Lipschitz mappings with no bilipschitz tangents in spaces with decent calculus

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Abstract

We show that even in an Ahlfors–regular space admitting a Poincaré inequality a David–Semmes -regular mapping need not have a bilipschitz restriction to a set of positive measure, nor a bilipschitz weak tangent.

David–Semmes regular mappings are natural finite–to–one analogs of bilipschitz mappings. As we shall see, they are anyhow very distinct even in good spaces.

Definition 1. A mapping $f : X \to Y$ between metric spaces is called *David–Semmes regular with data* (L, M, N) if f is L–Lipschitz and if for each closed ball B in Y the preimage $f^{-1}(B)$ can be covered by M closed balls in X each of diameter at most N times the diameter of B.

In [S] Stephen Semmes conjectured that in an Ahlfors–regular space with Poincaré inequality David–Semmes -regular mappings are always bilipschitz while restricted to a suitable set of positive measure. This conjecture had a weaker version which claimed only the existence of weak bilipschitz tangents. See [DS, page 71] for the definition.

Let us recall the construction from [L] with minor alterations. The Hausdorff dimension Q is fixed here, so the explanation is somewhat simpler than in [L].

Denote by K the standard "middle–third" Cantor set. Set $K_0 = K/3$ and $K_1 = K/3+2/3$. Denote by K_{00} and K_{01} the similar parts of K_0, i.e. $3^{-2}K$ and $(K/3+2/3)/3$ respectively. Continue this naming inductively. For K_a, $|a|$ denotes the length of the binary string a and $a1$ denotes the concatenation of the strings a and "1".

We construct our space O from $I \times K$ as follows. Set $w(m_1,\dots,m_k) = \sum_{i=1}^{k} m_i 3^{-i}$, where $0 \le m_i < 3$. Call sets $w(m_1,\dots,m_k) \times K$ in $I \times K$ *identification levels* of order k provided that $m_k > 0$. The requirement $m_k > 0$ is needed to exclude the overlapping of several levels. In a identification level of order k identify $w(m_1,\dots,m_k) \times K_{a0}$ with $w(m_1,\dots,m_k) \times K_{a1}$ for each binary string a of length $k-1$. A point (x_1,x_2) in $w(m_1,\dots,m_k) \times K_{a0}$ is thus identified with $(x_1, x_2 + 3^{1-k}2/3)$. In addition to these identifications we did in [L] we further identify all four sets $\{0\} \times K_0$, $\{0\} \times K_1$, $\{1\} \times K_0$ and $\{1\} \times K_1$ in a similar manner. Call the resulting space *wheel space* and denote it by O. Let s be the natural projection from $I \times K$ to O.

In the space O we set

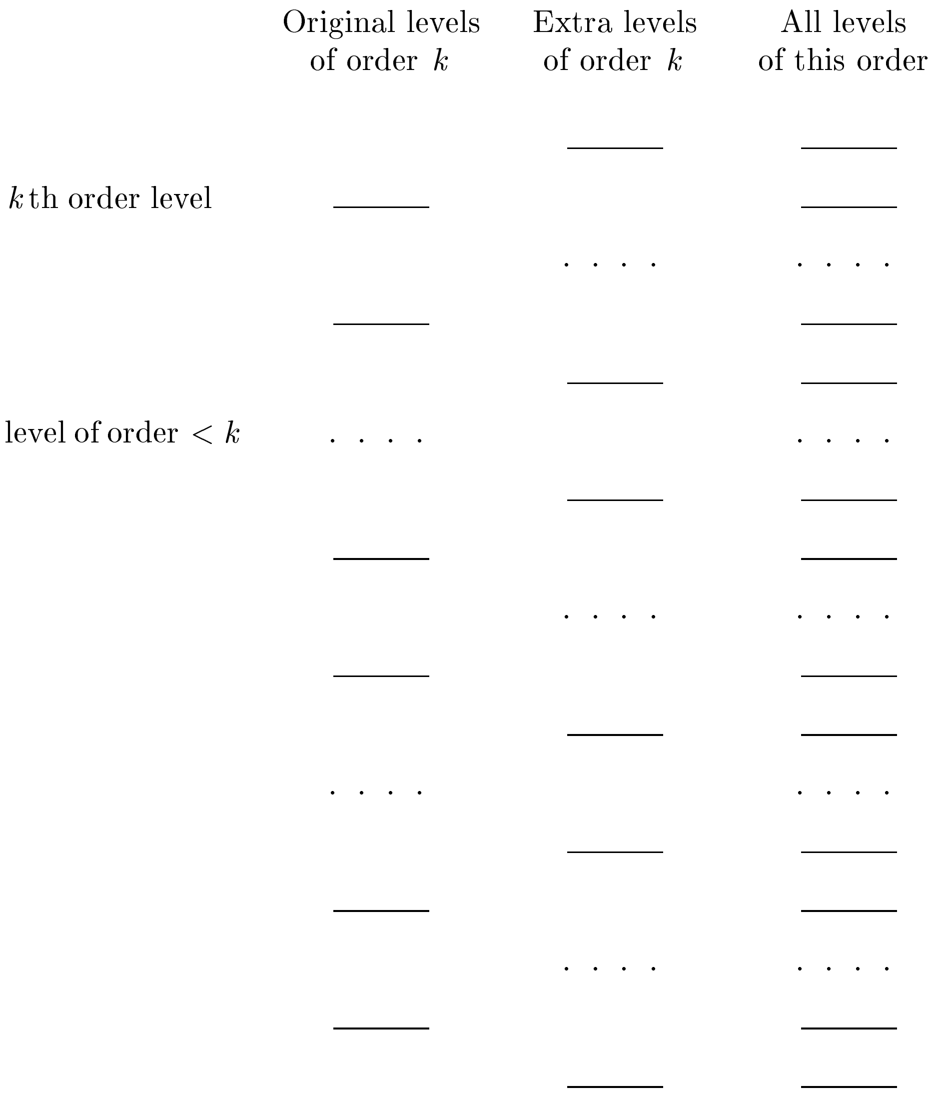
$$|x-y| = \inf\{H^1(\gamma) \mid \gamma \subset I \times K \ , \ s(\gamma) \text{ is a path from } x \text{ to } y\}.$$

Note that only the identifications from bottom to top affect to vertical distance. It makes therefore sense to speak in the modular fashion about the height $h(x)$ of a point $x \in O$, i.e. the I–component of its preimage in $I \times K$. Similarly we understand the concepts like upwards and downwards.

It was proved in [L], that s is David–Semmes -regular. Moreover, O is Ahlfors regular and admits a weak (1,1)–Poincaré inequality. We recall the argument for the second condition of David–Semmes -regularity here.

Consider $s^{-1}(B(x,r))$. Let m be such that $3^{-m} > 2r \ge 3^{-m-1}$. The minimal distance between identification levels of orders from 1 to $m+1$ is 3^{-m}, so $B(x,r)$ can reach only one identification level of those orders. On the other hand, identification levels of greater orders can only decrease distances of amount $\sum_{k=m+1}^{\infty} 3^{-k}$, because passing twice via identifications of the same order does not help at all. The preimage $s^{-1}(B(x,r))$ is therefore contained in at most two balls of radius $r + 3^{-m}/(2) \le 3r$.

Now, make a space O' by adding to O similar set of identifications shifted upwards a height of 1/2, but no new identifications from bottom to top. We understand this upwards shifting too in a modular fashion. In the resulting space we use the similar geodesic metric as in O. Denote the natural projection from O to O' by s'. This means that we add a new first order identification level in halfway between every first–order identification level. Something similar holds also for the greater order levels. Actually O' has the same identification pattern as O repeated twice in double density with respect to height. Namely the shift 1/2 is $\sum_{i=1}^{k} 3^{-i} + 3^{-k}/2$. Because the $k+1$th order identification pattern repeates itself with period 3^{-k}, only the last term matters for those identifications. This remaining shift has also the same proportion, 1/2, of the period, for each k. So the situation becomes clear, once and for all, in Picture 2.



Picture 2.

Therefore we can map O' onto O simply by wrapping O' twice around O with a piecewise bilipschitz map. The resulting onto self–mapping $f : O \to O$ is evidently Lipschitz, because s' is Lipschitz. Putting more identifications to the space with the geodesic metric does not increase distances. For the other side of David–Semmes -regularity we can use the same proof as for the projection s. We need there the fact that the maximal distance of points in two sides (of an image of a part of Cantor set K_a) which can be connected by kth order identification level is not more than $3^{-k}2$ in O.

Theorem 1. There exists an Ahlfors Q–regular space O admitting a weak (1,1)-Poincaré inequality, and a David–Semmes -regular onto self–mapping $f : O \to O$ such that there exists no set A with $H^Q(A) > 0$ such that $f|_A$ is bilipschitz. Moreover, f has no bilipschitz weak tangents.

Proof. Suppose, that we have $A \subset O$ such that $f|_A$ is L-bilipschitz. Each new identification of order k, connects points, that have distance 3^{-k} from each other. See Picture 2 again. Therefore, if $p \in O'$ is a point at such level with preimages p_1 and p_2 in O we see that one of the balls $B(p_1, 3^{-k}/(4L^2)\ )$ and $B(p_2, 3^{-k}/(4L^2)\ )$ does not meet points of A in its interior.

Let $B(x,r)$ be any ball in O. Take the smallest k such that $3^{-k}2$, the maximal distance of the extra kth order levels that s' adds to O, is at most $r/2$. Then $B(x,r)$ contains some of those extra levels, and even continues to the other side of it to distance $3^{-k}2$. Therefore it contains some ball $B(x', 3^{-k}/(4L^2)\ )$ whose interior is disjoint with A. This ball has radius uniformly comparable to $B(x,r)$. Hence A has no density points and $H^Q(A) = 0$. Note, that we can use the Lebesgue density theorem, because the space is Ahlfors–regular. See [DS, page 21], for example.

¿From [L] one knows that all the weak tangents of O are similar kind of spaces, and in addition, have the identification level pattern controlled by a sequence $\bar{j}$ that is a limit of shiftings of the indexes of the original j. See [L] for the role of j in the general construction. In our case $j_k = 3$ for all k, so the claim is even easier to see. Moreover, we know that identification levels converge to identification levels with the Gromov–Hausdorff convergence of the spaces. Therefore also these extra levels that s' adds to the space do converge. So, in the corresponding limit mapping, one has these extra levels in a similar pattern with respect to the original levels as in the original mapping. Then the same argument shows that the limit mapping cannot be bilipschitz when restricted to a set of positive measure.

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