FRACTIONAL BROWNIAN MOTION, RANDOM WALKS AND BINARY MARKET MODELS

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Abstract. We prove a Donsker type approximation theorem for the fractional Brownian motion in the case $H > 1/2$. Using this approximation we construct a binary market model that converges weakly to the fractional analogue of the Black–Scholes model. We show that there exist arbitrage opportunities in this model.

1. Introduction

The fractional Brownian motion is a continuous zero mean Gaussian process with stationary increments. The correlation of the the increments are characterized by means of the so-called Hurst index, H. Unlike the standard Brownian motion the fractional one has long-range dependency property when $H > 1/2$. In some empirical studies of financial time series it has been demonstrated that the log-returns have this long-range dependence, cf. Peters [11]. Mandelbrot [8] has also studied the long-range dependence in connection to financial data. One has also shown that the inputs of telecommunication networks exhibit long-range dependence. For the use of the fractional Brownian motion in telecommunication theory we refer to Norros [9].

It is well-known that the standard Brownian motion can be approximated as a random walk consisting of i.i.d. random variables. In section 2 we show that the fractional one can be approximated similarily by a "disturbed" random walk. We clarify how to use recent representations of the fractional Brownian motion in our approximation.

By using the approximation introduced in section 2 we in section 3 construct a fractional binary market model that approximates the fractional analogue of the Black–Scholes model. Moreover, we show that there exists arbitrage opportunities in our model. One such opportunity is constructed explicitly.

2. Fractional Brownian motion as a limit of a random walk

2.1. **Fractional Brownian motion.** The fractional Brownian motion Z with index $H \in (0,1)$ is a continuous zero mean Gaussian process with stationary increments and covariance function

$$\mathbf{E}Z_tZ_s = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

If $H < \frac{1}{2}$ the increments of the process are negatively correlated. In case of $H > \frac{1}{2}$ they are positively correlated. When $H = \frac{1}{2}$ we have the standard Brownian motion W, i.e. independent increments. We assume that the self similarity index H satisfies $H > \frac{1}{2}$. In this case we have the following kernel representation of Z with respect to the standard Brownian motion

$$Z_t = \int_0^t z(t,s)\, dW_s \tag{2.1}$$

with a deterministic kernel

$$\begin{aligned} z(t,s) &= c_H(H-\tfrac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,du \\ &= c_H(t-s)^{H-\frac{1}{2}}{}_2F_1(\tfrac{1}{2}-H, H-\tfrac{1}{2}, H+\tfrac{1}{2}, 1-\tfrac{t}{s}). \end{aligned}$$

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Here ${}_2F_1$ is the Gauss hypergeometric function and c_H is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}.$$

The integral (2.1) is defined in the pathwise sense in spite of the singularity of the kernel z at zero. This is possible because of the Hölder continuity of the paths of Z and the kernel z. For details we refer to [10]. We interpret the kernel $z(t,s)$ to be zero whenever $s \geq t$.

The integral representation (2.1) is due to Decreusefond and Üstünel [5]. Note that the representation above holds only when $H > \frac{1}{2}$. A similar representation for the general case $H \in (0,1)$ can be found in [5] and in [10]. However, since we are only considering the fractional Brownian motion with positively correlated increments we have no use for these general representations.

2.2. **Analogue of the Donsker's theorem.** Weak convergence to the fractional Brownian motion has already been investigated by Beran [1] and Taqqu [14]. Their approximation schemes involve normal random variables. Dasgupta [4] proved an approximation using binary random variables and Mandelbrot and Van Ness's [7] representation of the fractional Brownian motion. Cutland et. al. [3] also showed a result of this kind by using nonstandard analysis. However, using the integral representation (2.1) we are able to provide a very simple approximation in terms of i.i.d. square integrable random variable. Let us proceed with our approximation.

Let W be the standard Brownian motion and $\xi_i^{(n)}$ i.i.d. random variables. Assume further that the $\xi_i^{(n)}$'s are square integrable, centered and normalized, i.e.

$$\mathbf{E}\xi_i^{(n)} = 0 \quad \text{and} \quad \mathbf{D}^2\xi_i^{(n)} = 1.$$

Denote

$$W_t^{(n)} := \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^{(n)},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. According to Donsker's theorem $W^{(n)}$ converges weakly to W. Note that the weak convergence of stochastic processes is understood in the Skorohod space $D(0,T)$ of right continuous functions on the interval $[0,T]$ that have left limits. For details we refer to [2].

Denote now

$$Z_t^{(n)} := \int_0^t z^{(n)}(t,s)\, dW_s^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt \rfloor}{n}, s)\, ds\, \frac{1}{\sqrt{n}} \xi_i^{(n)},$$

where z is the kernel that transforms the standard Brownian motion into a fractional one and for all $t \in [0,T]$ the function $z^{(n)}(t,\cdot)$ is an approximation to $z(t,\cdot)$, viz.

$$z^{(n)}(t,s) := n \int_{s-\frac{1}{n}}^{s} z(\tfrac{\lfloor nt \rfloor}{n}, u)\, du.$$

Theorem 2.1. *The random walk $Z^{(n)}$ defined above converges weakly to the fractional Brownian motion.*

Proof. The proof consist of showing that the finite-dimensional distributions of $Z^{(n)}$ converge to those of Z and then showing that $Z^{(n)}$ is tight.

Let us consider the limiting finite-dimensional distributions using Lindeberg's theorem and the Cramér–Wold device. For arbitrary $a_1, \ldots, a_d \in \mathbb{R}$ and $t_1, \ldots, t_d \in [0,T]$ we want to show that

$$Y^{(n)} := \sum_{k=1}^{d} a_k Z_{t_k}^{(n)}$$

converges to a normal distribution with variance

$$\mathbf{E}(\sum_{k=1}^{d} a_k Z_{t_k})^2.$$

Let us calculate the limiting variance of $Y^{(n)}$.

$$\begin{aligned}
(\sigma^{(n)})^2 &:= \mathbf{D}^2 Y^{(n)} \\
&= \mathbf{E}(\sum_{k=1}^{d} a_k Z^{(n)}_{t_k})^2 \\
&= \sum_{k,l=1}^{d} a_k a_l \mathbf{E} Z^{(n)}_{t_k} Z^{(n)}_{t_l} \\
&= \sum_{k,l=1}^{d} a_k a_l\, n \sum_{i=1}^{\lfloor nT \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_l \rfloor}{n}, s)\, ds \qquad (2.2)
\end{aligned}$$

By the mean value theorem (2.2) is equal to

$$\sum_{k,l=1}^{d} a_k a_l \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} z(\tfrac{\lfloor nt_k \rfloor}{n}, s^{(n)}_{i,k}) z(\tfrac{\lfloor nt_l \rfloor}{n}, s^{(n)}_{i,l}) \qquad (2.3)$$

for some $s^{(n)}_{i,k}, s^{(n)}_{i,l} \in (\frac{i-1}{n}, \frac{i}{n}]$. By using the fact that for all $t \in [0,T]$ the functions $z(t,\cdot)$ are continuous and decreasing in $(0,T]$ we obtain that the inner sum in formula (2.3) is equal to

$$\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} z(\tfrac{\lfloor nt_k \rfloor}{n}, u^{(n)}_i) z(\tfrac{\lfloor nt_l \rfloor}{n}, u^{(n)}_i) \qquad (2.4)$$

for some

$$u^{(n)}_i \in \left[\min(s^{(n)}_{i,k}, s^{(n)}_{i,l}), \max(s^{(n)}_{i,k}, s^{(n)}_{i,l})\right] \subseteq \left(\frac{i-1}{n}, \frac{1}{n}\right].$$

By using the fact that z is continuous with respect to both arguments and that the maps $t \mapsto \frac{\lfloor nt \rfloor}{n}$ converge uniformly to the identity map in $[0,T]$ we see that (2.4) is a Riemann type sum. It follows that (2.3), and hence $(\sigma^{(n)})^2$, converges to

$$\sum_{k,l=1}^{d} a_k a_l \int_0^T z(t_k, s) z(t_l, s)\, ds = \mathbf{E}(\sum_{k=1}^{d} a_k Z_{t_k})^2.$$

Let us now write $Y^{(n)}$ as a sum in i.

$$\begin{aligned}
Y^{(n)} &= \sum_{k=1}^{d} a_k Z^{(n)}_{t_k} \\
&= \sum_{k=1}^{d} a_k \sum_{i=1}^{\lfloor nt_k \rfloor} \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds\, \xi^{(n)}_i \\
&= \sum_{i=1}^{\lfloor nT \rfloor} \sqrt{n} \xi^{(n)}_i \sum_{k=1}^{d} a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k \rfloor}{n}, s)\, ds \\
&=: \sum_{i=1}^{\lfloor nT \rfloor} Y^{(n)}_i.
\end{aligned}$$

Lindeberg's condition is satisfied if for all $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{(\sigma^{(n)})^2} \sum_{i=1}^{\lfloor nT \rfloor} \mathbf{E}(Y^{(n)}_i)^2 \mathbf{1}_{\left\{|Y^{(n)}_i| > \varepsilon \sigma^{(n)}\right\}} = 0. \qquad (2.5)$$

Let us first consider the set

$$\left\{|Y^{(n)}_i| > \varepsilon \sigma^{(n)}\right\} = \left\{(Y^{(n)}_i)^2 > \varepsilon^2 (\sigma^{(n)})^2\right\}$$

in detail. We give an upper bound for the random variables $(Y_i^{(n)})^2$. By Cauchy–Schwartz inequality and the facts that $z(\cdot,s)$ is increasing and $z(t,\cdot)$ is decreasing we obtain

$$\begin{aligned}(Y_i^{(n)})^2 &= n(\xi_i^{(n)})^2(\sum_{k=1}^d a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt_k\rfloor}{n},s)\,ds)^2 \\ &\le n(\xi_i^{(n)})^2 A(\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(T,s)\,ds)^2 \\ &\le (\xi_i^{(n)})^2 A\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(T,s)^2\,ds \\ &\le (\xi_i^{(n)})^2 A\int_0^{\frac{1}{n}} z(T,s)^2\,ds \\ &= (\xi_i^{(n)})^2 A\delta^{(n)},\end{aligned}$$

where we have denoted $A := (\sum_{k=1}^d a_k)^2$ and $\delta^{(n)} := \int_0^{1/n} z(T,s)^2\,ds$. We obtain

$$\left\{|Y_i^{(n)}| > \varepsilon\sigma^{(n)}\right\} \subseteq \left\{(\xi_i^{(n)})^2 A\delta^{(n)} > \varepsilon^2(\sigma^{(n)})^2\right\} =: D^{(n)}(\xi_i^{(n)}). \tag{2.6}$$

Using the inclusion (2.6) we obtain

$$\begin{aligned}\mathbf{E}(Y_i^{(n)})^2\mathbf{1}_{\left\{|Y_i^{(n)}|>\varepsilon\sigma^{(n)}\right\}} &= (\sum_{k=1}^d a_k\sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(t_k,s)\,ds)^2\mathbf{E}(\xi_i^{(n)})^2\mathbf{1}_{\left\{|Y_i^{(n)}|>\varepsilon\sigma^{(n)}\right\}} \\ &\le (\sigma_i^{(n)})^2\mathbf{E}(\xi_i^{(n)})^2\mathbf{1}_{D^{(n)}(\xi_i^{(n)})} \\ &= (\sigma_i^{(n)})^2\mathbf{E}\xi^2\mathbf{1}_{D^{(n)}},\end{aligned} \tag{2.7}$$

where $\xi := \xi_1^{(1)}$, $D^{(n)} := D^{(n)}(\xi_1^{(1)})$ and $(\sigma_i^{(n)})^2 := \mathbf{D}^2 Y_i^{(n)}$. By using the upper bound (2.7) to the Lindeberg's condition (2.5) we obtain

$$\begin{aligned}\frac{1}{(\sigma^{(n)})^2}\sum_{i=1}^{\lfloor nT\rfloor}\mathbf{E}(Y_i^{(n)})^2\mathbf{1}_{\left\{|Y_i^{(n)}|>\varepsilon\sigma^{(n)}\right\}} &\le \frac{(\sigma_1^{(n)})^2+\cdots+(\sigma_{\lfloor nT\rfloor}^{(n)})^2}{(\sigma^{(n)})^2}\mathbf{E}\xi^2\mathbf{1}_{D^{(n)}} \\ &= \mathbf{E}\xi^2\mathbf{1}_{D^{(n)}}.\end{aligned}$$

Since the function $z(T,\cdot)^2$ is integrable we obtain that $\delta^{(n)} > 0$ tends to zero as n tends to infinity. Consequently the expectation $\mathbf{E}\xi^2\mathbf{1}_{D^{(n)}}$ also tends to zero. Hence the Lindeberg's condition (2.5) is satisfied and the convergence of the finite-dimensional distributions follows.

It remains to prove the tightness. Let $s < t$ be arbitrary time points. By using Cauchy–Schwartz inequality we obtain

$$\begin{aligned}\mathbf{E}(Z_t^{(n)}-Z_s^{(n)})^2 &= \mathbf{E}(\sum_{i=1}^{\lfloor nt\rfloor}\sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt\rfloor}{n},u)-z(\tfrac{\lfloor ns\rfloor}{n},u)\,du\,\xi_i^{(n)})^2 \\ &= \sum_{i=1}^{\lfloor nt\rfloor}(\sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} z(\tfrac{\lfloor nt\rfloor}{n},u)-z(\tfrac{\lfloor ns\rfloor}{n},u)\,du)^2 \\ &\le \sum_{i=1}^{\lfloor nt\rfloor}\int_{\frac{i-1}{n}}^{\frac{i}{n}}(z(\tfrac{\lfloor nt\rfloor}{n},u)-z(\tfrac{\lfloor ns\rfloor}{n},u))^2\,du \\ &\le \int_0^t (z(\tfrac{\lfloor nt\rfloor}{n},u)-z(\tfrac{\lfloor ns\rfloor}{n},u))^2\,du \\ &= \left|\frac{\lfloor nt\rfloor}{n}-\frac{\lfloor ns\rfloor}{n}\right|^{2H}.\end{aligned} \tag{2.8}$$

Let now $s<t<u$ be arbitrary. Using Cauchy–Schwartz inequality again and the bound (2.8) we obtain

$$\begin{aligned}\mathbf{E}|Z_t^{(n)}-Z_s^{(n)}||Z_u^{(n)}-Z_t^{(n)}| &\le \left(\mathbf{E}(Z_t^{(n)}-Z_s^{(n)})^2\right)^{\frac12}\left(\mathbf{E}(Z_u^{(n)}-Z_t^{(n)})^2\right)^{\frac12}\\ &\le \left|\frac{\lfloor nt\rfloor}{n}-\frac{\lfloor ns\rfloor}{n}\right|^H\left|\frac{\lfloor nu\rfloor}{n}-\frac{\lfloor nt\rfloor}{n}\right|^H\\ &\le \left|\frac{\lfloor nu\rfloor}{n}-\frac{\lfloor ns\rfloor}{n}\right|^{2H}\end{aligned}$$

If now $u-s\ge\frac1n$ we have

$$\mathbf{E}|Z_t^{(n)}-Z_s^{(n)}||Z_u^{(n)}-Z_t^{(n)}|\le|2(u-s)|^{2H}. \tag{2.9}$$

If on the other hand $u-s<\frac1n$ then either s and t or t and u lie in the same subinterval $[\frac mn,\frac{m+1}{n})$ for some m. Thus the left hand side of (2.9) is zero. Therefore (2.9) holds for all $s<t<u$. Recalling now that $H>\frac12$ and by theorem 15.6 of Billingsley [2] we have the tightness of $Z^{(n)}$. This completes the proof of the theorem. □

We have shown that the fractional Brownian motion is a limit of a random walk consisting of independent, altough not identically distributed, square integrable and centered random variables. However, note that the increments of the random walk $Z^{(n)}$ are not independent. Also, note that the approximating kernel $z^{(n)}$ can be changed to

$$\tilde z^{(n)}(t,s) := \sum_{i=1}^{\lfloor nt\rfloor} z(\tfrac{\lfloor nt\rfloor}{n},s_i^{(n)})\mathbf{1}_{(\frac{i-1}{n},\frac in]}(s),$$

where $s_i^{(n)}$'s are fixed real numbers belonging to the intervals $(\frac{i-1}{n},\frac in]$, respectively.

Recall that the bracket $[X]$ of a stochastic process X is its quadratic variation, viz.

$$[X]_t := \lim_{n\to\infty}\sup_{\pi_n}\sum_{t_i\in\pi_n}\left(X_{t_i}-X_{t_{i-1}}\right)^2.$$

Here π_n is a partition $\{0=t_0<t_1<\cdots<t_n=t\}$ such that its mesh tends to zero as n tends to infinity. Denote the jump process of a stochastic process X by ΔX, viz.

$$\Delta X_t := X_t-X_{t-} := X_t-\lim_{s\uparrow t}X_s.$$

Theorem 2.2. *The bracket* $[Z^{(n)}]$ *converges to zero in* $L^1(\mathbf{P}\times\mathrm{Leb})$, *where* Leb *is the Lebesgue measure on the interval* $[0,T]$.

Proof. Using the bound (2.8) from the proof of the theorem 2.1 we obtain

$$\mathbf{E}(\Delta Z_t^{(n)})^2 \le \mathbf{E}(Z_t^{(n)}-Z_{t-\frac1n}^{(n)})^2 \le n^{-2H}.$$

Hence

$$\mathbf{E}[Z^{(n)}]_t = \sum_{s\le t}\mathbf{E}(\Delta Z_t^{(n)})^2 \le nt\cdot n^{-2H} = tn^{1-2H}.$$

Since the bracket is a non-negative and increasing process we obtain

$$\int_0^T\mathbf{E}[Z^{(n)}]_t\,dt \le \int_0^T\mathbf{E}[Z^{(n)}]_T\,dt \le T^2n^{1-2H}.$$

Recall that $H>\frac12$ and the claim follows. □

The following corollaries are now at hand.

Corollary 2.3. *The process* $[Z^{(n)}]$ *converges to zero in probability.*

Corollary 2.4. *The jump process* $\Delta Z^{(n)}$ *converges to zero in probability.*

Consider the process $S^{(n)}$ defined by

$$S_t^{(n)} := \prod_{s\le t}(1+\Delta Z_s^{(n)}).$$

Our aim is to prove that the process $S^{(n)}$ converges weakly to the geometric fractional Brownian motion. Since the big jumps of the process $Z^{(n)}$ are somewhat of a nuisance we shall consider them separately. Namely, define the processes $Z^{(1,n)}$ and $Z^{(2,n)}$ as follows

$$Z_t^{(1,n)} := \sum_{s\le t} \Delta Z^{(n)} \mathbf{1}_{\left\{|\Delta Z_s^{(n)}|<\frac{1}{2}\right\}} \tag{2.10}$$

$$Z_t^{(2,n)} := \sum_{s\le t} \Delta Z^{(n)} \mathbf{1}_{\left\{|\Delta Z_s^{(n)}|\ge\frac{1}{2}\right\}}. \tag{2.11}$$

In view of corollary 2.4 and theorem of 4.1 Billingsley [2] the following lemma is obvious.

Lemma 2.5. *The process $Z^{(2,n)}$ converges to zero in probability and hence $Z^{(1,n)}$ converges weakly to Z.*

Theorem 2.6. *The process $S^{(n)}$ defined above converges weakly to the geometric fractional Brownian motion e^Z.*

Proof. Write

$$S^{(n)} = S^{(1,n)}S^{(2,n)},$$

where

$$S_t^{(i,n)} := \prod_{s\le t}(1+\Delta Z^{(i,n)})$$

for $i=1,2$ and the processes $Z^{(i,n)}$ are as in (2.10) – (2.11).

By theorem 4.1 and problem 1 p. 28 of Billingsley [2] it is enough to show that $S^{(1,n)}$ converges weakly to e^Z and $S^{(2,n)}$ converges to 1 in probability.

Let us first consider the process $S^{(2,n)}$. Let $\varepsilon>0$. Then

$$\begin{aligned}
\mathbf{P}(\sup_{t\le T}|S_t^{(2,n)}-1|\ge\varepsilon) &= \mathbf{P}(\sup_{t\le T}|\prod_{s\le t}(1+\Delta Z_s^{(2,n)})-1|\ge\varepsilon) \\
&\le \mathbf{P}(\sup_{t\le T}|\Delta Z_t^{(2,n)}|>0) \\
&= \mathbf{P}(\sup_{t\le T}|\Delta Z_t^{(n)}|>\tfrac{1}{2}).
\end{aligned}$$

Since the process $\Delta Z^{(n)}$ converges to zero in probability the convergence of $S^{(2,n)}$ to one in probability follows.

It remains to prove that $S^{(1,n)}$ converges weakly to e^Z. Let us first prove that $\log S^{(1,n)}$ converges weakly to Z. Since $|\Delta Z_t^{(1,n)}|<\frac{1}{2}$ for all $t\le T$ the logarithm of $S^{(1,n)}$ is well defined. According to Taylor's theorem

$$\log(1+x) = x - \frac{1}{2}x^2 + r(x)x^2,$$

where $r(x)$ tends to zero as x tends to zero. Hence

$$\begin{aligned}
\log S_t^{(1,n)} &= \log\prod_{s\le t}(1+\Delta Z_s^{(1,n)}) \\
&= \sum_{s\le t}\left(\Delta Z_s^{(1,n)} - \frac{1}{2}(\Delta Z_s^{(1,n)})^2 + r(\Delta Z_s^{(1,n)})(\Delta Z_s^{(1,n)})^2\right) \\
&= Z_t^{(1,n)} - \frac{1}{2}[Z^{(1,n)}]_t + \sum_{s\le t} r(\Delta Z_s^{(1,n)})(\Delta Z_s^{(1,n)})^2.
\end{aligned}$$

According to corollary 2.3 the bracket $[Z^{(n)}]$ converges to zero in probability. Obviously $[Z^{(1,n)}]\le[Z^{(n)}]$. Hence $[Z^{(1,n)}]$ also converges to zero in probability. Since $\Delta Z_t^{(1,n)}$ is bounded by $\frac{1}{2}$ the remainder r is uniformly bounded. Hence the third term converges also to zero in probability by using the corollary 2.3 again. Hence we obtain, by using the theorem 4.1 of Billingley [2] and lemma 2.5, that $\log S^{(1,n)}$ converges weakly to Z

Finally, since the exponential is a continuous functional (in the Skorokhod topology) the theorem follows. □

3. Fractional Brownian motion and binary market models

3.1. **Fractional Black–Scholes model.** Let us briefly define what we mean by fractional Black–Scholes model.

Consider two assets, or securities, that are traded continuously over the time interval $[0,T]$. Here 0 is the current date and the terminal date T is fixed.

Denote by B the riskless asset, or *bond.* The dynamics of the asset B are given as follows

$$dB_t = r_t dt B_t, \tag{3.1}$$

where r is a deterministic interest rate.

The risky asset, *stock,* is denoted by S and has the dynamics

$$dS_t = (a_t dt + \sigma dZ_t)S_t, \tag{3.2}$$

where $\sigma > 0$ is a constant and Z is a fractional Brownian motion with index $H > \frac{1}{2}$. The function a is the deterministic drift of the stock.

Assume further that r and a are continuously differentiable on the interval $[0,T]$ then the solutions of the (stochastic) differential equations (3.1) – (3.2) are given by

$$B_t = B_0 \exp(\int_0^t r_s\, ds) \quad \text{and} \quad S_t = S_0 \exp(\int_0^t a_s\, ds + \sigma Z_t),$$

respectively. For details we refer to Zähle [15].

3.2. **Binary market models in general.** Let us briefly define what we mean by binary market models. For a detailed treatment of this subject we refer to [6].

Consider a securities market in which the two assets are traded at successive time periods $0 = t_0 < t_1 \cdots < t_N = T$. The dynamics of the bond and stock are now given as

$$B_n = (1 + r_n)B_{n-1} \tag{3.3}$$

and

$$S_n = (a_n + (1 + X_n))\, S_{n-1}, \tag{3.4}$$

respectively. Here B_n and S_n are is the values of the bond and stock, respectively, over the time interval $[t_n, t_{n+1})$. Similarily r_n and a_n are the interest rate and the drift of the stock, respectively, in the corresponding interval. The sequence $X = (X_n)$ is a stochastic process such that at each time point n the random variable X_n has two possible values u_n and d_n where $d_n < u_n$. Note that the values of u_n and d_n may depend on the path of X upto time $n-1$. So the the stock price S_n occupies one of the 2^n states at time n. Note that all the states are not necessarily different. However, there are 2^n different possible paths for the stock price to evolve upto time n.

Let us recall some basic concepts. Let Ψ_n and Φ_n denote the number of shares of the bond and stock, respectively, owned by the investor at the trading time t_n. The couple $\pi_n = (\Psi_n, \Phi_n)$ is called the investor's *portfolio* at time t_n. The whole sequence $\pi = (\pi_n)$ is called the *trading strategy.* Observe that the number of shares Ψ_n and Φ_n may depend on the prices B_l and S_l, where $l < n$, but not on the prices yet to be announced. The investor's *wealth* $V_n(\pi)$ at time t_n is naturally

$$V_n(\pi) := \Psi_n B_n + \Phi_n S_n.$$

Here V_0 is the *initial endownment.* The sequence $V(\pi) = (V_n(\pi))$ is called the *value process.* A trading strategy π is said to be *self-financing,* if the corresponding portfolio satisfies for all $n = 1, \ldots, N$ the condition

$$B_{n-1}\Delta\Psi_n + S_{n-1}\Delta\Phi_n = 0.$$

The condition means that the construction is founded only on the the initial endownment. No infusion of withdrawal of funds is allowed. A self-financing trading strategy π is called *admissible* if the corresponding wealth $V_n(\pi)$ remains non-negative for all n. Suppose that there exists an admissible strategy π such that the corresponding value process $V(\pi)$ satisfies

$$V_0(\pi) = 0 \quad \text{and} \quad \mathbf{P}(V_N(\pi) > 0) > 0.$$

Such strategy is called an *arbitrage opportunity.* By proposition 6.1.2 of Dzhaparidze and van Zuijlen [6] a binary market excludes arbitrage opportunities if and only if the condition

$$d_n < r_n - a_n < u_n \tag{3.5}$$

holds for all $n = 1, \dots, N$. Note that here d_n and u_n are random variables not necessarily independent of the random variables d_k and u_k, where $k < n$.

3.3. **Fractional binary market model.** Let us now define the fractional binary market model, i.e. a binary model that approximates the fractional analogue of the Black–Scholes model described in section 3.1. In particular, we construct sequences $B^{(N)}$ and $S^{(N)}$ corresponding the bond and stock dynamics given in formulas (3.3) – (3.4) that converge weakly to the corresponding dynamics in the fractional Black–Scholes model given in formulas (3.1) – (3.2) as N tends to infinity.

We introduce some notations. Fix an equidistant partition $\{t_k = \frac{Tk}{N} : k = 0, 1, \dots, N\}$. Denote

$$k(n,i) := k^{(N)}(n,i) := \sqrt{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N}, s)\, ds,$$

where z is the kernel introduced in section 2. Define the stochastic process X in formula (3.4) by

$$X_n := X_n^{(N)} := \sigma \Delta Z^{(N)}_{Tn/N},$$

where $Z^{(N)}$ is the approximation of fractional Brownian motion defined in section 2. The constant $\sigma > 0$ is the volatility of the stock in the formula (3.2). Let us write the process X by using the kernel k.

$$X_n = \sigma \sum_{i=1}^{n} \left(k(n,i) - k(n-1,i)\right) \xi_i, \tag{3.6}$$

where the random variables $\xi_i = \xi_i^{(N)}$ are i.i.d. We also assume that the ξ_i's are binary random variables, i.e.

$$\mathbf{P}(\xi_i = 1) \;=\; \frac{1}{2} \;=\; \mathbf{P}(\xi_i = -1)$$

for all i.

Setting $f_{n-1}(x_1, \dots, x_{n-1}) = \sigma \sum_{i=1}^{n-1} \left(k(n,i) - k(n-1,i)\right) x_i$ and $g_n(x) = \sigma k(n,n) x$ we obtain

$$X_n = f_{n-1}(\xi_1, \dots, \xi_{n-1}) + g_n(\xi_n),$$

when $n \geq 1$ using the conversion $f_0 = 0$. Since ξ_i's are a binary random variables we obtain

$$\begin{aligned} u_n &= u_n(\xi_1, \dots, \xi_{n-1}) = f_{n-1}(\xi_1, \dots, \xi_{n-1}) + g_n(1) \\ d_n &= d_n(\xi_1, \dots, \xi_{n-1}) = f_{n-1}(\xi_1, \dots, \xi_{n-1}) + g_n(-1). \end{aligned}$$

Define the deterministic sequences $r^{(N)}$ and $a^{(N)}$ in the formulas (3.3) – (3.4) by

$$r_n^{(N)} := \frac{1}{N} r_{t_n} \quad \text{and} \quad a_n^{(N)} := \frac{1}{N} a_{t_n},$$

respectively, where the functions r and a are as in the fractional Black–Scholes model.

In view of theorem 2.6 it is now clear that this model approximates the fractional Black–Scholes model in the following sense.

Theorem 3.1. *The price processes $B^{(N)}$ and $S^{(N)}$ defined above converge weakly to the corresponding price processes B and S in the fractional Black–Scholes model.*

Let us now consider arbitrage in this model. It is well-known that the limiting model admits arbitrage opportunities, cf. Dasgupta [4], Rogers [12] and Shiryaev [13]. It turns out that even our discrete approximation is not free of arbitrage.

Theorem 3.2. *The fractional binary market described above admits arbitrage opportunities.*

Proof. It is enough to show that condition (3.5) does not hold, i.e. the condition

$$f_{n-1}(\xi_1,\dots,\xi_{n-1})+g_n(-1)<r_n-a_n<f_{n-1}(\xi_1,\dots,\xi_{n-1})+g_n(1)$$

fails for some sequence $(\xi_1,\dots,\xi_n)$. In particular, by symmetry it is enough to show that

$$f_{n-1}(1,\dots,1)-g_n(1)\ \geq\ 0$$

for some $n\geq 2$ or

$$\sum_{i=1}^{n-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s)-z(\tfrac{n-1}{N},s)\,ds-\int_{\frac{n-1}{N}}^{\frac{n}{N}} z(\tfrac{n}{N},s)\,ds\ \geq\ 0. \tag{3.7}$$

Let us first give a lower bound for the integral term under the sum in inequality (3.7). Denote $C:=c_H(H-\frac{1}{2})$ and $\alpha:=H-\frac{1}{2}$.

$$\begin{aligned}
&\int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s)-z(\tfrac{n-1}{N},s)\,ds\\
&\qquad= C\int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha}\int_{\frac{n-1}{N}}^{\frac{n}{N}} u^{\alpha}(u-s)^{\alpha-1}\,du\,ds\\
&\qquad\geq \frac{C}{\alpha}\left(\frac{n-1}{N}\right)^{\alpha}\int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha}\left\{\left(\frac{n}{N}-s\right)^{\alpha}-\left(\frac{n-1}{N}-s\right)^{\alpha}\right\}ds\\
&\qquad\geq \frac{C}{\alpha}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n-i}{N}\right)^{\alpha}-\left(\frac{n-1-i}{N}\right)^{\alpha}\right\}\int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-\alpha}\,ds\\
&\qquad\geq \frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n+1-i}{i}\right)^{\alpha}-\left(\frac{n-i}{i}\right)^{\alpha}\right\}.
\end{aligned}\tag{3.8}$$

By using the lower bound (3.8) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}} z(\tfrac{n}{N},s)-z(\tfrac{n-1}{N},s)\,ds\\
&\qquad\geq \sum_{i=1}^{n-1}\frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha}\left\{\left(\frac{n+1-i}{i}\right)^{\alpha}-\left(\frac{n-i}{i}\right)^{\alpha}\right\}\\
&\qquad\geq \frac{C}{\alpha N}\left(\frac{n-1}{N}\right)^{\alpha}(n-1)^{-\alpha}\sum_{i=1}^{n-1}\{(n+1-i)^{\alpha}-(n-i)^{\alpha}\}\\
&\qquad= \frac{C}{\alpha}N^{-\alpha-1}(n^{\alpha}-1).
\end{aligned}\tag{3.9}$$

Let us now give an upper bound for the latter term in the right hand side of the inequality (3.7).

$$\begin{aligned}
\int_{\frac{n-1}{N}}^{\frac{n}{N}} z(\tfrac{n}{N},s)\,ds &= C\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\int_{s}^{\frac{n}{N}} u^{\alpha}(u-s)^{\alpha-1}\,du\,ds\\
&\leq C\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\left(\frac{n}{N}\right)^{\alpha}\int_{s}^{\frac{n}{N}}(u-s)^{\alpha-1}\,du\,ds\\
&= \frac{C}{\alpha}\int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-\alpha}\left(\frac{n}{N}\right)^{\alpha}\left(\frac{n}{N}-s\right)^{\alpha}ds\\
&\leq \frac{C}{\alpha(\alpha+1)}\left(\frac{n}{n-1}\right)^{\alpha}N^{-\alpha-1}.
\end{aligned}\tag{3.10}$$

Since (3.9) tends to inifinity as n tends to infinity and (3.10) tends to

$$\frac{C}{\alpha(\alpha+1)}N^{-\alpha-1}$$

as n tends to infinity we obtain (3.7) for $n\geq n_H$ and the theorem follows. ∎

Note that n_H grows to infinity as H tends to $\frac{1}{2}$. In particular, when $H = 0.6$ we have $n_H \approx 350$. This estimate was obtained by using the bounds (3.8) and (3.10).

Let us construct one arbitrage opportunity explicitly. Suppose first that the discounted drift of the stock $\tilde{a} = a - r$ satisfies $\tilde{a}_{n_0} < 0$ for some $n_0 \geq n_H$. In this case construct the strategy as follows: if the stock price only take jumps down upto time $n_0 - 1$ short sell $M > 0$ stocks and put the money on the bond, i.e. $\pi_{n_0} = (MS_{n_0}, M)$. Otherwise do nothing. Since $u_{n_0}(-1, \ldots, -1)$ is negative by the proof of the theorem 3.2 we have

$$MS_{n_0+1} < MS_{n_0}(1 + \tilde{a}_n) < MS_{n_0}.$$

Hence $V_{n_0+1}(\pi) > 0$. If on the other hand the discounted drift $\tilde{a}$ remains non-negative construct the strategy as follows: if the stock price has taken only upward jumps upto time $n \geq n_H - 1$ construct the portfolio $\pi_n = (-MS_n, M)$. Otherwise do nothing. Since $d_n(1, \ldots, 1)$ is positive by the proof of the theorem 3.2 we obtain

$$MS_{n+1} > MS_n.$$

Hence $V_{n+1}(\pi) > 0$. Clearly the strategy we have constructed is admissible, hence we have constructed an arbitrage opportunity.

Finally, note that upto time n_H there are at least two arbitrage paths, viz. $(-1, \ldots, -1)$ and $(1, \ldots, 1)$. Since these paths do not depend on N, provided $N \geq n_H$, we obtain that

$$\lim_{N\to\infty} \frac{\#\text{ abritrage paths}}{2^N} \;\geq\; \frac{2}{2^{n_H-1}} \;=\; 2^{2-n_H} \;>\; 0.$$

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