Embeddings of weakly compact sets and $*$-paired Banach spaces

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ABSTRACT. We use properties of Day's norm on $c_0(\kappa)$ to prove that, for every Eberlein compact space K, there exists a separately continuous symmetric mapping $d : K \times K \to \mathbb{R}$ such that we have $d(x,y) < \frac{d(x,x)+d(y,y)}{2}$ for any two distinct points x and y of K. As a consequence, we have that every Eberlein compact space K can be embedded as a point–separating set in its own function space $C(K)$ equipped with pointwise (or weak) topology; in the terminology of A.V. Arhangel'skij, this means that every Eberlein compact space is "self–dual".

We consider the problem whether every Eberlein compact space K can be embedded as a generating set in $C(K)$ (equipped with the weak topology). We show that such an embedding exists for every uniformly Eberlein compact space. We also show that every Eberlein compact space can be embedded as a free generating set in some $c_0(\kappa)$. These results are obtained as special cases of properties of "$*$-paired Banach spaces", a notion generalizing the relation of a reflexive Banach space and its dual.

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1. Introduction. We only consider real linear spaces. For any Banach space X, we denote by X_w the topological vector space obtained by equipping the underlying vector space of X with the weak topology of X. For every compact Hausdorff space K, we consider the function space $C(K)$ as a Banach space, equipped with the supremum-norm, and we denote the space $C(K)_w$ by $C_w(K)$; moreover, we denote by $C_p(K)$ the linear space $C(K)$ equipped with the topology of pointwise convergence.

A topological space K is *Eberlein compact* provided that, for some Banach space X, K is homeomorphic with a compact subspace of X_w; we show below that we can choose here $X = C(K)$. Moreover, we answer a question of A.V. Arhangel'skii [Ar] by constructing the embedding $x \mapsto f_x$ from K into $C_w(K)$ so that the image–set $\{f_x : x \in K\}$ separates the points of K.

A topological space K is *uniformly Eberlein compact* provided that, for some Hilbert space H, K is homeomorphic with a compact subspace of H_w. We show below that every uniformly Eberlein compact space K can be embedded in $C_w(K)$ as a generating set, i.e., as a set whose linear span is dense in $C_w(K)$. We do not know if the same conclusion holds for every Eberlein compact space.

A point–separating embedding $x \mapsto f_x$ from K into $C_w(K)$ is very easy to construct for a uniformly Eberlein compact space K: if K is a subspace of H_w, where H is a Hilbert

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space with an inner product $(\cdot,\cdot)$, then we can define f_x by the condition $f_x(y)=(y,x)$. The same argument works for a compact subspace X_w, when X is a Banach space which admits a separately continuous inner product, in particular, for a weakly compact subspace of a separable Banach space; however, for a Banach space admitting a separately continuous inner product, the weakly compact subspaces are all uniformly Eberlein compact, and hence the argument does not work with all Eberlein compact spaces. Below we show that the function $(x,y)\mapsto \|x\,y\|$, where $\|\cdot\|$ denotes Day's norm on $c_0(\kappa)$, has some properties analogous to those of an inner product, and we derive the embedding result for all Eberlein compact spaces using those properties.

For a uniformly Eberlein compact space K, the space $C(K)^*$ has a separately weak*-continuous inner product, and the embedding of K as a generating set in $C(K)_w$ can be derived from such an inner product. We characterize below uniform Eberlein compactness of a compact Hausdorff space K by the condition that $C(K)$ is "$*$-paired" with some Hilbert space. In a similar way, Eberlein compactness of K is characterized by the condition that $C(K)$ is $*$-paired with some reflexive space.

2. Embedding K as a point–separating set in $C_w(K)$.

According to a fundamental result of D. Amir and J. Lindenstrauss [AL], every Eberlein compact space can be embedded, for a suitable cardinal κ, in the Banach space $c_0(\kappa)$ equipped with its weak topology. This makes it possible to obtain the desired embedding of an Eberlein compact space as a point-separating set in its own function space by using properties of $c_0(\kappa)$-spaces. The space $c_0(\kappa)$ does not have a separately continuous inner product for $\kappa>\omega$ (because there exist non-uniformly Eberlein compact spaces of weight ω_1; for example, the modification of [Ju, Example 5.8], where $\mathbb{I}$ is replaced by a subset of power ω_1), but we can use instead the function $(x,y)\mapsto\|x\,y\|$, where $\|\cdot\|$ is a special norm of $c_0(\kappa)$.

Let κ be an infinite cardinal. Let us recall a definition of the equivalent strictly convex norm $\|\cdot\|$ for $c_0(\kappa)$ introduced by M. Day in [Da]:

$$\|x\|^2=\sup_{<\alpha_1,\dots,\alpha_n>}\sum_{i=1}^{n}4^{-i}(x_{\alpha_i})^2\ , \tag{1}$$

where the supremum is taken over all finite sequences $<\alpha_1,...,\alpha_n>$ of distinct elements of κ.

In his original definition, Day first defined a mapping $D: c_0(\kappa)\to\ell_2(\kappa)$ as follows: for $x\in c_0$, let $(\alpha_n^x)_{n=1}^\infty$ be a sequence of distinct elements of κ such that we have $\{\alpha\in\kappa: x_\alpha\neq 0\}\subset\{\alpha_n^x : n\in\mathbb{N}\}$ and the sequence $(|x_{\alpha_n^x}|)_{n=1}^\infty$ is non–increasing; the element $D(x)$ of $\ell_2(\kappa)$ is then defined by setting $D(x)_{\alpha_n^x}=2^{-n}\cdot x_{\alpha_n^x}$ for every n and $D(x)_\alpha=0$ for $\alpha\in D\setminus\{\alpha_n^x : n\in\mathbb{N}\}$. Even though it is possible to make the mapping D homogeneous, the mapping is not linear; nevertheless, Day showed in [Da] that we obtain a norm $\|\cdot\|$ of

$c_0(\kappa)$ by setting

$$\|x\| = \|D(x)\|_{\ell_2} = \sqrt{\sum_{\alpha\in\kappa} D(x)_\alpha^2} = \sqrt{\sum_{n\in\omega} 4^{-n}\ x_{\alpha_n^x}^2}$$

for every $x \in c_0(\kappa)$. Day showed that the norm $\|\cdot\|$ is strictly convex, and later J. Rainwater [Ra] established that $\|\cdot\|$ is locally uniformly convex.

As mentioned above, $c_0(\kappa)$ does not have any separately continuous inner product when $\kappa > \omega$. However, with Day's norm, the function $(x,y) \mapsto \|x\,y\|$ has some properties similar to those of an inner product.

2.1 Proposition *The following conditions hold for all $x, y \in c_0(\kappa)$ and $r \in \mathbb{R}$:*
(a) $\|x\,y\|^2 \le \|x^2\|\cdot\|y^2\|$.
(b) $\|x\,y\|^2 = \|x^2\|\cdot\|y^2\|$ *only if* $x \equiv 0$ *or* $y^2 = t\,x^2$ *for some* $t \in \mathbb{R}$.
(c) $\|x\,y\| < \frac{\|x^2\|+\|y^2\|}{2}$ *if* $x^2 \ne y^2$.
(d) *The function* $(x,y)\mapsto \|x\,y\|$ *is separately weakly sequentially continuous.*

Proof. **(a)** and **(b)**: We have, by Schwarz inequality, that

$$\|x\,y\|^4 = \left(\sum_{n=1}^{\infty} 4^{-n} x_{\alpha_n^{xy}}^2\, y_{\alpha_n^{xy}}^2\right)^2 \le \sum_{n=1}^{\infty} 4^{-n}\, x_{\alpha_n^{xy}}^4 \cdot \sum_{n=1}^{\infty} 4^{-n}\, y_{\alpha_n^{xy}}^4 \le \|x^2\|^2\cdot\|y^2\|^2 .$$

As a consequence, $\|x\,y\|^2 \le \|x^2\|\cdot\|y^2\|$.

Let us now assume that $x \not\equiv 0$ and $\|x\,y\|^2 = \|x^2\|\cdot\|y^2\|$. Then the inequalities in the above sequence are equalities. By the condition for equality in the Schwarz inequality, we get that there exists $t\in\mathbb{R}$ such that $y_{\alpha_n^{xy}}^2 = t\,x_{\alpha_n^{xy}}^2$ for every $n\in\mathbb{N}$. Moreover, we get that $\sum_{n=1}^\infty 4^{-n}x_{\alpha_n^{xy}}^4 = \|x^2\|^2$ and $\sum_{n=1}^\infty 4^{-n}y_{\alpha_n^{xy}}^4 = \|y^2\|^2$, and this implies that all the α's such that either $x_\alpha \ne 0$ or $y_\alpha \ne 0$ are among the α_n^{xy}'s. As a consequence, we have that $y^2 = t\,x^2$.
(c): We assume that $\|x\,y\| \ge \frac{\|x^2\|+\|y^2\|}{2}$, and we show that $x^2 = y^2$. The left side of the assumed inequality is zero if $x\equiv 0$. On the other hand, the right side of the inequality is strictly positive if $y \not\equiv 0$. As a consequence, if we have $x\equiv 0$, then we also must have $y\equiv 0$. We now assume that $x\not\equiv 0$. By Condition (a), we have that $\|x\,y\| \le \sqrt{\|x^2\|\cdot\|y^2\|} \le \frac{\|x^2\|+\|y^2\|}{2}$. It follows from the assumed inequality that $\|x\,y\| = \sqrt{\|x^2\|\cdot\|y^2\|} = \frac{\|x^2\|+\|y^2\|}{2}$. By Condition (b), there exists $t\ge 0$ such that $y^2 = t^2\,x^2$. We now have that $\|x\,y\| = t\|x^2\|$ and $\|y^2\| = t^2\|x^2\|$, and it follows that $t\|x^2\| = \frac{\|x^2\|+t^2\|x^2\|}{2}$. As a consequence, we have that $t^2+1 = 2t$; this shows that $t=1$ and $y^2 = x^2$.
(d): Let $y\in c_0(\kappa)$. To show that the mapping $x\mapsto \|x\,y\|$ is weakly sequentially continuous, first note that this is obvious if y is the zero function; hence we may assume that y is not identically zero. Let x and x^i, $i\in\mathbb{N}$, be members of $c_0(\kappa)$ such that the sequence $(x^i)_{i=1}^\infty$ converges weakly to x. We show that $\|x^i\,y\| \to \|x\,y\|$. Let $\epsilon > 0$. Then there exists $0 < \epsilon' < \min(\epsilon, 1, \|y\|_\infty)$ such that $\epsilon' \notin |xy|(\kappa)$. We show that there

exists $i_0 \in \omega$ such that $\left|\|x^i y\|^2 - \|x y\|^2\right| \leq 2\epsilon'$ for every $i > i_0$. The weakly converging sequence $(x^i)_{i=1}^{\infty}$ is bounded; let $M \geq 1$ be such that $\|x^i\|_\infty \leq M$ for every i. Denote by A the finite set $\{\alpha \in \kappa : |y_\alpha| > \frac{\epsilon'}{M}\}$. For every $\alpha \in \kappa \setminus A$, we have $|x_\alpha y_\alpha| \leq \epsilon'$ and $|x^i_\alpha y_\alpha| \leq \epsilon'$ for every i. Let $B = \{\alpha \in A : |x_\alpha y_\alpha| > \epsilon'\}$. Note that there exists $j \in \omega$ such that we have $\{\alpha \in A : |x^i_\alpha y_\alpha| > \epsilon'\} = B$ for every $i > j$. We have that $\sum_{\alpha \in \kappa \setminus B} D(zy)^2_\alpha \leq \epsilon'$ for every $z \in \{x\} \cup \{x^i : i > j\}$, and it follows that if $B = \emptyset$, then we have that $\left|\|x^i y\|^2 - \|x y\|^2\right| \leq 2\epsilon'$ for every $i > j$. Assume that $B \neq \emptyset$. Now it suffices to show that there exists $k \in \omega$ such that $|\sum_{\alpha \in B} D(x^i y)^2_\alpha - \sum_{\alpha \in B} D(xy)^2_\alpha| \leq \epsilon'$ for every $i > k$. Denote by r the minimum positive difference of two numbers from the set $\{|xy|_\alpha : \alpha \in B\} \cup \{0, \frac{\epsilon'}{M}\}$, and let $r' = \frac{r}{2}$. Let $k \geq j$ be such that we have $|(x^i y)_\alpha - (xy)_\alpha| < r'$ for all $\alpha \in B$ and $i > k$. Let $m = |B|$, and note that we have $\{\alpha^{x^i y}_n : 0 \leq n < m\} = B$ for every $i > k$.

It follows from the foregoing that, for each $i > k$, we have

$$\sum_{\alpha \in B} D(x^i y)^2_\alpha - \sum_{\alpha \in B} D(xy)^2_\alpha = \sum_{n=0}^{m-1} 4^{-n} (x^i y)^2_{\alpha^{x^i y}_n} - \sum_{n=0}^{m-1} 4^{-n} (xy)^2_{\alpha^{xy}_n}$$

$$= \sum_{n=0}^{m-1} 4^{-n} \left[(x^i y)^2_{\alpha^{x^i y}_n} - (xy)^2_{\alpha^{xy}_n} \right] .$$

Hence it suffices to show that, for $i > k$, we have $|(x^i y)^2_{\alpha^{x^i y}_n} - (xy)^2_{\alpha^{xy}_n}| < \epsilon'$ for every $n < m$. So let $i > k$. Let $t_0, ..., t_p$ be a decreasing enumeration of the numbers $|xy|_\alpha$ for $\alpha \in B$. For every $q < p$, let $C_q = \{\alpha \in \kappa : |xy|_\alpha \geq t_q\}$ and let $c_q = |C_q|$. Since $i > k$, it follows from the definitions of k and r' that $C_q = \{\alpha \in \kappa : |x^i y|_\alpha > t_q - r'\}$. As a consequence, we have that $C_q = \{\alpha^{xy}_n : n < c_q\} = \{\alpha^{x^i y}_n : n < c_q\}$. It follows by induction that we have, for every $q < p$, that

$$\begin{aligned}\{\alpha \in \kappa : |xy|_\alpha = t_q\} &= \{\alpha^{xy}_n : c_{q-1} \leq n < c_q\} \\ &= \{\alpha \in \kappa : ||x^i y|(\alpha) - t_q| < r'\} = \{\alpha^{x^i y}_n : c_{q-1} \leq n < c_q\} .\end{aligned}$$

It follows that, for every $n < m$, we have $\left||x^i y|_{\alpha^{x^i y}_n} - |xy|_{\alpha^{xy}_n}\right| < r'$ and hence

$$\begin{aligned}|(x^i y)^2_{\alpha^{x^i y}_n} - (xy)^2_{\alpha^{xy}_n}| &= \left||x^i y|_{\alpha^{x^i y}_n} + |xy|_{\alpha^{xy}_n}\right| \cdot \left||x^i y|_{\alpha^{x^i y}_n} - |xy|_{\alpha^{xy}_n}\right| \\ &\leq 2M\left||x^i y|_{\alpha^{x^i y}_n} - |xy|_{\alpha^{xy}_n}\right| < 2Mr' \leq \epsilon' . \qquad \square\end{aligned}$$

We shall obtain an embedding $K \to C_w(K)$, for an Eberlein compact space K, from a separately continuous function $\rho : K \times K \to \mathbb{R}$ which satisfies the condition that $\rho(x, y) < \frac{\rho(x,x)+\rho(y,y)}{2}$ whenever $x \neq y$; we call this condition the *subdiagonal inequality* and a function satisfying the condition a *subdiagonal function*. A simple example of a subdiagonal function is $-d$, where d is a metric on K.

2.2 Lemma *Let K be a compact space, and let $\rho : K \times K \to \mathbb{R}$ be a separately continuous subdiagonal function. For every $x \in K$, denote by ρ_x the function $y \mapsto \rho(x, y)$. Then the following hold:*

(a) *K is a Hausdorff space.*

(b) *For all $x, y \in K$, if $x \neq y$, then either $\rho_x(x) \neq \rho_y(x)$ or $\rho_x(y) \neq \rho_y(y)$.*

(c) *The mapping $x \mapsto \rho_x$ is an embedding of K into $C_p(K)$.*

(d) *If the function ρ is bounded, then $x \mapsto \rho_x$ is an embedding of K into $C_w(K)$.*

Proof. Condition **(b)** follows directly from the subdiagonal inequality, and **(a)** follows directly from **(b)**. To prove **(c)**, note that it follows from the separate continuity of ρ that, for each $x \in K$, the function ρ_x is continuous, and that the mapping $x \mapsto \rho_x$ is continuous from K to $C_p(K)$. By Condition **(b)**, the mapping $x \mapsto \rho_x$ is one-to-one, and it follows, since K is a compact Hausdorff space, that the mapping is an embedding of K in $C_p(K)$. To prove **(d)**, assume that ρ is a bounded function. Then the image of K under the mapping $x \mapsto \rho_x$ is a bounded subset of $C(K)$; it follows that in this case the mapping is also an embedding of K in $C_w(K)$, since $C_p(K)$ and $C_w(K)$ have the same bounded compact subsets, by a result of A. Grothendieck (see [Di], Lemma 5.3.1). □

2.3 Theorem *The following conditions are equivalent for a compact space K:*

A. *K is Eberlein compact.*

B. *$K \times K$ admits a separately continuous symmetric subdiagonal function.*

C. *K can be embedded in $C_p(K)$ as a point–separating set.*

D. *K can be embedded in $C_w(K)$.*

Proof. **A**$\Rightarrow$**B**: Assume that K is Eberlein compact. By a fundamental result of D. Amir and J. Lindenstrauss [AL], K can be embedded in $c_0(\kappa)_w$ for some cardinal κ. As is well known, we can assume that the image of K in the embedding is contained in the set $c_0(\kappa)^+ = \{x \in c_0(\kappa) : x_\alpha \geq 0 \text{ for each } \alpha\}$ (we can, for example, accomplish this by using the homeomorphism $\Theta : c_0(\kappa)_w \to c_0(\kappa)^+_w$, defined by the rule $\Theta(x)_{\lambda+2n+i} = \max(0, (-1)^i x_{\lambda+n})$, where λ is a limit ordinal, $n \in \omega$ and $i = 0, 1$). Hence we can assume that K is a subspace of $c_0(\kappa)^+_w$. We define the function $\rho : K \times K \to \mathbb{R}$ by the formula $\rho(x, y) = \|x\, y\|$, where $\|\cdot\|$ denotes Day's norm on $c_0(\kappa)$. Then ρ is clearly symmetric, and ρ satisfies the subdiagonal inequality by Condition (c) of Proposition 2.1; moreover, by Condition (d) of the same proposition, the function ρ is separately sequentially continuous; it follows, since K is Eberlein compact, that ρ is separately continuous on $K \times K$.

B$\Rightarrow$**C**: This follows from Lemma 2.2 (c).

C$\Rightarrow$**D**: Define $\theta : C_p(K) \to C_p(K)$ by the formula $\theta(f)(x) = \frac{f(x)}{1+f(x)^2}$. Then θ is a homeomorphism from $C_p(K)$ onto a bounded subspace of $C_p(K)$. Thus we see that every compact subspace of $C_p(K)$ is homeomorphic with a bounded subspace of $C_p(K)$. The implication $C \Rightarrow D$ now follows by the result of Grothendieck mentioned above that the weak and pointwise topologies coincide on bounded compact subsets of $C_p(K)$.

D$\Rightarrow$**A**: This follows directly from the definition of Eberlein compactness. □

If $x \mapsto \rho_x$ is the embedding of an Eberlein compact space K into $C_p(K)$ provided by Theorem 2.3 and Lemma 2.2, then part (b) of the lemma shows that the set $\{\rho_x : x \in K\}$ of functions separates the points of K. Hence we see that every Eberlein compact space is "paired" with itself, in the terminology of I. Namioka and E. Wheeler [NW], or "dual" to itself, in the terminology of A.V. Arhangel'skij [Ar]; this answers a question made in [Ar].

If $\langle \cdot, \cdot \rangle$ is an inner product on a linear space X, then the function ρ defined by the formula $\rho(x, y) = 2 \cdot \langle x, y \rangle$ is a subdiagonal function on $X \times X$. If $\| \cdot \|$ is the norm of X associated with $\langle \cdot, \cdot \rangle$, then we can write

$$\rho(x, y) = \|x + y\|^2 - \|x\|^2 - \|y\|^2 .$$

Now assume that X is a Banach space with norm $\|\cdot\|$. We can use the above formula to define a function ρ on $X \times X$. It is easy to see that we always have $\rho(x, y) \leq \frac{\rho(x,x)+\rho(y,y)}{2}$. Moreover, we see by [DGZ, Proposition II.1.3] that ρ is a subdiagonal function if, and only if, the norm $\| \cdot \|$ is strictly convex. Even though we cannot produce a separately weakly continuous subdiagonal function for $c_0(\kappa)$ from an inner product when κ is uncountable, it might be possible to obtain one from the above formula.

2.4 Problem *Does $c_0(\kappa)$ have an equivalent strictly convex norm $\| \cdot \|$ such that the function $(x, y) \mapsto \|x + y\|^2 - \|x\|^2 - \|y\|^2$ is separately weakly continuous?*

Note that a norm satisfying the above continuity condition is a *Kadec norm*, i.e., the weak and norm topologies agree on the unit sphere.

To close this section, we describe an alternative, more topological, construction of a separately continuous subdiagonal function for an Eberlein compact space K. For this construction, we need the following topological characterization of Eberlein compact spaces due to H.P. Rosenthal. Recall that a family $\mathcal{U}$ of subsets of a set K is a *point-separating family* provided that, for any two distinct points x and y of K, there exists $U \in \mathcal{U}$ such that U contains exactly one of the points x and y. Note that $\mathcal{U}$ is point-separating if, and only if, the set of characteristic functions $\{\chi_U : U \in \mathcal{U}\}$ separates the points of K.

2.4 Theorem *([Ro]) A compact Hausdorff space K is Eberlein compact if, and only if, K has a σ–point-finite point-separating family of open F_σ–subsets.*

To be able to use point-finite families in constructing the function $\rho : K \times K \to \mathbb{R}$, we need the following notion. We say that a set $F \subset C(K)$ is *point-finitely supported* provided that, for every $x \in K$, the set $\{f \in F : f(x) \neq 0\}$ is finite.

2.5 Lemma *Let $F \subset C(K, \mathbb{I})$ be a point-finitely supported set. Then the function d : $K \times K \to \mathbb{I}$, defined by the formula $d(x, y) = \sup_{f \in F} f(x)f(y)$ is separately continuous.*

Proof. Since d is symmetric, it suffices to observe that, for each $x \in K$, the function $y \mapsto d(x, y)$ is continuous. Let $x \in K$, and let F' be the finite set $\{f \in F : f(x) > 0\}$. Then we have, for every $y \in K$, that $d(x, y) = \max_{f \in F'} f(x)f(y)$. As a consequence, the function $y \mapsto d(x, y)$ is continuous. □

We will now sketch the alternative construction of a separately continuous subdiagonal function for an Eberlein compact space K. By Rosenthal's result, K has a point-separating family $\mathcal{U} = \bigcup_{n=0}^{\infty} \mathcal{U}_n$ of open F_σ–sets such that each $\mathcal{U}_n$ is point-finite. Let $\mathcal{U}' = \{\bigcap \mathcal{V} : \mathcal{V} \subset \mathcal{U} \text{ and } \mathcal{V} \text{ is finite}\}$, and note that $\mathcal{U}'$ consists of open F_σ–sets. For each $V \in \mathcal{U}'$, let f_V be a continuous function $K \to \mathbb{I}$ such that $V = \{x \in K : f(x) > 0\}$. For all $n, k \in \omega$, let $F_{n,k} = \{f_{\cap \mathcal{V}} : \mathcal{V} \subset \mathcal{U}_n \text{ and } |\mathcal{V}| = k\}$, and note that $F_{n,k}$ is point-finitely supported.

For all $n, k \in \omega$ define $\rho_{n,k} : K \times K \to \mathbb{I}$ by the formula $\rho_{n,k}(x,y) = \sup_{f \in F_{n,k}} f(x)f(y)$. By Lemma 2.5, the functions $\rho_{n,k}$ are separately continuous. It follows that the function $\rho = \sum_{n,k=0}^{\infty} 2^{-n-k} \cdot \rho_{n,k}$ is separately continuous. Since each $\rho_{n,k}$ is symmetric, the function ρ is symmetric. We leave it to the reader to verify that, for any two distinct points x and y of K, we have that $\rho(x,y)^2 < \rho(x,x)\rho(y,y)$, and hence also that $\rho(x,y) < \frac{\rho(x,x)+\rho(y,y)}{2}$; the proof of the first inequality depends on the observation that there exist n and k such that $f(x)f(y) = 0$ for every $f \in F_{n,k}$ while $g(x) + g(y) > 0$ for some $g \in F_{n,k}$.

3. Embedding K as a generating set in $C_w(K)$. Above, we showed that every Eberlein compact space K can be embedded in $C_w(K)$ as a point-separating set. The construction described at the end of the preceding section can easily be modified so that the resulting embedding maps some point of K to a non-zero constant function; with this extra condition, the Stone-Weierstrass Theorem implies that the image of K in the embedding generates a dense subalgebra of $C(K)$. In this section, we shall indicate some situations in which an Eberlein compact space K can be embedded in $C_w(K)$ *as a generating set*, i.e., in such a way that the image of K in the embedding generates a dense linear subspace of $C(K)$.

3.1 Problem *Can every Eberlein compact space K be embedded in $C_w(K)$ as a generating set?*

According to one of the results in [AL], every weakly compactly generated Banach space is generated by the (topologically) simplest possible kind of a weakly compact set: a set homeomorphic with the one–point compactification of a discrete space. More precisely: a weakly compactly generated Banach space of density κ is generated by a subspace which is homeomorphic, in its relative weak topology, with $A(\kappa)$, the one–point compactification of the discrete space on κ. According to another result of [AL], a Banach space $C(K)$ is weakly compactly generated if, and only if, the compact space K is Eberlein compact. Moreover, the density of $C(K)$ is κ for an infinite compact Hausdorff space K of weight κ. It follows that if K is an infinite Eberlein compact space of weight κ, then $C(K)$ is generated by a subspace of $C_w(K)$ homeomorphic with $A(\kappa)$. As a consequence, we have the following partial solution to Problem 3.1: for every cardinal κ, the space $A(\kappa)$ can be embedded in $C_w(A(\kappa))$ as a generating set. With the help of Theorem 2.3, we can generalize this result considerably.

3.2 Proposition *Let K be an infinite Eberlein compact space of weight κ such that K contains a clopen copy of the space $A(\kappa)$. Then K can be embedded in $C_w(K)$ as a generating set.*

Proof. By the result of Amir and Lindenstrauss mentioned above, $C_w(K)$ is generated by a subspace A which is homeomorphic with $A(\kappa)$. By Theorem 2.3, $C_w(K)$ contains a subspace K' homeomorphic with K. Since the compact subsets A and K' of $C_w(K)$ are bounded, we can move K' by a translation (if necessary), and hence we can assume that $A \cap K' = \emptyset$. Let A' be a clopen subspace of K' homeomorphic with $A(\kappa)$. Then $A \cup (K' \setminus A')$ is a generating set of $C_w(K)$ which is homeomorphic with K. $\square$

It follows from the above that every Eberlein compact space can be embedded in a "self-generating" space (of the same weight). As a consequence, we see that a self-generating space need not be either strongly or uniformly Eberlein compact. We do not know whether all strongly Eberlein compact spaces are self-generating, but we can show that all uniformly Eberlein compact spaces have this property. Before proving that result, we shall consider a more general situation, a sort of duality between two Banach spaces X and Y, which allows us to obtain "generating embeddings" of compact generating subsets of the space $X^*_{w^*}$ in the space Y_w.

3.3 Definition Two Banach spaces X and Y are *$*$-paired* if there exists a weak*-to-weak continuous, one-to-one and dense-range operator $X^* \to Y$; we call such an operator a *$*$-pairing* mapping.

Note that a $*$-pairing mapping is a weakly compact operator. It follows from known results on transpose mappings that if $T : X^* \to Y$ is $*$-pairing, then so is the transpose $T^* : Y^* \to X \subset X^{**}$. As a consequence, the relation of $*$-pairedness is symmetric. This symmetry is very apparent in the following result which shows that two Banach spaces are $*$-paired if, and only if, they form a "$*$-dual pair".

3.4 Lemma *Two Banach spaces X and Y are $*$-paired with each other if, and only if, the duals X^* and Y^* can be put in a separated duality by a separately weak*-continuous bilinear form.*

Proof. *Necessity.* If $T : X^* \to Y$ is a $*$-pairing mapping, then we can define the bilinear form $\langle \cdot, \cdot \rangle$ on $X^* \times Y^*$ by the formula $\langle \phi, \psi \rangle = \psi(T(\phi))$.
Sufficiency. Assume that $\langle \cdot, \cdot \rangle$ is a separately weak*-continuous bilinear form which puts X^* and Y^* in a separated duality. For every $\phi \in X^*$, since the linear functional $\psi \mapsto \langle \phi, \psi \rangle$ of Y^* is weak*-continuous, there exists $y_\phi \in Y$ such that $\langle \phi, \psi \rangle = \psi(y_\phi)$ for every $\psi \in C^*$. Then the mapping $T : X^* \to Y$, obtained by setting $T(\phi) = y_\phi$ for every $\phi \in X^*$, is $*$-pairing. $\square$

A familiar instance of two $*$-paired spaces is that of a reflexive space R and its dual R^*; in this case, the identity mapping $R^* \to R^*$ is $*$-pairing. Here the $*$-pairing mapping is onto, and this is essentially the only situation where that can occur: if $T : X^* \to Y$ is an onto $*$-pairing mapping, then T is an isomorphism, and it follows (see the proof of (b)$\Rightarrow$(a) below) that the unit ball of Y is weakly compact; as a consequence Y is reflexive and X is isomorphic with the dual of Y. The following result shows that not all $*$-paired Banach spaces are reflexive and not all Banach spaces are $*$-paired with other spaces.

3.5 Proposition *The following conditions are equivalent for a Banach space X:*

(a) *X is weakly compactly generated.*

(b) *X is $*$-paired with some Banach space.*

(c) *X is $*$-paired with some reflexive Banach space.*

Proof. (a)$\Rightarrow$(c) Assume that X is weakly compactly generated. By ([DFJP], Corollary 3) there exists a reflexive space R and a continuous one-to-one dense-range operator $T : R \to X$. We identify R with R^{**} and we consider T as a mapping $(R^*)^* \to X$. By reflexivity, the weak and weak* topologies coincide on $(R^*)^*$. As consequence, T is a $*$-pairing mapping.

(b)$\Rightarrow$(a) Assume that Y is a Banach space and that $T : Y^* \to X$ is a $*$-pairing mapping. By the Alaoglu Theorem, the closed unit ball B_Y of Y is compact in the weak*-topology. Since T is weak*-to-weak continuous, the set $T(B_Y)$ is compact in the weak topology of X. Since T has dense range in X, the set $T(B_Y)$ generates a dense linear subspace of X. □

Next we specify the Banach space in Condition (b) above by showing that all weakly compactly generated Banach spaces of a given density character are $*$-paired with one single space. Note that it is known (see, e.g., [Di], p.190) that the density character of a weakly compactly generated Banach space coincides with the weak*-density character of the dual of the space; it follows from this and Proposition 3.5 that any two $*$-paired Banach spaces have the same density character.

According to ([AL], Proposition 2), for every weakly compactly generated Banach space X, there exists a one-to-one and weak*-to-weak continuous operator $X^* \to c_0(\kappa)$ for some κ; the next result shows that we can make the operator also to have dense range.

3.6 Proposition *An infinite-dimensional weakly compactly generated Banach space of density character κ is $*$-paired with $c_0(\kappa)$.*

Proof. Let X be an infinite-dimensional weakly compactly generated Banach space of density κ. We show that there exists a $*$-pairing mapping $T : X^* \to c_0(\kappa)$. By ([VWZ], Theorem 3.1) there exists a biorthogonal system $\{(x_\alpha, x_\alpha^*) : \alpha < \lambda\}$ such that the set $A = \{x_\alpha : \alpha < \lambda\} \cup \{0\}$ is a weakly compact generating subset of X. Without loss of generality, we may assume that $\|x_\alpha\| \leq 1$ for every α. Note that the set A, in its relative weak topology, is homeomorphic with $A(\lambda)$. Since the density of X is κ, we have that $\lambda = \kappa$.

For every $\phi \in X^*$, denote by $T(\phi)$ the (transfinite) sequence $\langle \phi(x_\alpha) \rangle_{\alpha<\kappa}$; it is easy to see that $T(\phi)$ is a member of $c_0(\kappa)$. We show that the mapping $T : X^* \to c_0(\kappa)$ is $*$-pairing. The mapping T is clearly linear, and T is one-to-one because the set A generates X. Also, it is easily seen that T is continuous and $\|T\| \leq 1$.

Next we show that, for every $\psi \in c_0(\kappa)^*$, $\psi \neq 0$, there exists $y \in X$, $y \neq 0$, such that the mapping $\psi \circ T : X^* \to \mathbb{R}$ coincides with the evaluation map $\phi \mapsto \phi(y)$. So let $\psi \in c_0(\kappa)^*$, $\psi \neq 0$. There exists $\langle r_\alpha \rangle_{\alpha<\kappa} \in \ell^1(\kappa)$ such that $\psi(\langle p_\alpha \rangle_{\alpha<\kappa}) = \sum_{\alpha<\kappa} r_\alpha p_\alpha$ for every $\langle p_\alpha \rangle_{\alpha<\kappa} \in c_0(\kappa)$. Since $\psi \neq 0$, we must have $r_\alpha \neq 0$ for some α. Denote by y the element $\sum_{\alpha<\kappa} r_\alpha x_\alpha$ of X, and note that it follows from properties of a biorthogonal system that $y \neq 0$. Let $\phi \in X^*$. Then we have that $\phi(y) = \phi(\sum_{\alpha<\kappa} r_\alpha x_\alpha) = \sum_{\alpha<\kappa} r_\alpha \phi(x_\alpha) = \psi(T(\phi))$. We have shown that the mapping $\psi \circ T$ is the evaluation $\phi \mapsto \phi(y)$.

By the foregoing, the transpose T^* of the mapping $T : X^* \to c_0(\kappa)$ maps $c_0(\kappa)^*$ into $X \subset X^{**}$. As a consequence, the mapping T is weak*-to-weak continuous. To show that the range of T is dense in $c_0(\kappa)$, assume that this is not the case. Then there exists $\psi \in c_0(\kappa)^*$ such that $\psi \neq 0$ but $\psi(T(\phi)) = 0$ for every $\phi \in X^*$. Let $y \in X$, $y \neq 0$, be such that the mapping $\psi \circ T : X^* \to \mathbb{R}$ is the evaluation $\phi \mapsto \phi(y)$. We now have, for every $\phi \in X^*$, that $\phi(y) = \psi(T(\phi)) = 0$, and it follows that $y = 0$ – a contradiction. □

Next we consider $*$-pairedness of $C(K)$-spaces. Here we can restrict our consideration to Eberlein compact spaces, because it follows from Proposition 3.5 and a result of [AL] that $C(K)$ is $*$-paired with some Banach space if, and only if, K is Eberlein compact. Our first result shows that the condition that $C(K)$ is $*$-paired with a Banach space X can be characterized by the existence of a special embedding of K as a generating set in X equipped with its weak topology.

3.7 Theorem *Let K be a compact Hausdorff space and X a Banach space. The space $C(K)$ is $*$-paired with X if, and only if, there exists a generating subset $\hat{K}$ of X_w such that $\hat{K}$ is homeomorphic with K and the subspace $\{\phi|\hat{K} : \phi \in X^*\}$ is dense in $C(\hat{K})$.*

Proof. *Sufficiency.* If $\hat{K}$ has the property mentioned in the theorem, then it is easy to see that the mapping $\phi \mapsto \phi|\hat{K}$ from X^* to $C(\hat{K})$ is $*$-pairing.

Necessity. Assume that $C(K)$ and X are $*$-paired, and let $T : C(K)^* \to X$ be a $*$-pairing mapping. We denote by θ the natural embedding of K in $C(K)^*_{w^*}$, i.e., $\theta(k)$ is the evaluation map $f \mapsto f(k)$ defined on $C(K)$. For every $k \in K$, we set $\hat{k} = T(\theta(k))$. Further, we set $\hat{K} = \{\hat{k} : k \in K\} = T(\theta(K))$.

The mapping $T \circ \theta$ is continuous $K \to X_w$ and one-to-one, and hence it follows from compactness of K that $\hat{K}$, in its relative weak topology, is homeomorphic with K. To see that $\hat{K}$ generates X, note that the linear subspace of $C(K)^*$ generated by $\theta(K)$ corresponds to the set of all finitely supported measures on $C(K)$, and this subspace is dense in the weak*-topology of $C(K)^*$. The conclusion now follows since the mapping $T : C(K)^* \to c_0(\kappa)$ is weak*-to-weak continuous and has dense range.

It remains to show that $\{\phi|\hat{K} : \phi \in X^*\}$ is dense in $C(\hat{K})$. Note that this condition is equivalent with the condition that $\{(\phi \circ T)|\theta(K) : \phi \in X^*\}$ is dense in $C(\theta(K))$, in other words, with the condition that $\{T^*(\phi)|\theta(K) : \phi \in X^*\}$ is dense in $C(\theta(K))$ (here $\theta(K)$ is considered as a topological subspace of $C(K)^*_{w^*}$). Since the mapping $T^* : X^* \to C(K)$ is $*$-pairing, the set $T^*(X^*)$ is dense in in the space $C(K) \subset C(K)^{**}$. Hence our proof is complete once we show that, for all $\phi \in X^*$ and $k \in K$, the equation $T^*(\phi)(k) = T^*(\phi)(\theta(k))$ holds. So let $\phi \in X^*$ and $k \in K$. Since $T^*(\phi) \in C(K)$, there exists $f \in C(K)$ such that we have $T^*(\phi)(\mu) = \mu(f)$ for every $\mu \in C(X)^*$. Now we have that $T^*(\phi)(k) = f(k) = \theta(k)(f) = T^*(\phi)(\theta(k))$, as desired. $\square$

Remarks. **(a)** Note that it follows from the conditions satisfied by $\hat{K}$ above that $\hat{K}$ is a free set in X. As a consequence, every function $\hat{K} \to \mathbb{R}$ is the restriction of some linear function $X \to \mathbb{R}$. However, even though the continuous functionals are dense in $C(\hat{K})$, the equation $\{\phi|\hat{K} : \phi \in X^*\} = C(\hat{K})$ can hold only for a finite K: the $*$–pairing mapping $\phi \mapsto \phi|\hat{K}$ from X^* to $C(\hat{K})$ cannot be onto if $\hat{K}$ is infinite.
(b) The above proof shows that the set $\hat{K}$ constructed there has the following property, stronger than that stated in the theorem: if $E \subset X^*$ generates a weak*-dense subspace of X^*, then $\{\phi|\hat{K} : \phi \in E\}$ generates a dense subspace of $C(\hat{K})$.

By the above theorem and Proposition 3.6, we have the following result.

3.8 Corollary *Let K be an Eberlein compact space of weight κ. Then there is a generating subset $\hat{K}$ of $c_0(\kappa)_w$ such that $\hat{K}$ is homeomorphic with K and the subspace $\{\phi|\hat{K} : \phi \in c_0(\kappa)^*\}$ is dense in $C(\hat{K})$.*

For the remainder of this paper, we shall consider Banach spaces which are $*$-paired with themselves. A typical instance is that of a Hilbert space H, where the inner product of H^* provides a "$*$-pairing form", and the natural isomorphism $H^* \to H$ provides a $*$-pairing mapping. Another case is that of the space $c_0(\kappa)$. If we identify the dual of $c_0(\kappa)$ with $\ell^1(\kappa)$, then the formal identity map $\ell^1(\kappa) \to c_0(\kappa)$ $*$-pairs $c_0(\kappa)$ with itself. On the other hand, not every reflexive space is $*$-paired with itself: it follows from Pitt's Theorem that there is no continuous one-to-one linear map from $\ell^q(\kappa)$ to $\ell^p(\kappa)$ when κ is uncountable and $1 \leq p < q < \infty$ (see [HHZ], Exercise 15 on p.209).

Our next result gives simple characterizations for a Banach space to be $*$-paired with some Hilbert space, and it shows that the above observation on "self-$*$-pairedness" of Hilbert spaces can be generalized to all Banach spaces $*$-paired with Hilbert spaces.

3.9 Proposition *The following are equivalent for a Banach space X:*
(a) *X is $*$-paired with some Hilbert space.*
(b) *There exists a continuous, dense-range operator from some Hilbert space to X.*
(c) *There exists a weak*-to-weak continuous one-to-one operator from X^* into some Hilbert space.*
(d) *X^* has a separately weak*-continuous inner product.*

Proof. The implication **(a)**$\Longrightarrow$**(b)** holds trivially, and **(b)**$\Longrightarrow$**(c)** by well-known properties of transpose operators.
(c)$\Longrightarrow$**(d)**: Let T be a weak*-to-weak continuous one-to-one operator from X^* into a Hilbert space H. Since T is linear and one-to-one, we can define an inner product $\langle \cdot\,,\cdot\rangle$ on X^* by the formula $\langle \phi, \psi\rangle = (T(\phi), T(\psi))$, where $(\cdot,\cdot)$ denotes the inner product of H. The inner product $\langle \cdot\,,\cdot\rangle$ is separately weak*-continuous because T is weak*-to-weak continuous.
(d)$\Longrightarrow$**(a)** If X^* has a separately weak*-continuous inner product $\langle \cdot\,,\cdot\rangle$, then the completion $(H, (\cdot,\cdot))$ of the inner product space $(X^*, \langle \cdot\,,\cdot\rangle)$ is a Hilbert space, and the inclusion map from X^* to H is $*$-pairing. □

3.10 Corollary *A Banach space is $*$-paired with itself provided the space is $*$-paired with some Hilbert space*

Proof. This follows from the above result and Lemma 3.4, since a separately continuous inner product clearly puts a Banach space in a separated duality with itself. □

The symmetry of the "$*$-pairing form" obtained from an inner product has a natural characterization in terms of the associated $*$-pairing map. Let us call a weak*-to-weak continuous linear mapping $T : X^* \to X$ *self-adjoint* if the equation $T^* = T$ holds (this, of course, makes sense only after the identification of X with the corresponding subspace of X^{**}).

3.11 Lemma *Let X be a Banach space. The following are equivalent for a weak*-to-weak continuous linear map $T : X^* \to X$.*
(a) *T is self-adjoint.*
(b) *$\phi(T(\psi)) = \psi(T(\phi))$ for all $\phi, \psi \in X^*$.*
(c) *The bilinear form $\langle \cdot\,,\cdot\rangle$ on X^*, defined by $\langle \phi, \psi\rangle = \psi(T(\phi))$, is symmetric.*

Proof. The equivalence of conditions **(b)** and **(c)** is obvious. Moreover, T is self-adjoint if, and only if, we have, for all $\phi, \psi \in X^*$ that $\psi(T(\phi)) = T^*(\phi)(\psi)$, i.e., that $\psi(T(\phi)) = \phi(T(\psi))$. □

It follows from Lemma 3.11 and the proof of Corollary 3.10, that a Banach space is $*$-paired with itself by a self-adjoint mapping provided the space is $*$-paired with some Hilbert space. According to ([BRW], Theorem 3.2) a compact Hausdorff space K is uniformly Eberlein compact if, and only if, $C(K)$ is $*$-paired with some Hilbert space. As a consequence, we have the following result which exhibits a significant class of non-Hilbertian self-$*$-paired spaces.

3.12 Proposition *If K is a uniformly Eberlein compact space, then the space $C(K)$ is $*$-paired with itself by a self-adjoint mapping.*

The above results leave the following questions on self-$*$-paired spaces unanswered:

3.13 Problems *Let X be a Banach space which is $*$-paired with itself.*
A. *Can X be $*$-paired with itself by a self-adjoint mapping?*
B. *If X is $*$-paired with itself by a self-adjoint mapping, is X then $*$-paired with some Hilbert space?*
C. *If $X = C(K)$, is K uniformly Eberlein compact?*

With the following result we return to the original problem on the possibility of embedding compact spaces as generating sets in their own function spaces.

3.14 Proposition *Let K be a compact Hausdorff space. The space $C(K)$ is $*$-paired with itself if, and only if, $C_w(K)$ has a generating subset $\hat{K}$ such that $\hat{K}$ is homeomorphic with K and the evaluations $\hat{k} \mapsto \hat{k}(x)$, for $x \in K$, generate a dense subspace of $C(\hat{K})$.*

Proof. Sufficiency follows directly from Theorem 3.7. Necessity follows from Theorem 3.7 and Remark (b) following the theorem, because the evaluations $f \mapsto f(x)$, for $x \in K$, generate a weak*-dense subspace of $C(K)^*$. $\square$

In particular, it follows by the result of Proposition 3.12 that every uniformly Eberlein compact space K can be embedded as a generating subset in $C_w(K)$. Moreover, we can choose the embedding in a special way. According to Theorem 2.3, every Eberlein compact space K has a "symmetric point-separating embedding" into $C_w(K)$; if K is uniformly Eberlein compact, then it follows from Proposition 3.12 that we can strenghten "point-separating" to "generating".

3.15 Corollary *Let K be a uniformly Eberlein compact space. Then there exists a separately continuous symmetric function $\rho : K \times K \to \mathbb{R}$ such that the functions ρ_x, $x \in K$, generate $C(K)$, where $\rho_x(y) = \rho(x, y)$ for all $x, y \in K$.*

Proof. By Proposition 3.12 and Lemma 3.11, there exists a self-adjoint $*$-pairing map $T : C(K)^* \to C(K)$. Let θ be the natural embedding of K in $C(K)^*_{w^*}$. We define $\rho : K \times K \to \mathbb{R}$ by setting $\rho(x, y) = T(\theta(x))(y) = \theta(y)(T(\theta(x)))$ for all $x, y \in K$. By self-adjointness of T, the function ρ is symmetric. For every $x \in K$, we have that $T(\theta(x) \in C(K)$, and hence the function $y \mapsto \rho(x, y)$ is continuous on K; it follows by the symmetry that ρ is separately continuous. Finally, because the set $\theta(K)$ generates a weak*-dense subspace of $C(K)^*$ and since T is weak*-to-weak continuous and has dense range, $T(\theta(K))$ is a generating set in $C(K)$; this means that the functions ρ_x, $x \in K$, generate $C(K)$. $\square$

By Proposition 3.9 and ([BRW], Theorem 3.2), a compact Hausdorff space K is uniformly Eberlein compact if, and only if, the space $C(K)^*$ has a separately weak*-continuous inner product. One can obtain a function ρ with the properties stated in the above corollary also from such an inner product $(\cdot, \cdot)$ by the formula $\rho(x, y) = (\theta(x), \theta(y))$, where θ is again the natural embedding of K in $C(K)^*_{w^*}$.

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