiable curves are to be found. For most of our stronger results the assumption that we have a sense-preserving map is necessary; the proofs often rely on the path-lifting property of sense-preserving branched covers between oriented generalized manifolds as explained in 3.3. On the other hand, many results could be true in more generality. There is room for further research here.

In Part I of this paper, we study the relationship between regular and BLD–maps on generalized manifolds. It turns out that under quite general circumstances, the sense-preserving regular maps are precisely the BLD–maps of finite maximal multiplicity. (See Theorem 4.5.) We give examples that illustrate the relationship between various classes of maps. In Part I, we also introduce a particular kind of generalized metric manifolds, called spaces of "type A", and show that BLD–maps between such spaces behave in many respects like BLD–maps between Riemannian manifolds. In particular, for maps from spaces of type A into R^n, an analytic description akin to the one used by Martio and Väisälä in R^n is available. We point out some further analytic and metric properties of BLD–maps between spaces of type A, including a result on value distribution (Theorem 6.8).

Part II contains a proof of the generalized Berstein–Edmonds theorem, Theorem 0.3, together with some examples and applications. In particular, we study carefully the geometric decomposition spaces of Semmes, and show how they admit BLD branched coordinates in R^3. We also point out how one obtains bounded quasiregular maps in the open 3-ball in R^3 that fail to have radial limits in a large set on the boundary. Moreover, we construct Lipschitz maps $S^3 \to S^3$ with non-zero degree having as fibers some strange continua, e.g. the Whitehead continuum, but with dilatation function in a local Lebesgue space L^p for p arbitrarily close to 2. These examples are quite interesting in view of the fact that each Lipschitz map $S^3 \to S^3$ is a branched cover if the integrability exponent for the dilatation exceeds 2. (See 9.4 for the details.) Finally, we discuss how BLD–maps preserve Poincaré inequalities.

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Part I. BLD and regular maps

1. Generalized manifolds

All spaces in this paper are assumed to be locally compact, Hausdorff, connected, and locally connected. $H_c^*(X)$ denotes the Alexander–Spanier cohomology groups of a space X with compact supports and coefficients in $\mathbb{Z}$.

1.1. **Definition.** A space X is called an n–dimensional, $n \geq 2$, *cohomology manifold* (over $\mathbb{Z}$), or a *cohomology n–manifold*, if

(a) the cohomological dimension $\dim_{\mathbb{Z}} X$ is at most n, and
(b) the local cohomology groups of X are equivalent to $\mathbb{Z}$ in degree n and to zero in degree $n-1$.

Condition (a) means that $H_c^p(U) = 0$ for all open $U \subset X$ and $p \geq n+1$ (in general, $\dim_{\mathbb{Z}} X$ is the least integer n for which this is true; see [Br, II.16.14]). Condition (b) means that for each point $x \in X$, and for each open neighborhood U of x, there is another open neighborhood V of x contained in U such that

$$H_c^p(V) = \begin{cases} \mathbb{Z}, & \text{if } p = n, \\ 0, & \text{if } p = n-1, \end{cases}$$

and the standard homomorphism

$$H_c^n(W) \to H_c^n(V) \tag{1.2}$$

is a surjection whenever W is an open neighborhood of x contained in V.

1.3. **Remarks.** (a) Our definition for a cohomology n-manifold is weaker than what typically appears in the literature, for we require that the local groups are similar to that of an n-manifold in degrees $n-1$ and higher only. Cohomology manifolds as defined in [Br, V.16.8] and [Bo, I.3.3], for example, satisfy our requirements.

(b) The cohomological dimension $\dim_{\mathbb{Z}} X$ (of a separable metric X) equals the topological dimension of X provided the latter is finite. This is proved in [HW, VIII 3, p. 151] for X compact; in general, $\dim_{\mathbb{Z}} X$ is the supremum of dimensions $\dim_{\mathbb{Z}} Y$, where Y ranges over compact subspaces of X [Br, II.16.7].

(c) An n–dimensional cohomology manifold X need not be finite–dimensional, but $\dim_{\mathbb{Z}} X = n$ always [Br, p. 115 and p. 122], [Bo, I.2.3].

(d) A (connected) open subset of a cohomology manifold is a cohomology manifold of the same dimension. If $V \subset X$ is open, connected and satisfies $H_c^n(V) = \mathbb{Z}$, then the standard homomorphism in (1.2) is an isomorphism whenever W is connected; see [Br, V.16.16] or [Bo, I.4.3(2)].

1.4. **Examples.** (a) Every topological n–manifold is a cohomology manifold. Conversely, every (separable, metric) cohomology 2–manifold is an actual 2–manifold; see [Br, V.16.32] or [W, IX.5.5, p. 277]. For each $n \geq 3$ there are compact subsets of Euclidean space that are n–cohomology manifolds but not manifolds.

(b) A locally finite simplicial complex is a cohomology n–manifold if the link of every vertex has the homology of an $(n-1)$–sphere.

(c) Some authors call an ENR space X a *homology n–manifold* if, for each $x \in X$,

$$H_*(X, X \setminus \{x\}) \simeq H_*(R^n, R^n \setminus \{0\}), \tag{1.5}$$

where H_* denotes the singular homology with coefficients in $\mathbb{Z}$. Every homology n–manifold according to this definition is a finite dimensional cohomology n–manifold. This follows from [Br, V.16.8]. On the other hand, the latter need not be locally contractible. (See the example in [W, p. 245].) Recall that an ENR space can be characterized as a locally compact, finite–dimensional, and locally contractible metric space.

(d) If G is a cell–like decomposition of an n–manifold M, then the decomposition space M/G is a homology n–manifold as defined in (c) provided it is finite dimensional. In particular, M/G is a cohomology n–manifold. See [Da, p. 191], and also Section 8 below.

Next we define a generalized manifold in a way that suits our purposes in this paper.

1.6. **Definition.** A space X is called a *generalized n–manifold*, $n \geq 2$, if it is a finite–dimensional cohomology n–manifold.

It follows from 1.3 (b) that a generalized n–manifold has topological dimension n. Thus, a space X is a generalized n-manifold if and only if it is n-dimensional and conditions 1.1 (a) and (b) hold.

The geometric conditions imposed on most spaces in this paper automatically imply that the dimension is finite. Because of the assumption on finite–dimensionality, a generalized manifold is tacitly assumed separable. If not otherwise stated, we assume that a generalized manifold is always given a metric.

2. Orientation and local degree

If a generalized n–manifold X satisfies $H_c^n(X) \simeq \mathbb{Z}$, then X is said to be *orientable* and a choice of a generator g_X in $H_c^n(X)$ is called an *orientation*; X together with g_X is an *oriented* generalized n–manifold. If X is oriented, we can simultaneously choose an orientation g_U for all connected open subsets U of X via the isomorphisms

$$H_c^n(U) \to H_c^n(X);$$

see 1.3 (d).

Next, let X and Y be oriented generalized n–manifolds, and let $f : X \to Y$ be continuous. For each relatively compact domain D in X (a *domain* is an open connected set) and for each component V of $Y \setminus f(\partial D)$ the map

$$f|f^{-1}(V) \cap D : f^{-1}(V) \cap D \to V$$

is proper. Hence we have a sequence of maps

$$H_c^n(V) \to H_c^n(f^{-1}(V) \cap D) \to H_c^n(D), \tag{2.1}$$

where the first map is induced by f and the second map is the standard homomorphism as in (1.2). The composition of these two maps sends the generator g_V to an integer multiple of the generator g_D; this integer, denoted by $\mu(y, D, f)$, is called the *local degree* of f at a point $y \in V$ with respect to D. The local degree is an integer valued locally constant function

$$y \mapsto \mu(y, D, f)$$

defined in $Y \setminus f(\partial D)$. If $V \cap f(D) = \emptyset$, then $\mu(y, D, f) = 0$ for $y \in V$.

2.2. **Definition.** A continuous map $f : X \to Y$ between two oriented generalized n–manifolds is said to be *sense-preserving* if

$$\mu(y, D, f) > 0$$

whenever $D \subset X$ is a relatively compact domain and $y \in f(D) \setminus f(\partial D)$.

2.3. **Basic properties of the local degree.** The following properties of the local degree are easily ascertained.

(a) If $f, g : X \to Y$ are homotopic through proper maps h_t, $0 \le t \le 1$, such that $y \in Y \setminus h_t(\partial D)$ for all t, then

$$\mu(y, D, f) = \mu(y, D, g).$$

(b) If $y \in Y \setminus f(\partial D)$ and if $f^{-1}(y) \subset D_1 \cup \ldots \cup D_p$, where D_i are all disjoint domains and contained in D such that $y \in Y \setminus f(\partial D_i)$, then

$$\mu(y, D, f) = \sum_{i=1}^{p} \mu(y, D_i, f).$$

(c) If $f : D \to f(D)$ is a homeomorphism, then $\mu(y, D, f) = \pm 1$ for each $y \in f(D)$. In particular, if f is a local homeomorphism, there is for each $x \in X$ a connected neighborhood D such that $\mu(f(x), D, f) = \pm 1$. More generally, if f is discrete and open and $x \in X$, then there is a relatively compact neighborhood D of x such that $\{f^{-1}(f(x))\} \cap \overline{D} = \{x\}$; the number $\mu(f(x), D, f) = i(x, f)$ is independent of D and called the *local index* of f at x. (Recall that X is locally connected by assumption.)

(d) If f is open, discrete, and sense-preserving, then for each $x \in X$ there is a connected neighborhood D as in (c) above such that $f(\partial D) = \partial(f(D))$; D is called a *normal neighborhood* of x, and

$$i(x,f) = \max_{y \in f(D)} \operatorname{card}\,\{f^{-1}(y) \cap D\}. \tag{2.4}$$

If D is any domain such that $f(\partial D) = \partial(f(D))$, then D is called a *normal domain*.

3. Sense-preserving branched covers

Let $f : X \to Y$ be a continuous map between generalized n–manifolds. Recall from the introduction that f is said to be a branched cover if f is a discrete and open map.

3.1. **Branch set.** The *branch set* B_f of f is the closed set of points in X where f does not define a local homeomorphism. A theorem of Chernavskiĭ [C1], [C2] and Väisälä [V2] implies that for a branched cover $f : X \to Y$ the branch set B_f has topological dimension at most $n-2$. (This result is often stated for maps between manifolds but Väisälä's proof in particular extends to the present situation.) It follows that $X \setminus B_f$ is connected if $f : X \to Y$ is a branched cover by the proofs in [Bo, I.4.7], [Br, V.16.20].

3.2. **Remark.** If $f : X \to Y$ is a branched cover between two oriented generalized n–manifolds, then f is either sense-preserving or sense–reversing. The latter means that $\mu(y, D, f) < 0$ for all $y \in f(D) \setminus f(\partial D)$. For a proof, see [V1, 5.2].

3.3. **Path–lifting.** We next explain a path–lifting property of branched covers. Let $f : X \to Y$ be a sense-preserving branched cover between oriented generalized manifolds, and let $\beta : [a,b) \to Y$ be a path. A *maximal f–lifting* of β starting at a point $x \in f^{-1}(\beta(a))$ is a path $\alpha : [a,c) \to X, a < c \leq b$, such that (i) $\alpha(a) = x$, that (ii) $f \circ \alpha = \beta|[a,c)$, and that (iii) if $c < c' \leq b$, then there does not exist a path $\alpha' : [a,c') \to X$ such that $\alpha = \alpha'|[a,c)$ and $f \circ \alpha' = \beta|[a,c')$.

If $x_1, \ldots, x_k$ are k distinct preimages of $\beta(a)$, then there exists a *maximal sequence* $\alpha_1, \ldots, \alpha_m$ *of f–liftings* of β starting at the points $x_1, \ldots, x_k$. This means that

$$m = \sum_{i=1}^{k} i(x_i, f), \tag{3.4}$$

$$\text{each } \ \alpha_j : [a, c_j) \to X \ \text{ is a maximal } f\text{–lifting of } \ \beta, \tag{3.5}$$

$$\operatorname{card}\,\{j : \alpha_j(a) = x_i\} = i(x_i, f), \tag{3.6}$$

and

$$\operatorname{card}\,\{j : \alpha_j(s) = x\} \leq i(x,f) \ \text{ for all } \ x \in X \ \text{ and all } \ s. \tag{3.7}$$

The existence of a maximal sequence of f–liftings in the present generality can be proved as in [Ri2, II.3]; the original idea of that proof is due to Poletskiĭ [P].

4. BLD and regular maps

In this section, we study the elementary properties of BLD and regular maps. Recall the definitions from 0.1 and 0.5.

4.1. **Notation.** Throughout this paper, we denote by $B(x,r)$ an open ball of radius r centered at x in a metric space X, and by $\overline{B}(x,r)$ the closure of $B(x,r)$. By a *closed ball* (of radius r centered at x) we mean the set $B_c(x,r) = \{y \in X : |x-y| \le r\}$. Note that $B(x,r) \subset \overline{B}(x,r) \subset B_c(x,r)$, and that all inclusions can be strict.

4.2. **Lemma.** *A regular map $f : X \to Y$ between arbitrary metric spaces with data (L,M,N) is at most M–to–one.*

Proof. Suppose that $x_1, \ldots, x_{M+1}$ are distinct points in X with image y. Let

$$0 < \varepsilon < \frac{1}{2N} \min_{i \neq j} |x_i - x_j|.$$

Then diam $B_c(y,\varepsilon) = d \le 2\varepsilon$, and by assumption the points $x_1, \ldots, x_{M+1}$ belong to M balls in X each of diameter at most

$$Nd \le 2N\varepsilon < \min_{i \neq j} \; |x_i - x_j|.$$

This is a contradiction, and the lemma follows.

4.3. **Remarks.** (a) By Lemma 4.2, regular maps are always discrete. They need not be branched covers according to our definition; the folding $R^2 \to R^2$, $(x,y) \mapsto (x,|y|)$, is a regular map but not open.

(b) An open regular map between oriented generalized manifolds is either sense-preserving or sense–reversing by 3.2. In fact, a regular map between oriented generalized manifolds is sense-preserving or sense–reversing if and only if it is open. See the proof in [Ri2, VI.5.6].

(c) A BLD–map $f : X \to Y$ need not be regular. For example, a Riemannian covering map is BLD and has infinite multiplicity in general. More generally, if $f : X \to Y$ is locally uniformly bi–Lipschitz, then f is BLD if it is sense-preserving. We shall show in 4.5 below that under certain conditions on X and Y, BLD–maps that have finite multiplicity are regular maps.

4.4. **Quasiconvex spaces.** A metric space is said to be c*–quasiconvex*, $c \ge 1$, if every pair of points x, y in it can be joined by a curve whose length does not exceed $c|x-y|$.

4.5. **Theorem.** *Let X and Y be quasiconvex oriented generalized n–manifolds, let X be complete, and let $f : X \to Y$ be sense-preserving. Then f is regular if and only if it is a BLD–map of finite maximal multiplicity, quantitatively.*

4.6. **Remark.** None of the conditions (a) X is quasiconvex, (b) Y is quasiconvex, or (c) X is complete, can be left out in 4.5. We show this by the following simple examples.

First take X to be the open strip $\{(x_1, x_2) \in R^2 : x_1 > 1, |x_2| < 1/x_1^2\}$ and let $f_1 : X \to R^2 = Y$ be a bounded injective map such that f_1 is locally 2–bi–Lipschitz. Then (a) and (b) hold, but not (c). The map f_1 is BLD but not regular. Next consider the infinite cylinder $X = \{(x_1, x_2, x_3) \in R^3 : x_2^2 + x_3^2 = 1/(1+x_1^2)^2\}$, and let $f_2 : X \to f_2(X) = Y \subset R^3$ be a bounded, injective locally 2–bi–Lipschitz map. Then (a) and (c) hold, but not (b). The map f_2 is BLD but not regular. Finally, to get a case where (b) and (c) are fulfilled but not (a), one takes $f_3 = \hat{f}_2^{-1}$, where $\hat{f}_2 : X \to \hat{f}_2(X) \subset R^3$ is an unbounded version of the above f_2; the image $\hat{f}_2(X)$ wiggles more and more the further out it goes. Then f_3 is BLD but not regular because it is not Lipschitz.

On the other hand, if we know *a priori* that f is Lipschitz in Theorem 4.5, then the assumption that X be quasiconvex is not needed. This will be clear from the proof.

A regular map always satisfies (0.2):

4.7. **Lemma.** *A regular map $f : X \to Y$ between arbitrary metric spaces satisfies*

$$\frac{1}{C} \text{ length } \alpha \leq \text{length } f \circ \alpha \leq C \text{ length } \alpha \tag{4.8}$$

for all paths α in X, where $C \geq 1$ only depends on the data of f.

Naturally, condition (4.8) is interesting only if X and Y have nontrivial rectifiable paths. The proof of Theorem 4.5 will show that neither completeness nor quasiconvexity is needed in the "only if" direction.

Proof of Lemma 4.7: Let $f : X \to Y$ be a regular map between arbitrary metric spaces with data (L, M, N). Because f is L–Lipschitz, we always have that

$$\text{length } f \circ \alpha \leq L \text{ length } \alpha \tag{4.9}$$

for any path α in X, whether finite or infinite length. Next we shall show that the left inequality in (4.8) holds for a given (nonconstant) path $\alpha : [0,1] \to X$ with $C = C(L, M, N) \geq 1$.

To this end, let $0 = t_0 \leq \ldots \leq t_k = 1$ be a partition, where without loss of generality

$$|t_i - t_{i+1}| = \varepsilon$$

for all $i \in \{0, \ldots, k-1\}$. Let α_i be the subpath $\alpha|[t_i, t_{i+1}]$ and $|\alpha_i|$ its image. From Lemma 4.10 below it follows that

$$\text{diam } f(|\alpha_i|) \geq \frac{1}{2MN} \text{ diam } |\alpha_i| \geq \frac{1}{2MN} |\alpha(t_i) - \alpha(t_{i+1})|.$$

Therefore we have for some $t' < t''$ in $[t_i, t_{i+1}]$ that

$$\begin{aligned} |\alpha(t_i) - \alpha(t_{i+1})| &\leq 2MN |f \circ \alpha(t') - f \circ \alpha(t'')| \\ &\leq 2MN(|f \circ \alpha(t_i) - f \circ \alpha(t')| + |f \circ \alpha(t') - f \circ \alpha(t'')| \\ &\quad + |f \circ \alpha(t'') - f \circ \alpha(t_{i+1})|), \end{aligned}$$

whence

$$\sum_{i=0}^{k-1} |\alpha(t_i) - \alpha(t_{i+1})| \leq 2MN \text{ length } f \circ \alpha.$$

Because the partition was arbitrary, this and (4.9) imply (4.8) with $C = \max\{2MN, L\}$.

4.10. **Lemma.** ([DS3, 12.4]) *Let $f : X \to Y$ be a regular map between arbitrary metric spaces with data (L, M, N) and let $\gamma \subset X$ be a connected set. Then*

$$\mathrm{diam}\gamma \leq 2MN \mathrm{diam} f(\gamma).$$

Proof. We can assume that $\mathrm{diam} f(\gamma) = d > 0$. Then $f(\gamma)$ belongs to the closed ball $B_c(y, d)$ for some $y \in f(\gamma)$. We have diam $B_c(y, d) \leq 2d$ and by regularity we can cover γ by M closed balls of diameter at most $2Nd$. This proves the lemma.

Proof of Theorem 4.5. We have already seen that a regular map $f : X \to Y$ satisfies (4.8) for arbitrary metric spaces X and Y. If, in addition, the spaces are oriented generalized n–manifolds and f is sense-preserving, then f is open by 4.3 (b), and hence a BLD–map. The proof shows that if f has data (L, M, N), then we can take $\max\{2MN, L\}$ for the BLD–constant.

Next, assume that $f : X \to Y$ is an L–BLD map onto Y with maximal multiplicity

$$p = \max_{y \in Y} \text{ card } f^{-1}(y) < \infty,$$

and that Y is c_Y–quasiconvex. Let $B_c(y_0, R)$ be a closed ball in Y. We may assume that there exists $z_0 \in B_c(y_0, R)$ such that

$$\emptyset \neq \{x_1, \ldots, x_q\} = f^{-1}(z_0), \quad 1 \leq q \leq p.$$

We claim that

$$f^{-1}(B_c(y_0, R)) \subset \bigcup_{i=1}^{q} B_c(x_i, 2Lc_Y R). \tag{4.11}$$

Because we may assume that diam $B_c(y_0, R) \geq R/2$, it follows from (4.11) and (4.12) below that f is regular with data $(Lc_X, p, 8Lc_Y)$, if X is c_X–quasiconvex.

To prove (4.11), let $x \in X$ be such that

$$|f(x) - y_0| \leq R.$$

Let $\gamma : [0, 1] \to Y$ be a path with $\gamma(0) = f(x), \gamma(1) = z_0$ such that

$$\text{length } \gamma \leq c_Y |f(x) - z_0| \leq c_Y 2R.$$

By the path–lifting property of sense-preserving branched covers, explained in 3.3, there is a maximal f–lifting α of γ starting at x. Because f is a BLD–map, and because γ

has finite length, α stays bounded; moreover, by completeness of X, α ends at a point $u \in X$, with $f(u) = z_0$. Thus $u = x_i$ for some $i = 1, \ldots, q$, which implies

$$|x - x_i| \leq \text{length } \alpha \leq L \text{ length } \gamma \leq 2Lc_Y R.$$

Thus (4.11) follows.

To show that f is Lipschitz, fix $a, b \in X$ and pick a curve γ from a to b such that length $\gamma \leq c_X |a - b|$. Then

$$|f(a) - f(b)| \leq \text{length } f \circ \gamma \leq L \text{ length } \gamma \leq Lc_X |a - b|, \tag{4.12}$$

which shows that f is Lc_X–Lipschitz if it is L–BLD and X is c_X–quasiconvex. This completes the proof of Theorem 4.5.

In the next proposition, we establish a useful geometric description of BLD–maps; it is essentially a corollary to Theorem 4.5.

4.13. **Proposition.** *Let X and Y be quasiconvex oriented generalized n-manifolds, let X be complete, and let $f : X \to Y$ be a surjective BLD–map of finite multiplicity. Then there is a constant $C \geq 1$, depending only on the data, with the following property: for every ball $B = B(x, r)$ in X we have that*

$$B(f(x), r/C) \subset f(B) \subset B(f(x), Cr)\,, \tag{4.14}$$

and for every ball $B = B(y, r)$ in Y we have that

$$\bigcup_{x \in f^{-1}(y)} B(x, r/C) \subset f^{-1}(B) \subset \bigcup_{x \in f^{-1}(y)} B(x, Cr)\,. \tag{4.15}$$

Proof. The second inclusion in (4.14) is clear because f is Lipschitz by Theorem 4.5. To prove the first inclusion, let ℓ denote the distance from $f(x)$ to $\partial f(B)$, where $B = B(x, r) \subset X$. Note that $\ell > 0$ because f is open and that there is nothing to prove if $\partial f(B) = \emptyset$. Let β be a path from $f(x)$ to $\partial f(B)$ of length no more than a fixed constant times ℓ. Then any lift α of β, starting from x, must hit the boundary of B by the completeness of X, cf. 3.3. Thus

$$r \,\leq\, \text{length}\, \alpha \,\leq\, C\, \text{length}\, \beta \,\leq\, C\, \ell\,,$$

as desired. This proves (4.14).

Now the first inclusion in (4.15) is an easy consequence of the fact that f is Lipschitz (Theorem 4.5), while the second inclusion follows from (4.11) (note that we can take $z_0 = y$ in the proof of (4.11)). The proposition is thereby proved.

4.16. **Remarks.** (a) The quasiconvexity of X is not needed in Proposition 4.13 if f is assumed to be Lipschitz, cf. the comment at the end of Remark 4.6.

(b) It is clear that Proposition 4.13 has a local version, valid for BLD–maps of arbitrary multiplicity, and for source spaces X that need not be complete. (Recall our standing assumption that all spaces are locally compact.)

(c) It follows from (4.14) in Proposition 4.13 that BLD–maps between quasiconvex, complete, oriented generalized n-manifolds are *Lipschitz quotient maps* in the sense of Bates et. al. [BJLPS]. Indeed, the assumption on finite multiplicity is not needed for (4.14), as the proof shows.

The definition for a BLD–map $f : X \to Y$ consists of four conditions: the map has to be open, discrete, and sense-preserving, and it has to satisfy (0.2). Of these, only the requirement "sense-preserving" needs X and Y to be generalized manifolds. We next discuss what happens if these conditions are relaxed, or replaced with other similar conditions.

4.17. **Light maps.** A continuous map between topological spaces is *light* if the preimage of every point is totally disconnected, i.e., all its components are points.

4.18. **Proposition.** *A sense-preserving and light map between generalized manifolds is discrete and open.*

In the case of manifolds, Proposition 4.18 is due to Titus and Young [TY]. The proof extends easily; see [Ri2, IV.5.6].

4.19. **Corollary.** *A sense-preserving map between generalized manifolds is a BLD–map if and only if it is light and (0.2) holds.*

4.20. **Remarks.** (a) Light maps between manifolds need not be discrete, even if they satisfy (0.2); thus the assumption "sense-preserving" in Corollary 4.19 is necessary.

(b) There is a light map of the two sphere onto itself that satisfies (0.2) and maps a Cantor set to a point. To construct such a map, consider the standard Cantor ternary set C on $[0,1]$ and define a map $f : [0,1] \to [0,1]$ by $f(x) = 6\,\mathrm{dist}(x, C)$. By projecting f to the unit circle, we obtain an onto map $S^1 \to S^1$ that takes a Cantor set to a point. To get a desired map on S^2, we suspend.

(c) We do not know whether a continuous map between path–connected metric spaces that satisfies (0.2) is always light. Naturally, no fiber of such a map can contain a non–constant path.

(d) Väisälä [V4] (for $1 \le n \le 3$) and Church [Ch] (all $n \ge 1$) have shown that every countable map between n–manifolds is a local homeomorphism in a dense set. (A map is *countable* if it has countable fibers.) In particular, if $f : X \to Y$ is a regular map between metric quasiconvex manifolds, then there is an open dense set $\Omega_f \subset X$ such that f defines a locally bi–Lipschitz map, either sense-preserving or sense–reversing, in each component of Ω_f. Recall from 3.1 that Ω_f is connected if f is in addition open.

We study the metric size of $B_f = X \setminus \Omega_f$ in Section 6.

5. Spaces of type A

In this section, we isolate a class of generalized manifolds whose Lipschitz analysis is similar to that on Riemannian spaces. (Compare [HS].)

5.1. **Definition.** Let X be a metric space, and let $n \geq 2$ be an integer. Consider the following properties that X may or may not possess:

(A1) X is n–rectifiable and has locally finite $\mathcal{H}_n$–measure;
(A2) X is locally Ahlfors n–regular;
(A3) X is locally bi–Lipschitz embeddable in Euclidean space;
(A4) X is locally linearly contractible.

We call X a *space of type* $A1, A2, \ldots, A12, \ldots$, if it satisfies axioms $(A1), (A2), \ldots$, $(A1)$ and $(A2), \ldots$. If X satisfies all four axioms $(A1)$–$(A4)$, then we simply call X of *type* A. Note that in this terminology, the dimension n should be understood as fixed.

The first condition in $(A1)$ means, by definition, that X is a countable union of Lipschitz images of subsets of R^n plus a set of Hausdorff n–measure zero. (Observe the differing terminology in the literature: for subsets X of Euclidean space, the above condition is called *countable* $(\mathcal{H}_n, n)$*–rectifiability* in [F, 3.2.14].)

Condition $(A2)$ means, by definition, that for each compact set $K \subset X$ there are constants $C_K \geq 1$ and $r_K > 0$ such that the Hausdorff n–measure $\mathcal{H}_n$ in X satisfies

$$\frac{1}{C_K} r^n \leq \mathcal{H}_n(B_r) \leq C_K r^n \tag{5.2}$$

for each ball B_r of radius $0 < r \leq r_K$ centered at a point in K.

The meaning of $(A3)$ is clear. The dimension of the receiving Euclidean space is allowed to depend on the local embedding.

The last condition $(A4)$ means, by definition, that for each compact set $K \subset X$ there are constants $C_K \geq 1$ and $r_K > 0$ such that for each $0 < r \leq r_K$ and for each $x \in K$ the ball $B(x,r)$ is contractible in $B(x, C_K r)$. (Note that our requirement is weaker than the one bearing a similar term in [S4].)

Obviously, every Riemannian manifold is a space of type A. It is also easy to see that every Lipschitz n–manifold admits metrics that make the manifold into a space of type A. In particular, by Sullivan's theorem [Su], every topological n–manifold, $n \neq 4$, admits type A metrics. We shall show in a future note that more generally every quasiconformal n–manifold admits metrics that turn the manifold into a space of type A. This is less obvious.

5.3. **Change of variables formula.** Let X^n and Y^n be two spaces of type A1, and let $f : X \to Y$ be a Lipschitz map. Under these assumptions, we have the following proposition:

5.4. **Proposition.** *There is a locally integrable nonnegative function $J(f,x)$ on X such that*

$$\int_X u \circ f(x) J(f,x)\, dx = \int_Y u(y) N(f,y)\, dy \tag{5.5}$$

for each nonnegative measurable function u on Y.

Throughout this paper, we use the notation

$$
\begin{aligned}
&(5.6) \qquad N(f,y) = \operatorname{card}\{f^{-1}(y) : y \in Y\},\\
&\qquad\qquad N(f,A,y) = \operatorname{card}\{f^{-1}(y) \cap A : y \in Y\}, \qquad A \subset X,
\end{aligned}
$$

if $f : X \to Y$ is a map. Moreover, if the underlying Borel measure in a space is understood, we write $dx, dy, \ldots$ inside integrals; for instance, in (5.5), we have $dx = d\mathcal{H}_n(x)$ and $dy = d\mathcal{H}_n(y)$.

Proposition 5.4 is essentially a result of Kirchheim's [K], who proved that if $A \subset R^n$ is (Lebesgue) measurable, and if g is a Lipschitz map of A into an n–rectifiable metric space Z (in particular Z can be a space of type $A1$ as defined in 5.1), then there is a nonnegative locally integrable function $\mathcal{J}(g,\cdot)$ on A such that

$$
(5.7) \qquad \int_A v \circ g(a)\mathcal{J}(g,a)da = \int_Z v(z)N(g,A,z)dz
$$

for each nonnegative measurable function v on Z. See [K, Corollary 8, (ii)]. The "Jacobian" $\mathcal{J}(g,\cdot)$ is explicitly described in [K], where the notation $\mathcal{J}(MD(g,\cdot))$ is used. If Z is a subset of some Euclidean space, $\mathcal{J}(g,\cdot)$ is the *approximate Jacobian* of g described in [F, 3.2.20]; in this case, formula (5.7) is proved in [F, 3.2.20] with the assumption $\mathcal{H}_n(Z) < \infty$. We deduce (5.5) from these known formulas. (In fact, we found Proposition 5.4 also in a recent preprint of Ambrosio and Kirchheim [AK].)

To this end, write X as a countable disjoint union of subsets $A'_1, A'_2, \ldots$ plus a set E of n–measure zero,

$$
X = \bigcup_{k=1}^{\infty} A'_k \cup E,
$$

such that $A'_k = \lambda_k(A_k)$ for some Borel set $A_k \subset R^n$ and for some bi–Lipschitz map $\lambda_k : A_k \to A'_k$, for each k. From Lemma 4 and from the proof of Theorem 7 in [K], together with assumption $(A1)$, such sets A'_k and maps λ_k can be found (see also [F, 3.2.18]). Fix a measurable function $u : Y \to [0,\infty)$. Define

$$
J(f,x) = \frac{\mathcal{J}(f \circ \lambda_k, \lambda_k^{-1}(x))}{\mathcal{J}(\lambda_k, \lambda_k^{-1}(x))}
$$

for $x \in A'_k$. Because λ_k is bi–Lipschitz, the function $\mathcal{J}(\lambda_k,\cdot)$ is bounded away from 0 and ∞ almost everywhere on A_k. Then, by Kirchheim's formula (5.7), we have that

$$
\begin{aligned}
(5.8) \qquad &\int_{A_k} u \circ f \circ \lambda_k(a)\mathcal{J}(f \circ \lambda_k, a)da\\
&= \int_Y u(y)N(f \circ \lambda_k, y)dy = \int_Y u(y)N(f, A'_k, y)dy.
\end{aligned}
$$

On the other hand, we can apply (5.7) again and obtain that the left hand side of (5.8) is equal to

$$\int_{A_k} u \circ f \circ \lambda_k(a) \frac{\mathcal{J}(f \circ \lambda_k, \lambda_k^{-1} \circ \lambda_k(a))}{\mathcal{J}(\lambda_k, \lambda_k^{-1} \circ \lambda_k(a))} \mathcal{J}(\lambda_k, a) da$$

$$= \int_{A'_k} u \circ f(x) J(f, x) dx.$$

Thus (5.5) follows.

5.9. **Metric orientation.** Assume that X is an oriented generalized n–manifold (with metric) as defined in 1.6, and that X is of type $A123$. Let U be an open connected neighborhood of a point in X that can be embedded bi–Lipschitzly in some R^N. Thus, without much loss of generality, we assume that $U \subset R^N$ and that U has finite Hausdorff n–measure. Because of properties $(A1)$ and $(A2)$, the set U has a tangent n–plane T_xU at a.e. point $x \in U$. The collection of these planes is called a *measurable tangent bundle* of U, and denoted by TU. Note in particular that $(A2)$ guarantees that the tangent cone is an n-plane at a.e. point in X. (See [F, 3.2.16, 3.2.19].)

Each n–plane T_xU, whenever exists, is an n–dimensional subspace of R^N, and a measurable choice of orientations $\xi = (\xi_x)$ on each T_xU is called an *orientation* of the tangent bundle TU. Such orientations always exist. Because X is an oriented generalized n–manifold, there is another orientation on U, provided U is connected; this is a generator g_U in the group $H^n_c(U) = \mathbb{Z}$ determined by the fixed orientation on X. (See Section 2).

Now fix a point $x \in U$ such that T_xU exists. Then the projection

$$\pi_x : R^N \to T_xU + x$$

satisfies $x \notin \pi_x(\partial D)$, whenever D is a sufficiently small open connected neighborhood of x in U. Thus, if V is the x–component of $(T_xU + x) \setminus \pi_x(\partial D)$, we have

$$\begin{aligned} H^n_c(T_xU) \leftarrow H^n_c(V) &\xrightarrow{\pi_x^*} H^n_c(\pi_x^{-1}(V) \cap D) \\ &\to H^n_c(D) \to H^n_c(U), \end{aligned} \tag{5.10}$$

where the three unnamed arrows represent a canonical isomorphim. If the orientations ξ_x and g_U correspond to each other under the map in (5.10), we say that T_xU and U are *coherently oriented* at x by ξ_x and g_U; if a measurable coherent orientation $\xi = (\xi_x)$ is chosen at a.e. point, we say that TU is *metrically oriented* by ξ and g_U.

We shall need the following fact:

5.11. **Proposition.** [HS, Example 3.9] *Every generalized n–manifold of type A can be (locally) metrically oriented.*

5.12. **Remark.** In the ensuing sections, we could often replace assumption $(A4)$ by the weaker requirement that X admit a Poincaré inequality as in (9.7). More precisely, the assumption on linear local contractibility $(A4)$ is only used to guarantee that such an inequality holds (via Semmes's work [S4]) and that spaces of type A can be locally metrically oriented (as in Proposition 5.11 above). Thus, if only the validity of a Poincaré inequality is assumed, then one needs in addition to stipulate the metric orientability of X. We have chosen to work with conditions $(A1)-(A4)$ for their relative simplicity. There is room left for research as to what conditions exactly are needed for the results in the next section. (Compare the discussion in [HS, Section 5].)

6. Analytic and geometric properties of BLD–maps

In [MVä], Martio and Väisälä established interesting analytic and geometric properties of BLD–maps in Euclidean space. In this section, we prove results along these lines for BLD–maps between generalized manifolds. Many properties of BLD–maps in [MVä] were deduced from the more general theory of quasiregular maps. In our case, no such theory is currently available.

All spaces in this section are assumed to be oriented generalized manifolds with metric, unless otherwise stated.

Our first proposition follows from Theorem 4.5 because (this is easy to see) regular maps satisfy (6.2), quantitatively (cf. Lemma 4.10).

6.1. **Proposition.** *Let f be a BLD–map of finite multiplicity of a complete quasiconvex space X into a quasiconvex space Y. Then*

$$\frac{1}{C}\mathcal{H}_p(A) \leq \mathcal{H}_p(f(A)) \leq C\mathcal{H}_p(A) \tag{6.2}$$

for every set $A \subset X$ and for every number $p > 0$, where $\mathcal{H}_p$ denotes the Hausdorff p–measure in respective spaces and $C \geq 1$ depends only on p and on the data associated with f, X, and Y. In particular, BLD–maps between (locally) quasiconvex spaces preserve the Hausdorff dimension of every set.

The next proposition follows from Propositions 6.1 and 4.13. Note that 6.3 admits a local version by Remark 4.16 (b).

6.3. **Proposition.** *Assume that f, X, and Y are as in Proposition 4.13. Then, if either of the spaces X or Y is Ahlfors n-regular, then so is the other one, quantitatively.*

Next we prove a theorem concerning the size of the branch set of a BLD–map. The fact that both B_f and $f(B_f)$ have measure zero is of great importance in the theory of quasiregular maps in Euclidean space. We do not know whether Theorem 6.4 remains valid for BLD–maps between more general spaces.

6.4. **Theorem.** *Let $f : X^n \to Y^n$ be a BLD–map between locally quasiconvex spaces of type A. Then $\mathcal{H}_n(B_f) = \mathcal{H}_n(f(B_f)) = 0$.*

Proof. By the proof of Theorem 4.5, f is locally Lipschitz. Thus it suffices to show that $\mathcal{H}_n(B_f) = 0$. Because the issue is local, and the conclusion is bi–Lipschitz invariant, we may assume that X and Y are metrically oriented subsets of some Euclidean space by the discussion in 5.9 and by Proposition 5.11. In this framework, f is approximately differentiable at $\mathcal{H}_n$ almost everywhere in X; see [F, 3.2.16 and 3.2.19] and 6.5 below. The theorem therefore follows from Proposition 6.6 below.

6.5. **Approximate differentiability.** Recall that a map f of an n–rectifiable subset X of R^N into R^M is *approximately differentiable* at $x_0 \in X$ if x_0 has a neighborhood in X of finite $\mathcal{H}_n$ measure and if there exists $y_0 \in R^M$ and a linear map $T : R^N \to R^M$ such that

$$\lim_{r\to 0} \mathcal{H}_n(\{x \in X : |x - x_0| < r \text{ and } |f(x) - y_0 - T(x - x_0)| \geq \varepsilon|x - x_0|\})r^{-n} = 0$$

for each $\varepsilon > 0$. If a tangent n–plane $T_{x_0}X$ exists at x_0, the restriction of T to $T_{x_0}X$ is the *approximate differential* of f at x_0, and denoted by $\mathrm{ap}Df(x_0)$.

Next, assume that X and Y are two metrically oriented generalized n–manifolds of type A lying in some Euclidean space. Denote the respective orientations by (ξ, g_X) and (η, g_Y). Let OT_X be the set of all points in X such that T_xX exists, that T_xX and X are coherently oriented by ξ_x and g_X, and that X has Hausdorff n–measure density 1 at x (see [F, 2.10.19, 3.2.22]). The set OT_X has full measure in X. We define OT_Y similarly.

If $f : X \to Y$ is as in Theorem 6.4, it follows from Proposition 6.1 that the sets $f^{-1}(OT_Y) \cap OT_X$ and $f(OT_X) \cap OT_Y$ have full measure in X and $f(X)$, respectively.

6.6. **Proposition.** *Let $f : X \to Y$ be as in Theorem 6.4, with the added assumption that X and Y are metrically oriented subsets of some Euclidean space. Then f cannot be approximately differentiable at any point in $B_f \cap f^{-1}(OT_Y) \cap OT_X$.*

Proof. Let $x_0 \in f^{-1}(OT_Y) \cap OT_X$ be a point where f is approximately differentiable. Because f is a regular map near x_0 by Theorem 4.5, it is easy to see that the approximate differential $\mathrm{ap}Df(x_0)$ cannot be degenerate; that is, it has to map the tangent plane $T_{x_0}X$ onto the tangent plane $T_{y_0}Y$, where $f(x_0) = y_0$. (Compare [F, 3.2.22].) Moreover if we fix a sufficiently small neighborhood D of x_0 in X, then it is easy to see that the maps $\mathrm{ap}\overline{Df}(x_0) \circ \pi_{x_0}$ and $\pi_{y_0} \circ f$ are homotopic as maps $D \to T_{y_0}Y + y_0$, where π_* denotes the projection map as in 5.9, and $\mathrm{ap}\overline{Df}(x_0)$ is the affine map between the planes $T_{x_0}X + x_0$ and $T_{y_0}Y + y_0$ determined by the approximate differential $\mathrm{ap}Df(x_0)$. By using this and the definition for coherent orientation, it is clear that f must have local index $i(x_0, f) = 1$ at x_0 (see 2.3). In particular, f is a local homeomorphism at x_0, and the proposition follows.

6.7. **Remarks.** (a) The assumption that f be sense-preserving in Proposition 6.6 is crucial; it is not hard to give an example of a regular map $f : R^n \to R^n$ that is differentiable at a point in the branch set.

(b) Theorem 6.4 is sharp in the following sense: for each $n \geq 3$ and $\varepsilon > 0$ there is a BLD–map $f : R^n \to R^n$ such that the Hausdorff dimension of B_f is at least $n - \varepsilon$. On the other hand, in R^n, the branch set of an L–BLD–map has Hausdorff dimension at most $c = c(n, L) < n$. We do not know in what generality this latter statement is true. (See [MVä, 4.27] for the above assertions.)

In the next theorem, we shall give a growth estimate for the multiplicity function of a BLD–map. A similar result in R^n was established in [MVä]. Recall the notation (5.6).

6.8. **Theorem.** *Let $f : X^n \to Y^n$ be an L-BLD-map between oriented generalized manifolds of type A. Assume in addition that X is c_X-quasiconvex and that Y is c_Y-quasiconvex. If $x_0 \in X$, $r > 0$, and if $\lambda > 1$ such that the ball $B(x_0, \lambda r)$ has compact closure in X, then*

$$N(f, B(x_0, r), y) \leq (Lc_X)^n \frac{\mathcal{H}_n(B(x_0, \lambda r))}{\mathcal{H}_n(B(y, \frac{(\lambda-1)}{Lc_Y} r))}$$

for each $y \in Y$.

The following corollary is noteworthy.

6.9. **Corollary.** *Let $f : X \to Y$ be as in Theorem 6.8. Assume, in addition, that closed balls in X are compact and that the following mass bounds are satisfied:*

$$\mathcal{H}_n(B(x, r)) \leq V_X r^n, \tag{6.10}$$

$$\mathcal{H}_n(B(y, r)) \geq V_Y r^n, \tag{6.11}$$

for all $x \in X$, $y \in Y$, and for all positive radii $r < 2\mathrm{diam}X$. Then the maximal multiplicity of f is finite, and is at most

$$2^n L^{2n} (c_X c_Y)^n V_X V_Y^{-1}.$$

Proof. Apply 6.8 to $x_0 \in X, \lambda = 2$, to $r = \frac{3}{2}\mathrm{diam}X$ in the case $\mathrm{diam}X < \infty$ and r arbitrary for $\mathrm{diam}X = \infty$.

Proof of Theorem 6.8. We follow the argument in [MVä, p. 436]. Pick $y \in Y$ and let

$$\{x_1, \ldots, x_k\} = f^{-1}(y) \cap B(x_0, r).$$

The preimage $f^{-1}(y)$ of y in $B(x_0, r)$ is finite because $B(x_0, \lambda r)$ is assumed to be precompact; we may also assume that $f^{-1}(y)$ is nonempty. Fix a radius

$$s \leq \frac{(\lambda - 1)r}{Lc_Y}, \tag{6.12}$$

and fix a point $e \in B(y, s) \backslash f(B_f)$. Because f is an open map and because $\mathcal{H}_n(f(B_f)) = 0$ by Theorem 6.4, the set $B(y, s) \backslash f(B_f)$ is indeed nonempty. Join y to e by a path γ with

$$\text{length } \gamma \leq c_Y s \leq \frac{(\lambda - 1)r}{L},$$

and consider a maximal sequence $\alpha_1, \ldots, \alpha_m$ of f-liftings of γ starting at the points $x_1, \ldots, x_k$ (see 3.3). Because

$$\text{length } \alpha_i \leq L \text{ length } f \circ \alpha_i \leq L \text{ length } \gamma \leq (\lambda - 1)r,$$

all the lifts stay in the ball $B(x_0, \lambda r)$. Therefore, because $B(x_0, \lambda r)$ is precompact in X, all the lifts α_i are defined on the full interval $[0, l(\gamma)]$ and $f \circ \alpha_i = \gamma,\ i = 1, \dots, m$. (We assume here that γ is parametrized by the arc length.)

It follows from property (3.7) of f–liftings that the point e gets hit from $B(x_0, \lambda r)$ at least

$$m = \sum_{i=1}^{k} i(x_i, f) \geq k$$

times, because $e \notin f(B_f)$. Thus

$$\int\limits_{B(x_0,\lambda r)} J(f,x)dx = \int\limits_{Y} N(f, B(x_0, \lambda r), w)dw$$

$$\geq \int\limits_{B(y,s)\setminus f(B_f)} N(f, B(x_0, \lambda r), w)dw$$

$$\geq k\mathcal{H}_n(B(y,s)).$$

On the other hand, because X is c_X–quasiconvex, f is Lc_X–Lipschitz, and it is easy to see that

$$J(f,x) \leq (Lc_X)^n.$$

By combining these two estimates, we have that

$$k \leq (Lc_X)^n \frac{\mathcal{H}_n(B(x_0, \lambda r))}{\mathcal{H}_n(B(y,s))}$$

for each s as in (6.12). The theorem follows.

6.13. **Analytic description of BLD–maps.** Let $f : X^n \to Y^n$ be an L–BLD–map between complete quasiconvex (oriented) generalized n–manifolds of type A. Then f is Lipschitz and in view of Propositions 5.4 and 6.1, the "Jacobian" $J(f,z)$ of f satisfies

$$J(f,z) \geq c > 0 \tag{6.14}$$

for a.e. z in X, and for some c depending only on the data associated with f and the spaces X and Y. The Jacobian in (6.14) was defined in Section 5.3, but can alternatively be described by

$$J(f,z) = \lim_{r \to 0} \frac{\mathcal{H}_n(f(B(z,r))}{\mathcal{H}_n(B(z,r))} \tag{6.15}$$

for a.e. $z \in X$, in view of (5.5) and 6.4.

Using local embeddings in some Euclidean space, and metric orientations as in 5.9, the *sign* of the determinant $\operatorname{detap} Dg$ is well defined almost everywhere for each Lipschitz map

$g : X \to Y$ such that $J(g,z) > 0$ for a.e. $z \in X$. (Compare 5.4 and 6.6.) In particular, for each such g we can define a *signed Jacobian* $\hat{J}(g,z)$ by

$$\hat{J}(g,z) = sgn \operatorname{detap} Dg(z) J(g,z), \tag{6.16}$$

for a.e. z. In the following, the notation $\hat{J}(g,z) > 0$ a.e. tacitly means that $J(g,z) > 0$ a.e.

6.17. **Proposition.** *Let $g : X^n \to Y^n$ be a Lipschitz map between two generalized n-manifolds of type A such that $\hat{J}(g,z) > 0$ a.e. Then g is sense-preserving.*

Proof. Let D be a precompact domain in X, and let $y \in g(D) \setminus g(\partial D)$. Fix $x \in D$ such that $g(x) = y$. It easily follows from the hypotheses, and from the change of variables formula (5.5), that there exists a set E of measure zero in D such that for some $b \in g(D) \setminus (g(E) \cup g(\partial D))$ we have

(1) $\#\{g^{-1}(b) \cap D\} < \infty$, and
(2) $\hat{J}(g,a) > 0$ for each $a \in g^{-1}(b) \cap D$.

Moreover, approximate tangent planes exist at every point in $D \setminus E$, and in $g(D) \setminus g(E)$. (This latter statement implicitly involves a use of local embeddings in Euclidean space.)

Let $\{a_1, \ldots, a_k\} = g^{-1}(b) \cap D$. It follows from (1) and (2) above, and from an easy homotopy argument, that there exist domains D_j in D, with pairwise disjoint closures, containing points a_j, $j = 1, \ldots, k$, such that $b \notin g(\partial D_j)$, and that $\mu(b, D_j, g) = 1$ for each j. In conclusion,

$$\mu(b, D, g) = \sum_{j=1}^{k} \mu(b, D_j, g) = k > 0,$$

as required. The proposition follows.

Martio and Väisälä [MVä] proved that condition (6.14) characterizes BLD–maps among sense-preserving, locally uniformly Lipschitz maps between Euclidean domains. The most difficult point in proving this characterization is Reshetnyak's theorem [Re1] which implies that (locally) Lipschitz maps in R^n, with Jacobian determinant positive and bounded away from zero, are discrete and open. A generalization of this result was recently given in [HS]. Thus, we are able to extend the characterization from [MVä] to the following general setting:

6.18. **Theorem.** *Let $f : X^n \to R^n$ be a Lipschitz map from a quasiconvex oriented generalized n–manifold X of type A into R^n such that the signed Jacobian (see (6.16)) is a.e. positive and that (6.14) holds. Then $f|W$ is a BLD–map for each precompact domain W in X, quantitatively. In particular, f is a sense-preserving, discrete and open map.*

In addition to the result from [HS], the proof of Theorem 6.18 requires some metric space analysis from [HK3], [S4]. First we need two lemmas.

For a map $f : X \to R^n$ the *linear dilatation* $H(x,f)$ and the *inverse linear dilatation* $H^*(x,f)$ are defined as

$$H(x,f) = \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)},$$

$$H^*(x,f) = \limsup_{s\to 0} \frac{L^*(x,s)}{l^*(x,s)},$$

where

$$L(x,r) = \sup_{|y-x|=r} |f(y) - f(x)|,$$

$$l(x,r) = \inf_{|y-x|=r} |f(y) - f(x)|,$$

$$L^*(x,s) = \sup_{z\in\partial U_s} |x - z|,$$

$$l^*(x,s) = \inf_{z\in\partial U_s} |x - z|,$$

where $U_s = U(x,s)$ is the x–component of $f^{-1}(B(f(x),s))$.

6.19. **Lemma.** *Let X be as in 6.18 with the additional assumption that X is geodesic. Let $f : X \to R^n$ be a sense-preserving, discrete and open M–Lipschitz map such that (6.14) holds a.e., and let $x \in X$. Then there are constants H_x, $H_x^* < \infty$, depending only on the data of X, on c, M, and the local index $i(x,f)$, such that the linear dilatation $H(x,f)$ and the inverse linear dilatation $H^*(x,f)$ satisfy*

$$H(x,f) < H_x, \tag{6.20}$$

$$H^*(x,f) < H_x^*. \tag{6.21}$$

.

Proof. Write $y = f(x)$. There exists $\sigma = \sigma_x > 0$ such that (see 2.3 (d))

(1) U_s is a normal neighborhood of x for $0 < s \le \sigma$ and $\mathrm{diam} U_s \to 0$ as $s \to 0$.
(2) $U_s = U_\sigma \cap f^{-1}(B(y,s))$ for $0 < s \le \sigma$,
(3) $\partial U_s = U_\sigma \cap f^{-1}(S(y,s))$ for $0 < s < \sigma$,
(4) $l^*(x, L(x,r)) = L^*(x, l(x,r)) = r$ for $0 < r < l^*(x,\sigma)$.

The proof of (1)–(4) given in [Ri2, II.4.1] for Euclidean space applies as such to X. In particular, because X is geodesic, open balls are connected which fact is sometimes used in [Ri2, II.4.1]. To prove (6.20) we modify the proof of [Ri2, II.4.2] to the present situation. Let $0 < r < l^*(x,\sigma)$ and suppose $l = l(x,r) < L = L(x,r)$. Then by (4), $F = \overline{U}_l$ is a continuum with $x \in F$ and $\mathrm{diam} F \ge r$, and the set $E = \overline{U}_\sigma \setminus U_L$ is a continuum with $\mathrm{dist}(E,F) < r$. The connectedness of E follows easily by lifting paths lying in the ring $B(y,\sigma) \setminus B(y,L)$. For small r we also have $\mathrm{diam} E \ge r$.

We want to apply a local version of [HK3, Theorem 3.6] to the *condenser* $(E', F; U_\sigma)$, where $E' = E \cap U_\sigma$. First define $v : B(y,\sigma) \to [0,1]$ by setting

$$v(z) = (\log \frac{L}{l})^{-1} \log \frac{|z-y|}{l}, \qquad l \le |z-y| \le L,$$

and by extending v as 0 in $B(y,l)$ and as 1 in $B(y,\sigma)\setminus\overline{B}(y,L)$. Then $u=v\circ f|U_\sigma$ satisfies $u|E'=1$ and $u|F=0$; hence u is admissible for the condenser $(E',F;U_\sigma)$. (For the terminology, see [HK3, Section 2].) Set

$$L_f(\xi)=\limsup_{|\zeta-\xi|\to 0}\frac{|f(\zeta)-f(\xi)|}{|\zeta-\xi|}.$$

We claim that the Borel function $\rho=(|\nabla v|\circ f)L_f$ is an upper gradient of u in U_σ, i.e.,

$$|u(\xi)-u(\zeta)|\le\int_{\gamma_{\xi\zeta}}\rho\,ds \tag{6.22}$$

whenever $\gamma_{\xi\zeta}$ is a rectifiable curve in U_σ from ξ to ζ. Since f is Lipschitz, $f\circ\gamma_{\xi\zeta}$ is rectifiable for each such $\gamma_{\xi\zeta}$, and, moreover, f is absolutely continuous on $\gamma_{\xi\zeta}$ (see [V3, 5.2]). Clearly

$$|u(\xi)-u(\zeta)|\le\int_{f\circ\gamma_{\xi\zeta}}|\nabla v|\,ds$$

for each $\gamma_{\xi\zeta}$, and by [V3, 5.3] we get

$$\int_{f\circ\gamma_{\xi\zeta}}|\nabla v|\,ds\le\int_{\gamma_{\xi\zeta}}\rho\,ds.$$

Thus (6.22) follows.

It follows from the assumptions that X satisfies locally the condition for a Loewner space as defined in [HK3, Section 3] (see [HK3, 5.7] and [S4, Theorem B.10]). In particular, this means that the n-capacity of the condenser $(E',F;U_\sigma)$ satisfies

$$0<\kappa\le\operatorname{cap}_n(E',F;U_\sigma),$$

where κ depends only on the data of X in U_σ. Clearly $L_f(\xi)\le M$ for all $\xi\in X$. By the definition [HK3, 2.11] of the n-capacity, by (6.14), and by Proposition 5.4 we obtain

$$\begin{aligned}\kappa&\le\int_{U_\sigma}\rho^n=\int_{U_\sigma}|\nabla v|^n(f(\xi))L_f^n(\xi)d\xi\\&\le c^{-1}M^n\int_{U_\sigma}|\nabla v|^n(f(\xi))J(f,\xi)d\xi\\&=c^{-1}M^n i(x,f)\int_{B(y,\sigma)}|\nabla v|^n(\eta)d\eta=c^{-1}M^n i(x,f)\omega_{n-1}(\log\frac{L}{l})^{1-n},\end{aligned}$$

which gives (6.20) with

$$H_x = [\exp(2\kappa^{-1}c^{-1}M^n i(x,f)\omega_{n-1}]^{1/(n-1)},$$

where ω_{n-1} is the $(n-1)$–measure of the unit sphere in R^n.

For the proof of (6.21) we use similarly a modification of the proof of [Ri2, II.4.4]. In particular, we obtain the bound

$$H_x^* = H_x^{2i(x,f)c^{-1}M^n}.$$

We leave the details to the reader, and conclude the proof of Lemma 6.19.

6.23. **Lemma.** *Let X be as in 6.18. Let $f : X \to R^n$ be a sense-preserving, discrete and open M–Lipschitz map such that (6.14) holds a.e., and let $x \in X$. There exists a constant $Q < \infty$, depending only on the data of X and on c, such that*

$$L^*(x,s) \le QH_x^* s, \quad 0 < s < \sigma = \sigma_x. \tag{6.24}$$

Proof. We may assume that X is geodesic because X is quasiconvex. Indeed, $L^*(x,f)$, $H^*(x,f)$, c, M, and the data given in the definition of type A change only up to a bounded factor when a bi–Lipschitz change is made on the metric of X. A (locally) geodesic bi-Lipschitz equivalent metric on X is obtained as the infimum of the lengths of the paths joining a given pair of points in X. (Recall that X is assumed to be locally compact.)

Let $\sigma = \sigma_x$ be as in the proof of 6.19. We may assume $f(x) = 0$. We apply the idea from [MVä, 2.12]. For a Borel set $E \subset S(s) = \partial B(0,s)$, let

$$\varphi(E) = \mathcal{H}_n(f^{-1}(C_E) \cap \overline{U}_s),$$

where C_E is the cone $\{tv : 0 \le t \le 1,\ v \in E\}$, and where $U_s = U(x,s)$ (the x-component of $f^{-1}(B(f(x),s))$. Fix a point v on $S(s)$ such that the derivative $\varphi'(v)$ exists and is finite; because the set function φ is completely additive and finite, almost every point on $S(s)$ is such a point. Write

$$\beta_v(t) = \frac{t}{s}v, \quad 0 \le t \le s,$$

and let $\alpha_v : [0,s] \to \overline{U}_s$ be any lifting of β_v starting at x. The path α_v defines a function $g_v : [0,s] \to [0,\infty)$ by

$$g_v(t) = |\alpha_v(t) - x|.$$

Slight modification of a part of the proof of [Ri2, II.5.3], starting on p. 41 there, gives that g_v is absolutely continuous. Here we need (6.21) and local n–regularity of X. In [Ri2, II.5.3] we also need the fact that $(z,\tau) \mapsto l^*(z,\tau)$ is locally upper semicontinuous and $(z,\tau) \mapsto L^*(z,\tau)$ locally lower semicontinuous. The proofs of these statements can be copied for our setting from the Euclidean case as presented in [Ri2, II.4.7].

Because f is Lipschitz and satisfies (6.14) a.e., the preceding discussion implies that for almost every $v \in S(s)$ the following two conditions hold for almost every $t \in [0, s]$:

(a) $J(\alpha_v(t))) \geq c$,
(b) g_v is absolutely continuous and has a finite derivative $g_v'(t)$.

Fix v and $0 < t < s$ satisfying (a) and (b) for a lifting α_v as above. Set $u = \alpha_v(t)$. By (6.15) and (6.19) there exists $\rho_u > 0$ such that

$$\frac{c}{2} \leq \frac{\mathcal{H}_n(f(B(u,\rho)))}{\mathcal{H}_n(B(u,\rho))},$$

$$L^*(u, L(u,\rho)) < H_u^* l^*(u, L(u,\rho))$$

for $0 < \rho \leq \rho_u$. Moreover, we can choose ρ_u so small that $\overline{B}(u,\rho_u) \subset U_\sigma$ and that $l^*(u, L(u,\rho)) = \rho$ for $0 < \rho \leq \rho_u$ (see (4) in the proof of 6.19). Fix $0 < \rho \leq \rho_u$ and write $L = L(u,\rho)$. Then, by the local n–regularity of X and by choosing ρ_u still smaller if necessary, there exists a constant c_x such that

$$\begin{aligned} \frac{c}{2} &\leq \frac{\mathcal{H}_n(f(B(u,\rho)))}{\mathcal{H}_n(B(u,\rho))} \leq c_x \frac{L^n}{\rho^n} = c_x \frac{L^n}{l^*(u,L)^n} \\ &\leq c_x H_u^{*n} \frac{L^n}{L^*(u,L)^n}, \end{aligned} \tag{6.25}$$

where H_u^* is the bound given in Lemma 6.19. It follows that

$$g_v'(t) \leq (\frac{2c_x}{c})^{1/n} H_u^*. \tag{6.26}$$

By (2.4), we have that $i(u,f) \leq i(x,f)$, and hence, by the proof of 6.19, H_u^* has an upper bound $C_0 H_x^*$, where C_0 only depends on the data on X in a neighborhood of x, and on c and M. From this and the absolute continuity of g_v we conclude that

$$g_v(s) \leq C_0 (\frac{2c_x}{c})^{1/n} H_x^* s. \tag{6.27}$$

By choosing suitable lifts α_v we find a sequence $h_{v_i}(v_i)$ tending to a point $z \in \partial U_s$ with $|z - x| = L^*(x,s)$ and such that (6.27) holds with $v = v_i$. This completes the proof with

$$Q = C_0 (\frac{2c_x}{c})^{1/n}, \tag{6.28}$$

and Lemma 6.23 follows.

Proof of Theorem 6.18. It follows from Proposition 6.17 that f is sense-preserving, and from [HS] that f is a discrete and open map. It thus remains to show that $f|W$ satisfies (0.2) for each precompact W, and for some $L \geq 1$ depending on the data.

Since f is M–Lipschitz for some M, the second inequality of (0.2) holds with constant M for all paths α in X. To prove the first inequality of (0.2), we use the idea from [MVä, 2.15]. To this end, let α be a path in W and set $\beta = f \circ \alpha$. We may assume that $l(\beta) = \text{length}\, \beta < \infty$ and let $\beta : [0, l(\beta)] \to R^n$ be a parametrization by arc length. Since $\overline{W}$ is compact, the numbers

$$c_W = \sup_{x \in \overline{W}} c_x,$$

$$H_W^* = \sup_{x \in \overline{W}} H_x^*,$$

are finite, where c_x is the constant in (6.25) and H_x^* is the bound in Lemma 6.23. For a given partition P of $[0, l(\beta)]$ there is a refinement $P' = \{t_0, \ldots, t_k\}$ of P and numbers $r_0, \ldots, r_k > 0$ such that $U_i = U(\alpha(t_i), r_i)$ is a normal neighborhood of $\alpha(t_i)$, $\alpha[t_{i-1}, t_i] \subset U_{i-1} \cup U_i$, and $r_i < \sigma_{\alpha(t_i)}$, where $\sigma_{\alpha(t_i)}$ is given in Lemma 6.23. For each $i = 1, \ldots, k$, choose $s_i \in [t_{i-1}, t_i]$ with $\alpha(s_i) \in U_{i-1} \cap U_i$. Then 6.23 implies

$$|\alpha(s_i) - \alpha(t_{i-1})| \le L'(s_i - t_{i-1}),$$

$$|\alpha(t_i) - \alpha(s_i)| \le L'(t_i - s_i),$$

where

$$L' = C_W \left(\frac{2c_W}{c}\right)^{1/n} H_W^*, \tag{6.29}$$

where C_W corresponds to C_0 in (6.28). Hence

$$\sum_{i=1}^{k} |\alpha(t_i) - \alpha(t_{i-1})| \le L' \sum_{i=1}^{k} ((t_i - s_i) + (s_i - t_{i-1})) = L' l(\beta),$$

and so $l(\alpha) \le L' l(\beta)$. (Note: The condition $\beta[t_{i-1}, t_i] \subset B(\beta(t_{i-1}), r_{i-1}) \cup B(\beta(t_i), r_i)$ in the proof of [MVä, 2.15] does not guarantee that $\alpha(s_i) \in U_{i-1} \cap U_i$, hence the presentation in [MVä] is not quite right and should be corrected along the lines given above.)

We have obtained that $f|W$ is L–BLD with $L = \max(M, L')$. In L' the factor H_W^* depends also on $\max\{i(x, f) : x \in \overline{W}\}$. To get rid of the dependence on the local index, we deduce from Theorem 6.4 that in the proof of Lemma 6.23 we can in addition to conditions (a) and (b) require that $\mathcal{H}_1(f(B_f) \cap [0, v]) = 0$. This gives by (6.26) a bound for $g_v'(t)$ almost everywhere on $[0, s]$ which is independent of $i(u, f)$. Consequently, (6.24) can be replaced by

$$L^*(x, s) \le P_x s,$$

where P_x depends only on the data of X in a neighborhood of x, on M, and on c. It follows that H_W^* in (6.29) can be replaced by a constant independent of the local index. This completes the proof of Theorem 6.18.

6.30. **BLD–maps into R^n.** In [HS], an oriented generalized n-manifold X was termed *locally BLD–Euclidean* if every point in X has a neighborhood U together with a Lipschitz

BLD–map $f : U \to R^n$. We close this section by pointing out the following property of locally BLD-Euclidean spaces. (For a further property of locally BLD-Euclidean spaces, see 9.6 below.)

6.31. **Proposition.** *A locally BLD–Euclidean space is of type $A12$ and is a Lipschitz manifold outside a closed singular set of topological dimension at most $n-2$.*

Proof. If X^n is a locally BLD–Euclidean space, then X is locally Ahlfors n-regular by Proposition 6.3 and Remark 4.16 (b). To prove that X is rectifiable, we may assume without loss of generality that there exists a Lipschitz BLD–map $f : X \to R^n$ of finite multiplicity. It is easy to see that f is locally bi-Lipschitz outside the branch set B_f, by the path-lifting property. Hence it suffices to show that B_f is n-rectifiable. To this end, fix a normal neighborhood U of a point $x_0 \in B_f$, and let

$$C_0 = \{x \in U : i(x, f) = i(x_0, f)\}.$$

(See 2.3 (c) and (d).) Then $f|C_0$ is injective, and it is easy to see that $f^{-1}|f(C_0) : f(C_0) \to C_0$ is locally Lipschitz, by the path-lifting property. Thus C_0 is n-rectifiable. Because the function $x \mapsto i(x, f)$ is upper semicontinuous, the set $U \setminus C_0$ is open. Hence, the proof is easily completed by exhausting $U \setminus C_0$ by a countable number of compact sets, and proceeding by downward induction with respect to the local index. It follows that X is of type $A12$.

The second assertion follows because $X \setminus B_f$ is connnected and because X is easily seen to be a Lipschitz manifold outside the branch set B_f. This completes the proof of Proposition 6.31.

6.32. **Remarks.** (a) We do not know whether locally BLD–Euclidean spaces are of type $A3$, that is, whether they are (loally) bi-Lipschitz embeddable in some finite dimensional Euclidean space.

(b) We do not know if a locally BLD–Euclidean space X of type $A3$ is locally metrically orientable. It is easy to see (cf. Theorem 6.4) that such an X is locally metrically orientable if for some (all) Lipschitz BLD-maps $f : X \to R^n$ we have that the branch set B_f has zero Hausdorff n-measure. We conjecture that this is the case.

(c) Locally BLD–Euclidean spaces need not be locally contractible, *a fortiori* not of type $A4$. (In particular, we cannot use 5.11 to solve the problem posed in (b) above.) An example to this effect can be given, for example, by using the example in [Wi, p. 245] and the generalized Berstein-Edmonds theorem 0.3.

Part II: Extension of BLD maps with applications

7. Proof of Theorem 0.3. This section is devoted to the proof of the generalized Berstein-Edmonds theorem 0.3.

7.1. **Notation.** We use [H] as a standard reference for PL theory. We consider complexes that are polyhedra in some Euclidean space. If A is a simplex, $\dot{A}$ denotes its boundary as a simplicial complex, $\mathring{A}$ its interior, and $\hat{A}$ its barycenter. The join of A with another simplex B is denoted by $A * B$. The latter notation is used also for joinable simplicial complexes (see [H, p. 6]). We write $I^k = [-1,1]^k$ for the closed k–cube. The interior of a closed manifold M is denoted by $\operatorname{int} M$, and cl denotes closure. If $v_0, \dots, v_k$ are points in some Euclidean space, we denote the (unoriented) k–simplex with vertices $v_0, \dots, v_k$ by $\langle v_0, \dots, v_k \rangle$.

7.2. **Separating complex.** The idea of the proof of Theorem 0.3 is based, on one hand, on the techniques from [HR, Section 5], which originate in [Ri1, 7.2, 8.3], and on the other hand, on [BE, Theorem (6.2)] in Subsection 7.7. Apart from [BE] our presentation is self-contained; in particular, no reference to [HR] or [Ri1] is necessary. In view of the Berstein-Edmonds theorem, we may assume $p \geq 3$.

We start out by defining boundary collars in W to $M_1, \dots, M_{p-1}$ as follows. We choose for $j \geq 1$ a PL embedding $h_j : M_j \times [0,1] \to W$ such that $h_j(x,0) = x$ and such that the images $W_j = h_j(M_j \times [0,1])$ are disjoint and do not meet M_0 (see [H, p. 36]). Set

$$\tilde{W} = cl(W \setminus \bigcup_{j=1}^{p-1} W_j),$$

$$M_j^1 = h_j(M_j \times \{1\}), \quad j = 1, \dots, p-1.$$

Next fix a closed PL ball B in int $\tilde{W}$ and a PL homeomorphism $\Phi : B \to B'$, where B' is the polyhedron in R^3 of the form

$$B' = Y \times I^1,$$

and $Y \subset R^2 = R^2 \times \{0\} \subset R^3$ is a regular $(2p-2)$–gon with 0 as center and diameter 2. Let the faces of Y be $E_1, \dots, E_{2p-2}$ in this order. We connect the (combinatorial) face $F_{2j} = \Phi^{-1}(E_{2j} \times I^1)$ (see [H, p. 30] for terminology) of B to M_j^1 by a PL cylinder A_j for $j = 1, \dots, p-1$, with a PL homeomorphism $\psi_j : [0,1] \times I^2 \to A_j$ such that $\psi_j(\{0\} \times I^2) \subset M_j^1$, $\psi_j(\{1\}) \times I^2) \subset F_{2j}$, $\psi_j((0,1) \times I^2) \subset \operatorname{int}(\tilde{W} \setminus B)$, and A_j's are all disjoint. The cylinder A_j can be constructed by taking some regular neighborhood of a polygonal arc joining M_j^1 and F_{2j} in $\operatorname{int}(\tilde{W} \setminus B)$.

Let C_j' be the join of $E_j \times I^1$ and $J = (0,0) \times I^1$, and set

$$C_j = \Phi^{-1}(C_j'), \quad j = 1, \dots, 2p-2,$$

$$Z = \tilde{W} \setminus \operatorname{int}(\bigcup_{j=1}^{p-1} (A_j \cup C_{2j})).$$

The set Z satisfies the condition for a PL 3–manifold at points in $Z \setminus \Phi^{-1}(J)$ and Z has two boundary components, M_0 and $|K|$, where K is a simplicial 2–complex and a subcomplex of a triangulation Q of W (see Fig. 1). The complex K has the following properties:

(a) $|K|$ satisfies the condition of a PL–manifold at points of $|K| \setminus \Phi^{-1}(J)$;

(b) $|K|$ separates $M_0, M_1, \ldots, M_{p-1}$ from each other in W;
(c) $W \setminus |K|$ consists of p components, $V_0, \ldots, V_{p-1}$, such that

$$M_j \subset V_j, \quad j = 0, \ldots, p-1;$$

(d) there exists a PL homeomorphism

$$g_j : M_j \times [0,1] \to \overline{V}_j, \quad g_j(x,0) = x, \quad j = 1, \ldots, p-1.$$

Figure 1

We may assume that the triangulation Q of W is so chosen that K satisfies the following additional properties:

(1) If v is a vertex of K and $v \notin \Phi^{-1}(J)$, then v belongs to an even number of 2–simplexes of K;
(2) If v is a vertex of K in $\Phi^{-1}(J)$, then v belongs to an even number of 2–simplexes of K in each $\Phi^{-1}(X)$, where X is any of the four 2–cells that are faces of C'_{2j} and meet J, $j = 1, \ldots, p-1$.

Conditions (1) and (2) are satisfied as soon as K is a barycentric subdivision of some triangulation of $|K|$. (In [HR, 5.4] much of the complications arise from certain symmetry requirements.)

If P is a simplicial complex, we let P^i be the set of i–simplexes in P. The subcomplex of K whose space is $|K| \cap \overline{V}_j$ is denoted by $K \cap \overline{V}_j$ or K_j. It follows from (1) and (2) that the number $\#K_j^2$ of 2–simplexes of K_j is even for $j = 0, \ldots, p-1$. Here we use the fact that $|K_j|$ is homeomorphic to an orientable closed 2–surface for $j \geq 1$, and, on the other hand,

$$\#K_0^2 = \sum_{j=1}^{p-1} \#K_j^2.$$

For each $j=1,\dots,p-1$, the complex K_j is a *map complex* in the terminology of [HR, 4.4], which means that each vertex of K_j belongs to an even number of 2–simplexes.

If $A \in K^2$, then there is a unique $j=1,\dots,p-1$ such that $A \subset \partial V_0 \cap \partial V_j$, and we say that A has *classes* $0, j$. Conditions (1) and (2) divide the vertices of K into three classes α, β, γ, which are uniquely determined if we fix the vertices a and b of a side so that $a \in \alpha$ and $b \in \beta$.

Next we define positive and negative simplexes in K^2 as follows (cf. [HR, 5.2]). Let $A \in K^2$ have classes $0, j$. There is exactly one 3–simplex C in $Q \cap \overline{V}_j$ having A as a face. Let v be the vertex of C not in A and let ω be the orientation preserving affine map of C into R^3 such that $\omega(a)=0$, $\omega(c)=e_1$, $\omega(b)=e_2$. Here a,b,c are the vertices of A in classes α,β,γ respectively and e_i is the standard ith basis vector in R^3. If $\omega(v)_3 > 0$, then A is *positive*, otherwise *negative*. It follows from the construction that in each K_j, $j \geq 1$, the number of positive and negative 2–simplexes is the same. Consequently, this is true for the whole K.

Fix a 1–simplex s of K in $\Phi^{-1}(J)$. Let $j=1,\dots,p-1$. In $|K_j|$ we first define a tree T'_j with the properties

(a) the vertices of T'_j are the barycenters $\hat{A}$ of the simplexes $A \in K_j^2$, and

(b) each side of T'_j is of the form $[\hat{A}, \hat{l}] \cup [\hat{l}, \hat{B}]$, where A and B are in K_j^2 with a common side l, which is not in $\Phi^{-1}(J)$. (Here $[\hat{A}, \hat{l}]$ is the line segment with endpoints $\hat{A}$ and $\hat{l}$.)

The existence of the trees T'_j follows easily since $|K_j|$ is a surface and $\Phi^{-1}(J)$ does not separate. To T'_j we add the the barycenter $\hat{s}$ of the fixed 1–simplex s and a side $[\hat{A}, \hat{s}]$, where A is one of the two 2–simplexes in K_j having s as a side. The resulting tree is denoted by T_j. From the trees T_j we build trees L_k, $k=1,\dots,p-1$, by setting

$$L_k = \bigcup_{\substack{j=1 \\ j \neq k}}^{p-1} T_j.$$

We give L_k a partial ordering by letting the vertex $\hat{s}$ be the last element. Hence, for each side of L_k with endpoints $\hat{A}$ and $(A')^\wedge$ we know which one of $\hat{A}$ and $(A')^\wedge$ precedes the other.

7.3. **Sheets.** Following the principles in [HR, 5.3], we now replace each 2–simplex A in K by a certain number of nearby PL disks, called *sheets*, the number depending on whether A is positive or negative. The teatment differs from [HR, 5.3] because this time we are dealing with an embedding into R^m and not just into R^3. Fix $j \in \{1,\dots,p-1\}$ and let $A \in K_j$. There is exactly one 3–simplex C in $Q \cap \overline{V}_j$ having A as a face. We choose a point q_A in the interior of the line segment $[\hat{A}, \hat{C}]$ joining the barycenters of A and C. Bounds for the distance from q_A to $\hat{A}$ are given later.

If A is positive, we replace A by $2p-j$ PL disks $A = A_0, \dots, A_{2p-j-1}$ defined by

$$A_\mu = \dot{A} * a_\mu,$$

where $\dot{A}$ is the boundary of A, $*$ is the join operation, and

$$a_\mu = \hat{A} + \frac{\mu}{2p-j-1}(q_A - \hat{A}).$$

If A is negative, we replace it by $p+j$ PL disks $A_0, \ldots, A_{p+j-1}$ defined by

$$A_\mu = \dot{A} * b_\mu$$

where

$$b_\mu = \hat{A} + \frac{\mu}{p+j-1}(q_A - \hat{A}).$$

We say that the A_μ's are *inherited* from A. With these definitions the sheets all lie in C. If $A' \in K_j^2$ is another 2–simplex, then the sheets that are inherited from A' can meet the sheets A_μ along their boundaries, but this happens only when A and A' share a side. Notice that A and A' can have a common 3–simplex C as above.

Figure 2

We next perform a gluing process among the sheets that are inherited from simplexes $A, A' \in K_j^2$ with common side. This process includes a way of opening certain connections among different layers between the sheets. After the gluing, the sheets have been deformed near their boundaries; we will continue to call them sheets inherited from A. After the gluings have been performed for all such pairs, and for all $j = 1, \ldots, p-1$, $|K|$ is replaced by a considerably more complicated branched surface, a 2–complex which we will call R. The gluing process is done so that the complement $W \setminus |R|$ consists of p components $V_0', \ldots, V_{p-1}'$ such that V_j' is PL homeomorphic to V_j (see 7.6), that

$$M_j \subset V_j', \quad j = 0, \ldots, p-1,$$

and that for each $A \in K$ and each $j = 0, \ldots, p-1$, there are (deformed) sheets that are inherited from A and are contained in $\overline{V}_j'$.

A more detailed description of the gluing process is as follows. (The precise technique is given in 7.4.) We use the partially ordered tree L_k given in 7.2 as a guideline to get for every $A \in K_j^2$, $j \geq 1$, $j \neq k$, a connection within V_k' from M_k to some of the sheets inherited from A. The principles of getting these connections (in the case where one has the space embedded in R^3) are described in [HR, 5.3] and [Ri1, 7.2, 8.3]. The sets $V_0', \ldots, V_{p-1}'$ appear in a cyclic order when one follows a line segment such as $[q_A, \hat{A}]$.

Figure 3

To illustrate the order of the sets V_j' near the chosen side s of K in $\Phi^{-1}(J)$, we consider the case $p = 4$. In Fig. 2 we have drawn schematically the sheets inherited from the six simplexes $A^1, \ldots, A^6$ in K^2 with s as a side and from six other simplexes $B^1, \ldots, B^6$ in K^2 such that A^i and B^i belong to a common K_j, $j \geq 1$, and share a side. The center of the picture represents the side s and the appearance of the sets V_j' are given by numbers. In addition, the curves with arrows show the way towards M_1 of connections from the layers between the sheets associated with V_1'. For sheets that are inherited from 2–simplexes A in some K_j, $j > 1$, the connections are determined by the tree L_1. The partial ordering of L_1 determines the direction of the arrows in Fig. 2. When there are two layers between the sheets inherited from the same simplex and these two layers have the same index, the tree L_k (In Fig. 2 look at L_1) does not tell the rule for the connections between these layers. In fact, there is a choice to be made. The principle is that no loops should appear, and this principle could be presented through a new tree with layers associated with one V_k' as vertices. To avoid still extra notation we will not introduce such a tree. Connections for the layers of sheets inherited from simplexes $A \in K_1$, and associated with V_1', are simpler. In Fig. 2, there are four curves with arrows to indicate these kind of connections. Fig. 3 represents (for $p = 4$) a generic case of sheets inherited from some $A, A' \in K_1^2$ with a common side together with the connections in each V_j' towards M_j. Note the different type of choices of connections of layers with index 2 and 3.

The gluing technique is also illustrated in [HR, Fig. 5]. Note, however, that the picture there represents the gluing in the so called cave refinement process which is not needed now; here we are dealing with the final gluing in the terminology of [Ri1, Section 7].

It follows from the construction that each $\overline{V}_j' \cap R$ has the same even number $2l$ of sheets and that $l = 3q$, where q is half of the number of 2–simplexes in K^2. To see this, let $j = 0, \dots, p-1$. If A, A' is a pair of adjacent 2–simplexes in some K_k, then $\overline{V}_j' \cap R$ always contains exactly six sheets inherited from A and A'. Hence $3\#K^2 = \#(R \cap \overline{V}_j')^2$, where we have extended the notation of the set of 2–simplexes to sheets. Recall that $\#K^2$ is even.

7.4. **PL gluings.** Since we are working inside R^m, we need some extra care in performing the gluings. Moreover, we will give a precise description of how to do the gluing in a PL manner. (In this respect our presentation differs from that of [Ri1] and [HR].) So let first A and A' be in $K_j^2, j \geq 1$, with a common 1–simplex l, which is not in $\Phi^{-1}(J)$. Let v be one of the vertices of l. Let $H = \overline{\mathrm{star}}(v, Q \cap \overline{V}_j)$ be the star of v in $Q \cap \overline{V}_j$ as a complex. (Recall that Q is a fixed triangulation of our 3–manifold W.) Then there is a PL homeomorphism $\psi : |H| \to I^2 \times [0,1]$ such that $\psi(v) = 0$ and $\psi(|H| \cap |K_j|) = I^2 \times \{0\}$. The sheets inherited from A and A' lie in 3–simplexes C and C' which both belong to H. (Here C and C' can coincide.) Therefore, we can first perform the gluings of the images of the sheets under the map ψ in $I^2 \times [0,1]$ near $\psi(l)$. The problem now is that the images of the sheets under ψ can in general not be projected to a 2–plane so that the projection is one-to-one.

Let H_1 be a subdivision of H such that ψ is simplicial with respect to H_1, let $l_1, \dots, l_r$ be the 1–simplexes of $H_1 \cap l$ in this order, let v_{i-1} and v_i be the vertices of l_i, and let $B_i \subset A$ and $B_i' \subset A'$ be the two 2–simplexes in $H_1 \cap |K_j|$ with l_i as a common side. As before, there are uniquely determined 3–simplexes C_i and C_i' in H_1 with B_i and B_i' as a face, where C_i and C_i' can coincide. The images under ψ are denoted by $l_i^* = \psi(l_i)$ etc. In l_i^* we choose poins u_i^* and w_i^* such that $|v_{i-1}^* - u_i^*| = |w_i^* - v_i^*| = \sigma/3$, where $\sigma = |v_{i-1}^* - v_i^*|$. For some $\tau \in (0, \sigma/6)$ the intersection $D_\tau \cap A_\mu^*$ is contained in C_i^* for every sheet A_μ inherited from A, where D_τ is the τ–neighborhood of the line segment $[u_i^*, w_i^*]$ in R^3, provided we have chosen q_A near enough $\hat{A}$ (see the beginning of 7.3). In such neighborhoods D_τ we have good control of the shape of the images of the sheets under ψ since each $\psi|C_i$ is affine. We take τ so small that the corresponding property holds for the sheets inherited from A', too. As l is not in $\Phi^{-1}(J)$, we can separate three different cases of connections to be performed by gluings (see Fig. 3):

(a) connection from the layer between sheets $A_{\mu-1}$ and A_μ to the layer between sheets $A_{\nu-1}'$ and A_ν';

(b) connection from the layer between sheets $A_{\mu-1}$ and A_μ to the layer between sheets $A_{\nu-1}$ and A_ν;

(c) connection from the layer between sheets $A_{\mu-1}$ and A_μ to some V_k, $k = 0,\dots,p-1$.

The meaning of these three cases will become clearer below.

Case (a). We will look at the image under ψ and perform the gluing in a neighborhood D_τ of some $[u_i^*, w_i^*]$. Let T_1 be the plane in R^3 containing l_i^* and orthogonal to $I^2 \times \{0\}$. We choose points $d, e \in [u_i^*, w_i^*]$ with $\delta = |d-e| < \tau/4$. Let T_2 be the plane $\{\xi \in R^3 : |\xi - d| = |\xi - e|\}$. Let $z^1 \in T_1 \cap T_2$, $\zeta \in A_\mu^* \cap T_2$, $\zeta' \in A_\nu'^* \cap T_2$ be points such that their distance to l_i^* is δ, and that the third coordinate z_3^1 of z^1 is positive. We assume that $|q_A - \hat{A}|$ is small enough so that the parts of the images of the sheets inherited from A and contained in D_τ both are projected injectively to $I^2 \times \{0\}$ by $x \mapsto (x_1, x_2, 0)$ and lie on "one side" of T_1, and similarly for A' in the opposite side of T_1. Let further z^0 be the midpoint of $[d,e]$ and let z^2 be a point in $T_1 \cap T_2$ with $|z^2 - z^0| = 2\delta$ and positive third coordinate. Let λ be the simplicial map of the 3–simplex $G = \langle d, e, z^2, \zeta\rangle$ which maps affinely $\langle d, z^0, z^2, \zeta\rangle$ onto $\langle d, z^1, z^2, \zeta\rangle$ and $\langle e, z^0, z^2, \zeta\rangle$ onto $\langle e, z^1, z^2, \zeta\rangle$, and preserves the order of vertices. Let λ' be the corresponding map with ζ replaced by ζ'.

Now the connection for Case (a) is obtained as follows. The images $A_0^*, \dots, A_{\mu-1}^*$, $A_0'^*, \dots, A_{\nu-1}'^*$ remain unchanged. The rest of the sheet images inherited from A, say $A_\mu^*, \dots, A_r^*$, we deform to $\check{A}_\mu^*, \dots, \check{A}_r^*$ by using λ and setting

$$\check{A}_h^* = (A_h^* \cap ((I^2 \times [0,1]) \setminus G)) \cup \lambda(G \cap A_h^*), \quad h \geq \mu.$$

Similarly we deform the rest of the image sheets inherited from A' by using λ'. As a result $A_{\mu-1}^* \cup \check{A}_\mu^* \cup A_{\nu-1}'^* \cup \check{A}_\nu'^*$ bounds a PL 3–ball.

Case (b). We may assume $\mu < \nu - 1$. Let d, e and z^0 be as in Case (a). Let $\zeta \in A_\mu^* \cap T_2$ and $\zeta^2 \in A_{\nu-1}^* \cap T_2$ be points at distance δ from l_i^*, and let ζ^1 be the midpoint of $[\zeta, z^0]$. We map the 3–simplex $G_1 = \langle d, e, \zeta^2, \zeta\rangle$ simplicially by a map λ_1 that takes affinely $\langle d, z^0, \zeta^2, \zeta\rangle$ onto $\langle d, \zeta^1, \zeta^2, \zeta\rangle$ and $\langle e, z^0, \zeta^2, \zeta\rangle$ onto $\langle e, \zeta^1, \zeta^2, \zeta\rangle$, preserving the order of vertices. The images $A_0^*, \dots, A_{\mu-1}^*$ and $A_\nu^*, \dots, A_r^*$ remain unchanged. The images $A_\mu^*, \dots, A_{\nu-1}^*$ we deform to $\check{A}_\mu^*, \dots, \check{A}_{\nu-1}^*$ by setting

$$\check{A}_h^* = (A_h^* \cap ((I^2 \times [0,1]) \setminus G_1)) \cup \lambda_1(G_1 \cap A_h^*), \quad \mu \leq h \leq \nu - 1,$$

which clearly gives the desired connection. Again, $A_{\mu-1}^* \cup \check{A}_\mu^* \cup \check{A}_{\nu-1}^* \cup A_\nu^*$ bounds a PL 3–ball.

Case (c). Suppose first that $k \neq 0$. Then we use exactly the rule from Case (b) with $\zeta^2 \in A_r^* \cap T_2$, where r is the last index. If $k = 0$, we use the rule from Case (b) with $\zeta \in A_0^* \cap T_2$, i.e., μ is then the first index.

Various connections in some neighborhood of l^* can be performed independently by choosing the line segments corresponding to $[d,e]$ above small and far enough from each other. Moreover, connections that are performed near different sides l do not interact, provided the line segments $[d,e]$ are small and stay away from the endpoints of the sides l.

It remains to consider connections at $\Phi^{-1}(J)$. The only 1–simplex of K that has to be considered is s, which we fixed in 7.2 in connection with the definition of the trees L_k. This time we make use of the PL map $\Phi : B \to B'$. For each pair $k, j \in \{1, \dots, p-1\}$, $k \neq j$, we have to build a connection from V_k to the layer between some $A_{\mu-1}$ and A_μ, where $A \in K_j^2$ and $[\hat{s}, \hat{A}]$ is a side in T_j (see 7.2). We use a technique similar to the one in Case (a) above. Let us use $*$ to denote images under Φ. As in Case (a) we choose points $d, e \in s^* \subset J$, and assume that $\delta = |d - e|$ is small enough. Let T_1 be the plane in R^3 through J such that C'_{2j} is the reflection of C'_{2k} with respect to T_1, and let $T_2 = \{\xi \in R^3 : |\xi - d| = |\xi - e|\}$. The line containing J divides T_1 into two half planes; let T_1^+ be the one of these such that the angle near $[d, e]$ between T_1^+ and $A_{\mu-1}^*$ is smaller than the angle between T_1^+ and A_μ^+.

Next choose $z^1 \in T_1^+ \cap T_2$ and $\zeta \in A_{\mu-1}^* \cap T_2$ with distance δ to J, and $z^2 \in T_1^+ \cap T_2$ with distance 2δ to J. Let z^0 be the midpoint of $[d, e]$. There are two elements in K_k^2 with s as a side. Let A' be the one with the smaller angle between A'^* and T_1^+ near J. Suppose $A'_0, \dots, A'_r$ are the sheets inherited from A'. Then let $\zeta' \in A_r'^* \cap T_2$ be the point with distance δ to J. Now we are in a position to define maps λ and λ' exactly as in Case (a). More precisely, except for the image sheets $A_0^*, \dots, A_{\mu-1}^*$ and $A_0'^*, \dots, A_r'^*$, which will be deformed by using λ and λ', all other image sheets containing s^* remain unchanged. This gives then the required connection. There will be no other types of connections at sides in $\Phi^{-1}(J)$ (see Fig. 2).

7.5. **PL homeomorphism of V'_k onto V_k.** Recall the definition of V'_k from 7.3, which is now completed through the gluing process in 7.4. The space $|R|$ of the 2–complex R is the union of all the new sheets. In the following, we use the word *connection* to mean any of the opening processes defined in 7.4. Fix $k = 1, \dots, p-1$. We start by collapsing inductively that "part" of V'_k, where the boundary of V'_k is inherited from 2–simplexes in some K_j, $j \geq 1$, $j \neq k$. After the gluing process, the original sheets have been deformed. According to our convention, we continue to call the deformed objects sheets. In particular, the new sheets inherited from some A are now denoted by $\tilde{A}_\mu$. Note that we used the notation $\breve{A}_\mu$ in 7.4, Case (a) and (b), but $\breve{A}_\mu$ was not the final deformed sheet because other connections are performed, too.

Let now $A \in K_j^2$, $j \geq 1$, $j \neq k$, be such that its barycenter $\hat{A}$ is an end of the tree L_k (see 7.2). Suppose first that V'_k appears only in one "layer" between $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$, i.e., $\tilde{A}_{\mu-1} \cup \tilde{A}_\mu \subset \partial V'_k$ only for one index μ. There is a unique $A' \in K_j^2$ such that L_k has a side connecting $\hat{A}$ and $(A')^\wedge$. The connection for V'_k to a layer between $\tilde{A}'_{\nu-1}$ and $\tilde{A}'_\nu$ is performed by Case (a) in 7.4. Recall that in 7.4 we denoted by A^* etc. the image of A etc. under maps ψ; let now $\#$ denote the image under ψ^{-1} (see the beginning of 7.4). Using the notation in Case (a), we first observe that the *open* set Ω bounded by $\tilde{A}_{\mu-1}^* \cup \tilde{A}_\mu^* \cup \langle d, e, z^1 \rangle$ is the interior of a PL ball. Hence V'_k collapses to $U = V'_k \setminus \Omega^\# \cup \langle d, e, z^1 \rangle^\#$, and there is a PL homeomorphism $\Lambda : V'_k \to U$; more precisely, there is a subdivision Q_1 of the original triangulation Q so that Λ is simplicial with respect to Q_1. We then continue similarly by collapsing U to $U_1 = U \setminus \langle d, e, z^1, \zeta' \rangle^\#$ and obtain a PL homeomorphism $\Lambda_1 : U \to U_1$. Combining these two operations we have reduced the number of sides of the tree L_k by one, since U_1 is precisely a "V'_k" if the side of L_k connecting $\hat{A}$ and $(A')^\wedge$ is not there.

Suppose then that V_k' appears in two "layers", one between $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$, and the other between $\tilde{A}_{\nu-1}$ and $\tilde{A}_\nu$ (there cannot be more such layers, see Fig. 2). Assume $\mu < \nu - 1$. Suppose first that from both these layers there are connections to exactly one layer in V_k' between $\tilde{A}_{h-1}'$ and $\tilde{A}_h'$. Collapsing both layers corresponding to A separately as above, we have again reduced the tree L_k by one side. The other choice is that the connection from the layer between $\tilde{A}_{\nu-1}$ and $\tilde{A}_\nu$ is performed to the layer between $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$ with the rule in Case (b) in 7.4. Then one of the layers, say the one between $\tilde{A}_{\nu-1}$ and $\tilde{A}_\nu$ must have a connection to exactly one layer in V_k' between $\tilde{A}_{h-1}'$ and $\tilde{A}_h'$. Using the notation in Case (b), let now Ω be the open PL ball bounded by $\tilde{A}_{\mu-1}^* \cup \tilde{A}_\mu^* \cup \langle d, e, \zeta^1 \rangle$. We first collapse V_k' to $U = V_k' \setminus \Omega^\# \cup \langle d, e, \zeta^1 \rangle^\#$ and then collapse U to $U \setminus \langle d, e, \zeta^1, \zeta^2 \rangle^\#$. After this we continue as in the case of only one layer, which then results again in the reduction of number of sides of L_k by one.

The induction step is similar to the process described above except for the last steps, namely, the removing of the sides of the form $[\hat{A}, \hat{s}]$. This we will consider a little later.

Let next $A \in K_k^2$ and suppose V_k' appears in the layer between the sheets $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$. There is always exactly one such layer (see Fig. 2). To "remove" that layer we use collapsing as above with the Case (b). This works because gluing is done according to Case (c) and this can be considered as a degenerate case of Case (b).

Finally, we turn to the last steps of removing the sides $[\hat{A}, \hat{s}]$ in L_k. If V_k' appears only in one layer between some sheets $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$, we use collapsing similarly as in Case (a) (see also the end of 7.4). If V_k' appears also in another layer, say between $\tilde{A}_{\nu-1}$ and $\tilde{A}_\nu$, there is a connection to the layer between $\tilde{A}_{\mu-1}$ and $\tilde{A}_\mu$. We use first collapsing associated with the connection as in Case (b), and then continue as in the case of only one layer.

With all the collapsings understood, we obtain a PL homeomorphism $\vartheta_1 : V_k' \to G_k$, where ∂G_k consists of M_k and another polyhedron N_k which is PL homeomorphic to $|K_k|$. In fact, N_k is a result of a number of small deformations of the simplexes of K_k. First each $A \in K_k$ is mapped onto the sheet A_r (r maximum index), then pullbacks by ψ of simple maps like λ, λ' in Case (a), λ_1 in Case (b) etc. are applied. Hence, we get easily a PL homeomorphism $\vartheta_2 : \overline{G}_k \to \overline{V}_k$ by a small adjustment of N_k. The required PL homeomorphism is then $\eta_k = \vartheta_2 \circ \vartheta_1 : V_k' \to V_k$.

Finally, let $k = 0$. This case is easier since we only have connections of Case (c) in 7.4. The corresponding collapsings were considered above. As a result we get a PL homeomorphism $\eta_0 : V_0' \to V_0$ with the help of an auxiliary adjustment corresponding to ϑ_2 above.

7.6. **Increasing $\#K^2$.** The construction of K in Subsection 7.2 gave an even number $2q = \#K^2$ of 2–simplexes in K. We shall perform two types of subdivisions of the simplicial complex Q giving an increase in $\#K^2$. Recall from 7.2 that $|Q| = W$ and K is a subcomplex of Q.

To perform the first subdivision, pick any 2–simplex A in K with vertices a, b, c, and then add three vertices a', b', c' as in Fig. 4. This gives a new complex which we call K'. To extend this subdivision to a subdivision Q' of Q, we simply use the join operation to d and e, where $A * d$ and $A * e$ are the simplexes in Q having A as a face. The complex

K' is still a map complex (every vertex belongs to an even number of 2–simplexes) and it clearly also satisfies conditions (1) and (2) in 7.2. The number $\#K^2$ is increased by six.

In the second subdivision we consider the case of a vertex v of K, which is not in $\Phi^{-1}(J)$ (see 7.2), such that v belongs to exactly four 2–simplexes of K. If $l = \langle v, w\rangle$ and $l_1 = \langle v, W_1\rangle$ are opposite 1–simplexes in K with common vertex v, we add two new vertices $\hat{l}$ and $\hat{l}_1$ to K and get the subdivision in Fig. 5. Again we extend this to Q by first using the join operation to adjacent 3–simplexes as in the first subdivision, and then also to those 3–simplexes in Q which are of the form $l * l^*$ and $l_1 * l_1^*$, where l^* and l_1^* are some 1–simplexes in $H_v = \overline{\text{star}}(v, Q)$. For example, if $l * l^*$ is such a 3–simplex in H_v, it is replaced by two 3–simplexes, $\langle v, \hat{l}\rangle * l^*$ and $\langle \hat{l}, w\rangle * l^*$. With this definition $|Q| \setminus \text{int}|H_v|$ remains untouched. This second subdivision has the effect of increasing $\#K^2$ by four.

Figure 4

Figure 5

We can always start with the first type subdivision and then continue with the second type, because after the first type subdivision the vertices a', b', c' each belong to exactly four 2–simpexes. Consequently, using these types alternatively we can have a new complex K' with $\frac{1}{2}\#K'^2$ any prescribed integer $i \geq i+5 = i_0$. (Note that if Q is obtained as a barycentric subdivision, then one can start with the second type.) By change of notation let K be the complex with $\#K^2 = 2i$.

7.7. **Definition of the map** φ. Let i_0 be the integer from 7.6 and assume $i \geq i_0$. We start by defining φ on $|R|$ (see the beginning of 7.5). We take S^3 embedded in R^4 as the unit sphere. In the equator S^2 of S^3, we consider the unit sphere $S^1 = (R^2 \times \{0\}) \cap S^3$. Let ω be a rotation of S^3 with dihedral angle $2\pi/p$ and fixed point set S^1. Let S^+ be one of the two components of $S^2 \setminus S^1$, and set

$$Y'_k = \omega^k \bar{S}^+, \quad k = 0, \ldots, p-1, \quad (\omega^0 = \mathrm{id}),$$

$$Y' = \bigcup_{k=0}^{p-1} Y'_k.$$

We may assume that S^3 is given a C^1 triangulation $g : N \to S^3$ in the sense of [M, p. 81] such that the sets Y'_k are subcomplexes of S^3. Write $Y_k = g^{-1}(Y'_k), Y = g^{-1}(Y')$, and $Y_p = Y_0$. The set $N \setminus Y$ consists of p components. Through a preliminary PL homeomorphism of N onto itself we may assume that the ball B_k lies in the component bounded by $Y_k \cup Y_{k+1}$, $k = 0, \ldots, p-1$.

On $g^{-1}(S^1)$ we fix three points a', b', c' in an order to be fixed below. Each sheet in $|R| \cap \overline{V}'_k$ is also a sheet of (i) $|R| \cap \overline{V}'_{k-1}$ or (ii) $|R| \cap \overline{V}'_{k+1}$. In the case (i) we map a given sheet in $|R| \cap \overline{V}'_k$ by a PL homeomorphism onto Y_k, and in the case (ii) onto Y_{k+1}, such that vertices of classes α, β, γ are taken to a', b', c' respectively. This defines φ on $|R|$.

Recall from the end of 7.5 the definition of the PL homeomorphism $\eta_k : V'_k \to V_k$, $k = 0, \ldots, p-1$. The definition of φ on $|R|$ induces via η_k, and its boundary correspondence, a map $\tilde{\varphi}_k$ of $|K_k|$ onto the PL 2–sphere $Y_k \cup Y_{k+1}$. Let $k \geq 1$. The set $|K_k|$ is one of the two boundary components of the PL 3–manifold $\overline{V}_k$, the other boundary component being M_k (see 7.2 (d)). The map $\tilde{\varphi}_k$ is a branched cover of degree $d = 3i$. By the Berstein-Edmonds theorem [BE, (6.2)], cited in the introduction, we can extend φ_k and $\tilde{\varphi}_k$ to a PL sense-preserving branched cover $\hat{\varphi}_k : \overline{V}_k \to X_k$ of degree d, where X_k is the closed PL ring bounded by $Y_k \cup Y_{k+1}$ and ∂B_k, provided we have chosen a', b', c' in the right order. The same order works for all $k \geq 1$. Using the map η_k again, we get the required extension of $\varphi||R|$ to $\overline{V}'_k$, i.e., we have defined $\varphi|\overline{V}'_k : \overline{V}'_k \to X_k$ for $k \geq 1$.

It remains to consider the case $k = 0$. The difference now is that $|K_0|(= |K|)$ is not a manifold. This is not a serious problem, however, because the only trouble is in a neighborhood of $\Phi^{-1}(J)$ (see 7.2). Using the PL homeomorphism $\Phi : B \to B'$, we can collapse V_0 to $U_0 = V_0 \setminus B$ by a PL homeomorphism $\zeta_0 : V_0 \to U_0$. The set $\overline{U}_0$ is a PL 3–manifold with boundary components M_0 and P_0. The maps $\zeta_0 \circ \eta_0 : V'_0 \to U_0$ and $\varphi||R|$ induce by boundary correspondence a branched cover $\tilde{\varphi}_0 : P_0 \to Y_0 \cup Y_1$ with degree d. The orientations are compatible so that we can again apply [BE, (6.2)] to extend $\tilde{\varphi}_0$ and φ_0 to a sense-peserving PL branched cover $\hat{\varphi}_0 : \overline{U}_0 \to X_0$. Finally, with the help of $\zeta_0 \circ \eta_0$ we get the required extension $\varphi|\overline{V}'_0 : \overline{V}'_0 \to X_0$ in the remaining part. This completes the construction of the required extension

$$\varphi : W \to N \setminus \bigcup_{k=0}^{p-1} \mathrm{int} B_k,$$

which is a sense-preserving PL branched cover. The proof of Theorem 0.3 is complete.

8. Geometric decomposition spaces and BLD–maps

In [S3], Semmes defined a class of decomposition spaces of R^3 and provided them with metrics with a special self–similarity property. In fact, his spaces are all subsets of R^4 with the induced metric, and they are constructed by the aid of similarities from what Semmes calls an excellent package (see 8.1).

Here we consider spaces that arise from an excellent package with an additional contractibility condition (see (h) in 8.1), which then leads to a cell–like decomposition. As a result, we are then in the framework of generalized 3–manifolds by [Da, Corollary 1A, p. 191]. Moreover, it follows from Semmes's work in [S3] that such spaces are of type A as defined in 5.1.

In this section, we show by means of Theorem 0.3 that the compact generalized 3-manifolds constructed from the excellent packages with a contractibility condition all admit BLD–maps onto S^3 (Theorem 8.17). The special example arising from the Bing doubling decomposition space leads to a space X that is homeomorphic to S^3 but not quasisymmetrically equivalent to S^3 [S3, p. 244]. This particular example shows a sharp contrast between finite–to–one and injective cases.

8.1. **Excellent packages.** We use the book [Da] as a general reference on decomposition spaces. We will be concerned with special decomposition spaces of R^3 arising through a *defining sequence* (C_i) in the terminology of [Da, p. 61]. This means that each $C_i \subset R^3$ is a compact (not necessarily connected) 3–manifold with boundary and $C_{i+1} \subset \mathrm{int}\, C_i$. The sequence (C_i) defines a *decomposition space* R^3/G, where the *decomposition* G is the set of components of $\cap_i C_i$ and the singletons of $R^3 \setminus \cap_i C_i$. See [Da, p. 8].

Following Semmes [S3, p. 200], an *initial package* consists of bounded smooth domains $D, D_1, \ldots, D_k \subset R^3$, with $\overline{D}_j \subset D$ and $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$, together with sense–preserving diffeomorphisms $\varphi_j : U \to U_j$, where U (resp. U_j) is a neighborhood of $\overline{D}$ (resp. $\overline{D}_j$) and where $\varphi_j(D) = D_j$. An initial package determines a defining sequence (C_i) by setting $C_0 = \overline{D}$, $C_1 = \cup_j \overline{D}_j$, $C_2 = \cup_j \varphi_j C_1$, etc. An *excellent package* [S3, p. 202] consists of an initial package as above, bounded smooth domains Ω, $\omega_1, \ldots, \omega_k, \Omega_1, \ldots, \Omega_k$ of R^4, sense–preserving similarities $\psi_j : R^4 \to R^4$, $j = 1, \ldots, k$, with dilation factor $\rho \in (0, 1/10)$, and a diffeomorphism $\theta : R^4 \to R^4$ such that

(a) $\overline{\omega}_j \subset \Omega$, $\overline{\Omega}_j \subset \Omega$, $\overline{\omega}_i \cap \overline{\omega}_j = \emptyset$, $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$;
(b) $\overline{\Omega} \cap R^3 = \overline{D}$, $\overline{\omega}_j \cap R^3 = \overline{D}_j$ and $\partial\Omega$, $\partial\omega_j$ intersect R^3 transversally;
(c) $\psi_j(R^3) = R^3$, $\psi_j(\Omega) = \Omega_j$;
(d) θ is the identity on a neighborhood of $R^4 \setminus \Omega$;
(e) $\theta(\omega_j) = \Omega_j$ and $\theta = \psi_j \circ \varphi_j^{-1}$ on a neighborhood of $\overline{D}_j$ in R^3.

Following [S3, p. 203], we set $S_0 = \{\emptyset\}$ and

$$S_l = \{(a_1, \dots, a_l) : a_j \in \{1, \dots, k\}\}, \qquad l \geq 1.$$

For $\alpha = (a_1, \dots, a_l) \in S_l$, $l \geq 1$, we define recursively a similarity $\psi_\alpha : R^4 \to R^4$ and a domain $\Omega_\alpha \subset R^4$ by setting $\psi_\alpha = \psi_{\alpha'} \circ \psi_{a_l}$, $\Omega_\alpha = \psi_{\alpha'}(\Omega_{a_l})$, where $\alpha' = (a_1, \dots, a_{l-1})$ if $l > 1$ and $\alpha' = \emptyset$ if $l = 1$. Here we have put $\psi_\emptyset =$ identity and $\Omega_\emptyset = \Omega$.

To build the decomposition space with a self–similarity property Semmes starts out with the compact 3–manifold

$$\Sigma = \theta(\overline{D} \setminus \bigcup_{j=1}^{k} D_j), \tag{8.2}$$

smoothly embedded in R^4 [S3, p. 205]. For $j \geq 1$ set

$$M^j = (R^3 \setminus D) \cup (\bigcup_{l=0}^{j-1} \bigcup_{\alpha \in S_l} \Sigma_\alpha) \cup (\bigcup_{\alpha \in S_j} \Omega_\alpha \cap R^3),$$

where $\Sigma_\alpha = \psi_\alpha(\Sigma), \Sigma_\emptyset = \Sigma$. The set M^j is a smoothly embedded 3–manifold in R^4 which converges in the Hausdorff topology to

$$M = (R^3 \setminus D) \cup (\bigcup_{l=0}^{\infty} \bigcup_{\alpha \in S_l} \Sigma_\alpha) \cup F, \tag{8.3}$$

where

$$F = \bigcap_{l=1}^{\infty} \bigcup_{\alpha \in S_l} \overline{\Omega}_\alpha.$$

The set F is a Cantor set in R^3 if $k \geq 2$, and a point if $k = 1$, and $M \setminus F$ is a smoothly embedded 3–manifold in R^4. By construction, M satisfies the self–similarity condition

$$\Omega_\alpha \cap M = \psi_\alpha(\Omega \cap M) \ \text{ for all } \ \alpha. \tag{8.4}$$

We provide M with the metric induced from R^4. The decomposition space R^3/G determined by the defining sequence (C_i) of the given initial package is homeomorphic to M by [S3, Lemma 3.21].

In order to get into the framework of BLD–maps with good analytic properties, we assume that the excellent package satisfies the following additional conditions:

(f) $4k\rho < 1$, where ρ is the dilation factor of ψ_j;
(g) the sets

$$R^3 \setminus D \ \text{ and } \ E = \overline{D} \setminus \bigcup_{j=1}^{k} D_j$$

are connected;

(h) $\overline{D}_j$ is contractible in D.

Condition (f) is a stronger condition than $k\rho^3 < 1$ which appears in [S3, Lemma 3.45, p. 212]. Condition (g) is [S3, (3.69), p. 218]. In the specific examples considered in [S3] (h) is satisfied. Note that (g) and (h) are conditions on the initial package only.

For future use we introduce the notation $E_\alpha = \varphi_\alpha(E)$, where φ_α is defined analogously to ψ_α from the maps $\varphi_1, \ldots, \varphi_k$.

8.5. **Properties of M.** We fix an excellent package and assume that the conditions (f)–(h) are also satisfied.

8.6. **Proposition.** *Let M be the space defined in (8.3). Then*

(1) *M is of type $A123$ (see 5.1), and*

(2) *M is quasiconvex.*

Proof. The condition $A1$ is clear with the observation that $\mathcal{H}_3(F) = 0$ for the set F in (8.4). The space M lies in R^4 with the induced metric, so $A3$ is true even globally. Lemma 3.45 in [S3, p. 212] shows that M is $A2$ with uniform bounds; here condition (f) is used. By [S3, Proposition 3.70, p. 218] M is quasiconvex; here condition (g) is used. The proposition follows.

8.7. **Cell-like sets.** Recall that a compact set C in an n–manifold Y is *cell–like* if for every neighborhood U of C there is another neighborhood V of C such that V is contractible in U. A decomposition G of Y is *cell–like* if G is an upper semicontinuous decomposition (see [Da, p. 13] for the definition) and if each $g \in G$ is a cell–like set.

The following proposition is an easy consequence of condition (h); see also [Da, Proposition 4, p. 248].

8.8. **Proposition**. *The decomposition G of R^3 defined by the given initial package is cell–like.*

If G is a cell–like decomposition of some n–manifold Y, then (1.5) holds for every $x \in X = Y/G$ by [Da, Proposition 1, p. 191]. If dim $(Y/G) < \infty$, then Y/G is ENR by [Da, Corollary 12B, p. 129]. If moreover $n = 3$, then dim $Y/G \leq 3$ by [Da, Theorem 12, p. 141]. From these facts, together with 8.8 and 1.4(c), we obtain the following proposition.

8.9. **Proposition.** *The space M defined in (8.3) is a locally contractible generalized 3–manifold.*

8.10. **Remarks.** (a) The space M satisfies in fact the stronger condition described in 1.4 (c) (for $n = 3$).

(b) From $M \subset R^4$ we get immediately the weaker estimate dim $R^3/G \leq 4$ for the dimension, which would suffice to prove 8.9. An initial package fits also into the setting of [Da, Proposition 3, p. 248] from which dim $R^3/G \leq 3$ could alternatively be concluded.

8.11. **Proposition.** *The space M is of type $A4$ (see 5.1).*

Proof. The idea is to use the local contractibility from 8.9 together with the self–similarity expressed in (8.4). Since Σ, as defined in (8.2), has a smooth open neighborhood in M, we can fix a smaller such neighborhood Σ' of Σ where the condition of local linearly contractibility holds, i.e. there exist $C > 1$ and $r_0 > 0$ such that for each $x \in \Sigma'$ and $0 < r \leq r_0$,

$$B^4(x,r) \cap M \text{ is contractible in } B^4(x,Cr) \cap M. \tag{8.12}$$

The complement $M \setminus \Omega$ is just $R^3 \setminus \Omega$, so we may assume (8.12) holds also for $x \in M \setminus \Omega$ and $0 \leq r \leq r_0$.

Let R_0 be the distance of $\partial\Omega$ to the set

$$K_0 = \bigcup_{j=1}^{k} \overline{\Omega}_j \cap M.$$

Then for each $x \in K_0$ there exists $r_x > 0$ such that $B^4(x, 2r_x) \cap M$ contracts in $B^4(x, R_0) \cap M$. We cover K_0 by sets $B^4(x_i, r_{x_i}) \cap M$, $i = 1, \ldots, q$. Let $r_0' = \min\{r_{x_1}, \ldots, r_{x_q}, r_0\}$. Then (8.12) is true for some $C > 1$ if

(i) $x \in (M \setminus \Omega) \cup \Sigma'$ and $0 < r \leq r_0'$, or
(ii) $x \in K_0$ and $\rho r_0' \leq r \leq r_0'$.

In order to get (8.12) for smaller r in K_0 we use the similarities ψ_α. To illustrate this, let us look at the first step. Condition (8.12) is clearly true for $x \in \psi_j(\Sigma')$, $j = 1, \ldots, k$, and $0 < r < \rho r_0'$ by (i). Condition (ii) implies that (8.12) is true for $x \in \psi_j(K_0)$, $j = 1, \ldots, k$, and $\rho^2 r_0' \leq r \leq \rho r_0'$. Since

$$K_0 \subset \bigcup_{j=1}^{k} \psi_j(\Sigma') \cup \sum_{j=1}^{k} \psi_j(K_0),$$

we have in the first step extended the range of r's to $\rho^2 r_0' \leq r \leq \rho r_0'$. By continuing similarly, we conclude the proof of the proposition.

8.13. **Summary.** We have now proved that M is a generalized 3–manifold of type A, and, in addition, quasiconvex. Moreover, M satisfies a weak (1,1)-Poincaré inequality as defined in [HK3, Section 5] (see Section 9.6 below). This latter property follows from [S3, Proposition 10.8].

8.14. **Remark.** In [S3] Semmes also considers a space $\tilde{M}$ obtained from M as follows. First we fix a sequence of disjoint 4–balls $B^4(z_j, 2r_j)$, $j = 1, 2, \ldots$, converging to the origin with $z_j \in R^3$. Choose a ball $B = B^4(0, R)$ containing $\overline{\Omega}$. Fix similarities $A_j : R^4 \to R^4$ such that $A_j(R^3) = R^3$, $A_j(B) = B_j = B(z_j, r_j)$. Then define

$$\tilde{M} = (R^3 \setminus \bigcup_{j=1}^{\infty} B_j) \cup \bigcup_{j=1}^{\infty} A_j(B \cap M^j).$$

The set $\tilde{M}$ is homeomorphic to R^3 by construction. Semmes uses examples of spaces $\tilde{M}$ to study various questions of parametrization by R^3 (see [S3, pp. 232–240]).

We now build a compact space X by using M. Let $B^4(0,R)$ be a ball containing Ω. We take the part $\overline{B}^4(0,2R)\cap M$ of M and attach to it the hemisphere

$$S^3_+(0,2R)=\{x\in R^4 : |x|=2R,\ x_4\geq 0\};$$

that is,

$$X=S^3_+(0,2R)\cup(\overline{B}^4(0,2R)\cap M) \tag{8.15}$$

with the metric induced from R^4. The space X has all the properties listed in 8.13.

8.16. **Proposition.** *The space X is orientable.*

Proof. As in the proof of 8.11, we make use of the self–similarity given in (8.4). Choose a point $x\in F$. Since X is a generalized 3–manifold (in the stronger sense of 1.4 (c); see 8.10 (a)), there is a small open neighborhood V of x such that

$$H^3_c(V)=\mathbb{Z},$$
$$H^q_c(V)=0,\ q\neq 3.$$

Since the standard homomorphism in (1.2) is an isomorphism if $W\subset V$ is connected, we may assume that $V=\Omega_\alpha\cap M$ for some α. Let $U=\Omega\cap M$ and $A=X\setminus U$. By (8.4),

$$H^3_c(U)=\mathbb{Z},$$
$$H^q_c(U)=0,\ q\neq 3.$$

Since $A=S^3_+(0,2R)\cup(\overline{B}^3(0,2R)\setminus D)$, A is homeomorphic to a compact subset A' of S^3 with $S^3\setminus A'$ a smooth domain. By Alexander duality (see [Do, p. 301]),

$$H^2_c(A')\simeq \tilde{H}_0(S^3\setminus A')=0,$$
$$H^3_c(A')\simeq \tilde{H}_{-1}(S^3\setminus A')=0,$$

where $\tilde{H}_i$ denotes reduced singular homology. From the exact sequence

$$\cdots\to H^2_c(A)\xrightarrow{\delta} H^3_c(U)\xrightarrow{\tau} H^3_c(X)\xrightarrow{i^*} H^3_c(A)\to\cdots$$

we obtain $\mathbb{Z}=H^3_c(U)\simeq H^3_c(X)$ as required.

We fix the orientation on X such that the orientation of the subset $B^3(0,2R)\setminus\overline{D}$ of X agrees with the standard orientation of R^3.

8.17. **Theorem.** *Let X be the space defined in (8.15). Then there exists an integer $d_0\geq 3$ with the following property: for each integer $d=3i\geq d_0$ there exists a BLD–map $f: X\to S^3$ of degree d.*

Proof. Let Σ be the compact oriented smooth 3–manifold defined in (8.2). By (g), the domain D has one boundary component. Hence Σ has $k+1$ boundary components, all diffeomorphic to each other. By [M, 10.6 and 10.8] there exists a C^1 triangulation $h : W \to \Sigma$ (in the sense of [M, p. 81]) of a polyhedron $W \subset R^4$ with the following property. If $M_0 = h^{-1}(\partial D)$, $M_j = h^{-1}(\partial(R^3 \cap \Omega_j))$, $j = 1, \ldots, k$, each boundary map $\psi_j' = h^{-1} \circ \psi_j \circ h : M_0 \to M_j$, $j \geq 1$, is simplicial with respect to the simplicial complexes M_0 and M_j. Here we recall the notations ψ_j and Ω_j from 8.1. Let $\tau : N \to S^3$ be a C^1 triangulation of S^3, where N is the space $|P|$ of a simplicial 3–complex P in R^4 such that P contains one (closed) regular 3–simplex A_0. We give W and N orientations induced by h and τ. Next, fix sense–preserving similarities η_j of A_0 into itself with dilation factor ρ (given in the definition of X) such that the simplexes $B_j = \eta_j A_0$ satisfy $\operatorname{dist}(B_j, B_k) \geq \rho \operatorname{diam} A_0$, $j \neq k$, and $\operatorname{dist}(\partial A_0, B_j) \geq \rho \operatorname{diam} A_0$, $j = 1, \ldots, k$. This is possible because of the condition (f) in 8.1. Write $B_0 = N \setminus \operatorname{int} A_0$.

We want to apply Theorem 0.3 to W and the PL balls $B_0, \ldots, B_k$ in N. To this end, let d_0 be the bound given in Theorem 0.3, and let $\varphi_0 : M_0 \to \partial B_0 = \partial A_0$ be a sense–preserving PL branched cover of degree $d = 3i \geq d_0$ (with the orientations given in Theorem 0.3). On each M_j, $j \geq 1$, define a sense–preserving PL branched cover $\varphi_j : M_j \to \partial B_j$ of the same degree d by

$$\varphi_j = \eta_j \circ \varphi_0 \circ h^{-1} \circ \psi_j^{-1} \circ h | M_j.$$

Theorem 0.3 gives an extension of the φ_j's to a sense–preserving PL branched cover $\varphi : W \to N \setminus \bigcup_{j=k}^{k} \operatorname{int} B_j$ of degree d. We transfer φ to Σ via h and get a map

$$\xi_0 = \varphi \circ h^{-1} : \Sigma \to N \setminus \bigcup_{j=0}^{k} \operatorname{int} B_j.$$

For $\alpha \in S_l$ we defined ψ_α in 8.1. From the η_j's we define similarly η_α. Then extend ξ_0 to $(\Omega \cap M) \setminus F$ as a map ξ by setting

$$\xi | \Sigma_\alpha = \eta_\alpha \circ \xi_0 \circ \psi_\alpha^{-1}. \tag{8.18}$$

The map ξ is BLD because ξ_0 is BLD and because ψ_α and η_α are similarities with the same dilatation factor $\rho^{|\alpha|}$. The map ξ extends by construction continuously to F. Let ξ_1 be this extension. In fact, it follows from (8.18) that $\xi_1 | F$ is bi–Lipschitz, accordingly ξ_1 is BLD.

The next step is to extend ξ_1 to $C = \overline{B}^3(0, 2R) \setminus D = B^4(0, 2R) \cap M \setminus (\Omega \cap M)$. Here we only need to use the theorem of Berstein and Edmonds [BE, (6.2)], because C has just two boundary components. Fix a closed PL 3–ball E in $N \setminus A_0$, a C^1 triangulation of C, and, with appropriate orientations, fix a PL branched cover $\sigma : S^2(0, 2R) \to \partial E$ of degree d. Then [BE, (6.2)] applied to C and the boundary maps σ and $\xi | \partial D$ gives a PL branched cover of degree d from C onto $N \setminus \operatorname{int}(E \cup A_0)$. Hence we obtained an extension of ξ_1 to $X \setminus \operatorname{int} S_+^3(0, 2R)$, call it ξ_2. It remains to extend ξ_2 to $S_+^3(0, 2R)$. This is done simply by coning the boundary map σ with respect to suitable vertex points in $\operatorname{int} S_+^3(0, 2R)$ and int$E$. We have obtained a sense–preserving PL branched cover $f' : X \to N$ of degree d. Composing with the triangulation τ gives the required map $f = \tau \circ f' : X \to S^3$, and the theorem follows.

8.19. **Remark.** The above proof shows that the space M (as defined in (8.3)) admits a finite-to-one BLD–map onto R^3.

9. Examples and applications

In this section, we display some examples and applications of the results in previous sections.

9.1. **Decomposition spaces.** Specific choices of excellent packages as in 8.1 lead to some familiar decomposition spaces of geometric topology. Three such spaces from the point of view of Section 8 were discussed by Semmes in [S3]: the Whitehead continuum space, Bing's dogbone space, and Bing's double. Only the last of the three is a manifold. Semmes shows in [S3] that the space X in (8.15), associated with the Bing double, is not bi-Lipschitzly or even quasiconformally homeomorphic to R^3, not even locally near the points in the Cantor set F. By Propositions 8.6, 8.11, and 8.16, X is an oriented generalized 3-manifold of type A, and by Theorem 8.17 there exists a BLD–map $f : X \to R^3$.

Sullivan and the first author discuss in [HS] the problem of finding (local) bi-Lipschitz coordinates for generalized manifolds. Examples like those above demonstrate the sharpness of the conditions given in [HS]. Namely, by pulling back the standard co-frame on S^3, we obtain differential 1-forms ρ_1, ρ_2, ρ_3 on X with bounded measurable coefficients that are closed (in the sense of distributions) and satisfy

$$\rho_1 \wedge \rho_2 \wedge \rho_3 \geq c > 0$$

almost everywhere in X. In [HS], a collection of forms as above is called a *Cartan-Whitney presentation* of the *metric gauge* determined by X. The existence of such a presentation is thus guaranteed by Proposition 8.16, and Semmes's result about the non-existence of local bi-Lipschitz parametrizations implies that the *residue*, $\mathrm{Res}(\rho, p)$, of no Cartan-Whitney presentation $\rho = (\rho_1, \rho_2, \rho_3)$ can be 1 at a point $p \in F$. For the terminology here, see [HS].

Similar remarks apply to the examples based on the Whitehead continuum, Bing's dogbone space, and other similar constructions, e.g. those mentioned in 8.14. The beautiful book of Daverman [Da] offers a thorough discussion of various decomposition spaces; see in particular Section II.9 in [Da]. We leave it to the reader to visualize the particular examples of BLD–maps that arise in this manner.

9.2. **Non-existence of radial limits.** Recall that a continuous map f from a domain in R^n into R^n is *quasiregular* if f is in the local Sobolev space $W^{1,n}_{loc}$ and if the (formal) differential $df(x)$ of f satisfies $|df(x)|^n \leq K \det df(x)$ a.e. x for some finite $K \geq 1$. See [Re2], [Ri2] for the basic theory of these maps.

BLD–maps are particular examples of quasiregular maps by 6.13. The boundary behavior of BLD–maps is often easy to describe, by using definition (0.2) only (compare [MVä]). In contrast, the boundary behavior of bounded quasiregular maps of the unit ball B^n, say, is not well understood. For instance, it is not known whether a quasiregular map $f : B^n \to B^n$ has a single radial limit, if $n \geq 3$. (Dimension $n = 2$ is special here because quasiregular maps then factor as quasiconformal maps composed with holomorphic functions by Bojarski's measurable Riemann mapping theorem. In particular, radial limits exist in a set of positive Hausdorff dimension.) Martio and Srebro [MS] have recently shown, by an explicit construction, that there exist bounded quasiregular maps of B^n with no radial limits in an exceptional set of any prescribed Hausdorff dimension less than $n-1$ on the boundary ∂B^n. The exceptional sets provided by Martio and Srebro are all totally disconnected. We shall next indicate how the generalized Berstein-Edmonds theorem 0.3 can be used to give a large family of similar examples. In our examples, the exceptional sets can be complicated topologically. The examples of Martio and Srebro correspond, in a sense, to the case $p = 0$ in the proof of Theorem 9.3. One should note, however, that the maps constructed in [MS] are locally injective; our examples are necessarily branched.

9.3. **Theorem.** *Let Γ be a geometrically finite torsion free Kleinian group without parabolic elements acting on the two sphere S^2 with the limit set Λ_Γ not the whole sphere. Then there exists a bounded quasiregular map $f : B^3 \to B^3$ such that f has no radial limit at points in Λ_Γ.*

Proof. The group Γ acts in the hyperbolic three space, in its ball model B^3, with the quotient $M^3 = B^3/\Gamma$ a hyperbolic three manifold. The action of Γ on $S^2 \setminus \Lambda_\Gamma$ uniformizes finitely many Riemann surfaces $\Sigma_0, \ldots, \Sigma_p$, $p \geq 0$, and we can compactify M by adding these surfaces to it. The resulting manifold W is a compact 3-manifold with boundary, and the assumption of geometric finiteness guarantees that the interior of W, in some fixed PL-metric, is quasiconformally equivalent to the hyperbolic metric in M. Thus, the map

$$f = F \circ \pi,$$

where $\pi : B^3 \to M$ is a locally isometric covering projection and $F : W \to R^3$ is a bounded BLD-map provided by Theorem 0.3, is a bounded quasiregular map. (Note that the case of one boundary component, $p = 0$, can be reduced to Theorem 0.3 as well; we leave the details to the reader.) Clearly the limit $f(rw)$, $r \to 1$, cannot exist for points $w \in \Lambda_\Gamma$. The theorem follows.

It is well known that the limit set Λ_Γ as in Theorem 9.3 must have Hausdorff dimension less than 2, but can be arbitrarily close to 2. See, for example, [BJ] or [Ca] for further discussion and references.

9.4. **Lipschitz maps with strange fibers.** In nonlinear elasticity, one is interested in finding topological properties in a mapping from its analytic data only. (See [Ba], [HK1], [IŠ], [MTY], for example.) Reshetnyak proved in 1966 [Re1] that quasiregular maps, as defined in 9.2, are branched covers or constants. One way to express Reshetnyak's theorem is to consider the *dilatation function*

$$K_f(x) = \frac{|df(x)|^n}{\det df(x)}$$

for a map $f \in W^{1,n}_{loc}$, where it is understood that $K(x, f) = 1$ at points x where the formal differential does not exist, or where $\det df(x) = 0$. Reshetnyak's theorem states that if K_f is in L^∞, then f is a discrete and open map or constant (locally). Iwaniec and Šverak [IŠ] proved that in dimension $n = 2$, it suffices to assume $K_f \in L^1_{loc}$ for the same conclusion. They also conjectured that $K_f \in L^{n-1}_{loc}$ suffices in all dimensions. It is now known that $K_f \in L^p_{loc}$ for some $p > n-1$ suffices; see [MVi].

Earlier, Ball [Ba] had exhibited examples of nonconstant maps $f : R^n \to R^n$ that take a line segment to a point, and have $K_f \in L^p_{loc}$ for all $p < n-1$. In the next theorem, we shall point out how the geometric decomposition space theory as in Section 8, and the generalized Berstein-Edmonds theorem 0.3, can be used to give examples of Lipschitz maps $f : S^3 \to S^3$, of non-zero degree, with more exotic fibers than line segments, and with K_f still in some local Lebesgue space.

9.5. **Theorem.** *For each excellent package as defined in 8.1, and for each $0 < p < 2$, there exist an equivalent excellent package with defining sequence (C_i) and a sense-preserving Lipschitz map $f : S^3 \to S^3$ such that*

(a) *$f|S^3 \setminus \cap_i C_i$ is a finite-to-one branched cover;*
(b) *each component of $\cap_i C_i$ is a fiber of f;*
(c) *$K_f \in L^p_{loc}$.*

We call two excellent packages *equivalent* if there is a diffeomorphism of R^4 that conjugates the pertinent maps in the packages. In (c), we use Lebesgue 3-measure and define K_f in Euclidean terms in R^3. Note that f has non-zero degree. The map f to be constructed depends on the particular choice of exponent p.

Proof. Fix a defining sequence (C_i) and $0 < p < 2$; we have the freedom to choose (C_i) up to a diffeomorphism. Here and below we use the notation from Section 8.

Each domain $D, D_1, \ldots, D_k$ in the package is a handle-body in R^3 with smooth boundary. We assume that D is a smooth regular neighborhood of its *core*, where by a core we mean a 1-polygon inside D. We assume that D is essentially a 1-neighborhood of γ, up to a small but fixed error. We also assume that the length ℓ_0 of the core is so large that we can choose D to satisfy

$$|D| \leq 4\pi\ell_0\,,$$

where $|\cdot|$ denotes Lebesgue 3-measure. Now fix $\varepsilon > 0$ small depending only on p and the package, to be defined later. The cores γ_i of D_i lie inside D subject to some topological constraints; in any case, we can choose sense-preserving diffeomorphisms $\varphi_i : D \to D_i$, $\varphi_i(\gamma) = \gamma_i$, such that

$$\det d\varphi_i \;\leq\; \eta\varepsilon^2\,,$$

and that

$$\mathrm{Lip}\varphi_i^{-1} \;\leq\; \varepsilon^{-1}\,,$$

where $\eta > 1$ is a fixed constant and Lip stands for the Lipschitz constant. Thus, the D_i's are essentially ε-neighborhoods of the γ_i's. The dilation factor ρ of the similarities $\psi_1, \ldots, \psi_k$ is at our disposal, and we choose $\rho = \varepsilon$. It follows that

$$\mathrm{Lip}(\psi_\alpha \circ \theta \circ \varphi_\alpha^{-1}) \;\leq\; C\,\rho^n\varepsilon^{-n} = C\,,$$

and that

$$\det d\psi_\alpha \det d\varphi_\alpha^{-1} \geq (\rho^3/\eta\varepsilon^2)^n = (\varepsilon/\eta)^n$$

for $|\alpha| = n$. Here and below, C denotes any positive constant independent of α. Now define

$$f = F \circ \psi_\alpha \circ \theta \circ \varphi_\alpha^{-1} \qquad \text{on} \qquad E_\alpha\,,$$

where F is a BLD-map from the decomposition space $M(= R^3/G)$ to R^3. The existence of F is guaranteed by the study in Section 8; see Remark 8.19. We claim that f has the desired properties.

To this end, first observe that f is globally Lipschitz. Second, we note that

$$\det df \geq C(\varepsilon/\eta)^n$$

on E_α if $|\alpha| = n$, and then compute

$$\begin{aligned}
&\int_D (\det df)^{-p}\,dx = \sum_\alpha \int_{E_\alpha} (\det df)^{-p}\,dx \\
&\leq C \sum_\alpha |E_\alpha| \varepsilon^{-p|\alpha|} \eta^{p|\alpha|} \\
&\leq C \sum_n k^n \eta^n \varepsilon^{2n-pn} \eta^{pn} < \infty,
\end{aligned}$$

provided we choose ε small enough so that $k\varepsilon^{2-p}\eta^{p+1} < 1$. Finally, outside the domain D the integrability requirement is clearly satisfied. This completes the proof of Theorem 9.5.

9.6. **BLD–maps and Poincaré inequalities.** Spaces that admit a Poincaré inequality have turned out to be a proper environment for abstract first order calculi. This fact was implicitly used in the proof of our Theorem 6.18. The validity of such an inequality is a bi-Lipschitz invariant property of a space. We next demonstrate that, to some extent, this invariance remains true when BLD–maps are used. For calculus in spaces with a Poincaré inequality, see [Che], [HaK], [HK3], [S5], [Sh], for example.

Recall (see [HK3]) that a metric measure space (X, μ) admits a (*weak*) $(1,1)$-*Poincaré inequality* if there exist constants $C \geq 1$ and $\tau \geq 1$ such that

$$\int_B |u - u_B|\,d\mu \leq C\,\mathrm{diam}B \int_{\tau B} \rho\,d\mu \tag{9.7}$$

for all balls B in X, for all bounded measurable functions u in the enlarged ball τB, and for all Borel functions $\rho : X \to [0,\infty]$ with the property that

$$|u(a) - u(b)| \leq \int_{\gamma_{ab}} \rho\,ds$$

whenever γ_{ab} is a rectifiable curve joining two points a and b in τB. Thus, ρ is an *upper gradient* of u in τB, cf. the proof of Lemma 6.19. In (9.7), u_B denotes the mean value of u in B.

9.8. **Theorem.** *Let X be a complete, quasiconvex oriented generalized n-manifold, and assume that there exists a BLD-map of finite multiplicity either from X onto R^n or from R^n onto X. Then X admits a weak $(1,1)$-Poincaré inequality with constants only depending on the data associated with X and f.*

The harder case in Theorem 9.8 is to establish the Poincaré inequality on X if X admits a BLD–map onto R^n. To this end, we rely on the work of Semmes [S4] in addition to the results from Part I of this paper. It is particularly crucial here to have the path-lifting property of BLD–maps as explained in 3.3. The case where R^n is the receiving space is also more interesting in light of the existing examples; compare Section 8.

Proof of Theorem 9.8. Assume first that there exists a surjective BLD–map $f : X \to R^n$ of finite multiplicity. Then X is Ahlfors n-regular by Proposition 6.3. Under these conditions, the validity of a weak $(1,1)$-Poincaré inequality essentially follows from the work [S4], but a few explanatory remarks are due. Semmes shows in [S4] that a weak $(1,1)$-Poincaré inequality holds in every complete Ahlfors n-regular metric space X provided there exists a constant $C > 1$ so that the following assertion is true: given $x, y \in X$, there exists a Lipschitz map

$$F : X \times (0, |x-y|) \to S^n$$

with Lipschitz constant satisfying

$$\mathrm{Lip}(F|X \times (\varepsilon, |x-y| - \varepsilon)) \leq C\,\varepsilon^{-1}, \qquad 0 < \varepsilon < |x-y|/2,$$

such that the maps $F_t = F(\cdot, t) : X \to S^n$, $0 < t < |x-y|$, all have nonzero degree and satisfy

$$F_t|X \setminus B(x, Ct) \equiv \sigma, \qquad 0 < t < |x-y|/2,$$

and

$$F_t|X \setminus B(y, C(|x-y| - t)) \equiv \sigma, \qquad |x-y|/2 < t < |x-y|,$$

for some fixed point $\sigma \in S^n$. See [S4, Section 12 and Appendix B].

To show the existence of a map F as above, we use Proposition 4.13, and postcompose our BLD–map $f : X \to R^n$ with an appropriate Lipschitz map $R^n \to S^n$. Indeed, if $B(x,r)$ is a ball in X, then there is a ball $B' = B(f(x), r')$ in R^n, with r' comparable to r, such that $B' \subset f(B)$ by Proposition 4.13; moreover, the x-component U of $f^{-1}(B')$ belongs to $B(x, r_1)$, and contains $B(x, r_2)$, with both r_1 and r_2 comparable to r. Choose a homotopically nontrivial C/r'-Lipschitz map $h_{B'} : R^n \to S^n$ such that $h_{B'}|R^n \setminus B' \equiv \sigma$ for some fixed point $\sigma \in S^n$. Then define a map F_B by $F_B = h_{B'} \circ f$ in U, and $F_B \equiv \sigma$ in $X \setminus U$. This map $F_B : X \to S^n$ is homotopically nontrivial, and takes the value σ outside U; in particular, $F|X \setminus B(x, r_1) \equiv \sigma$. Because the radii r', r_1, and r_2 are all comparable to r, it is easy to see how, given a pair of points x and y in X, the above construction of the map F_B provides a continuous family of Lipschitz maps F_t as required by Semmes's criterion. We leave the details to the interested reader. This proves the first part of Theorem 9.8.

Now assume that there exists a surjective BLD–map $f : R^n \to X$ of finite multiplicity. The validity of the Poincaré inequality in X is a rather straightforward application of the change of variables formula (5.5) (see also Section 6.13) and Proposition 4.13. We leave the details to the reader. This completes the proof of Theorem 9.8.

9.9. **Remarks** (a) The proof of Theorem 9.8 shows that R^n can often be replaced by a more general space in order to get similar results. On the other hand, the other hypotheses such as completeness, quasiconvexity, and finite multiplicity are generally needed; compare the examples in 4.6.

(b) If we assume in Theorem 9.8, in addition, that $f : X \to R^n$ be Lipschitz, then quasiconvexity of X need not be assumed; it follows as a consequence. Indeed, Proposition 4.13 remains valid as pointed out in 4.16 (a). Next, one shows without difficulty that X is path-wise connected, and (as in the proof of 9.8) that X admits a Poincaré inequality. Quasiconvexity of X now follows, as demonstrated in [HaK].

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