Topologically, quasiconformally or Lipschitz locally flat approximation of embeddings

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Abstract

Let CAT denote either the category LIP of locally bi-Lipschitz embeddings or the category LQC of locally quasiconformal embeddings. The following theorem consists of representative special cases of our main results: For range manifolds not of dimension 4, in codimensions different from 2 a closed topological embedding between CAT manifolds without boundary can be approximated by locally CAT flat CAT embeddings, these approximations being equivalent up to a small ambient CAT isotopy. The auxiliary analogous TOP results extend known results and are interesting in themselves.

Key words: Lipschitz; Quasisymmetric; Quasiconformal; Manifold; Locally flat; Embedding; Approximation; Isotopy

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1 Introduction

This paper deals with the category LIP of locally bi-Lipschitz embeddings and Lipschitz manifolds and the category LQS of locally quasisymmetric embeddings and quasiconformal manifolds. The category LQS is more general than the category LQC of locally quasiconformal embeddings, but here they are equivalent, with some obvious slight exceptions. For the usual definition of these concepts through atlases it suffices to recall that an embedding $f\colon X \to Y$ of metric spaces is said to be LIP or LQS, respectively, if each point $x_0 \in X$ has a neighborhood U for which there is a constant $L \geq 1$

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such that $f|U$ is L-bi-Lipschitz, that is, satisfies $d(x,y)/L \leq d(f(x),f(y)) \leq Ld(x,y)$ for all $x,y \in U$, or, respectively, for which there is a homeomorphism $\eta\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+$ such that $f|U$ is η-quasisymmetric, that is, satisfies $d(f(x),f(y))/d(f(x),f(z)) \leq \eta(d(x,y)/d(x,z))$ for all $x,y,z \in U$, $x \neq z$. We also recall that a homeomorphism between open subsets of $\mathbb{R}^n_+$ is LQS if and only if locally it is quasiconformal (read quasisymmetric if $q=1$) in the usual sense.

For CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ we prove fundamental results on the locally CAT flat CAT taming of locally flat topological embeddings and on the existence and equivalence of locally CAT flat CAT approximations of topological embeddings. The taming theorem is proved with the aid of results from [25] on CAT approximation of homeomorphisms between locally CAT flat CAT manifold pairs. The approximation theorem is deduced from its analogue in the category TOP of topological embeddings and topological manifolds and the taming theorem. The equivalence theorem is likewise deduced from its TOP analogue with the aid of [25]. We present these TOP results in a uniform manner and prove them by using various known results; they are also interesting in themselves as in some instances they slightly extend or complete these known results or give them new formulations. The following theorem consists of representative special cases of our main CAT results.

Theorem 1.1. *Let* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let* $0 \leq q \leq n$ *with* $n \neq 4$, *let* M *be a* CAT n*-manifold and* Q *a* CAT q*-manifold, both without boundary, and let* $f\colon Q \to M$ *be a topological embedding.*

(1) *Suppose that* f *is locally flat and that* $\varepsilon \in C_+(M)$. *Then there is an ambient* ε*-isotopy* $H = (h_t)_{t\in I}$ *of* M *such that* $h_1 f$ *is a locally* CAT *flat* CAT *embedding.*

(2) *Suppose that* $n \neq q+2$ *if* $q \geq 2$ *and that* $\varepsilon \in C_+(Q)$. *Then there is a locally* CAT *flat* CAT *embedding* $g\colon Q \to M$ *within* ε *of* f.

(3) *Suppose that* $n \neq q+2$ *if* $q \geq 1$, *that* f *is closed, and that* $\varepsilon \in C_+(M)$. *Then there is* $\delta \in C_+(Q)$ *with the following property*: *If* $f_i\colon Q \to M$ *is a locally* CAT *flat* CAT *embedding within* δ *of* f *for* $i = 0,1$, *then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t\in I}$ *of* M *such that* $h_1 f_0 = f_1$.

With $n \leq 3$ or $n \geq q+3$, part (2) improves on the absolute case of the result [26, 4.4] on relative approximation by CAT embeddings which are not required to be locally CAT flat. With a weaker definition of CAT isotopies, the case $n = q+1$ of the theorem is proved in [22]. Here it is given a new proof.

It follows from [13, Theorem 2] that part (2) of Theorem 1.1 holds neither for LQS nor for LIP if $n = q = 4$. Part (3) cannot hold for $n = q+2 \geq 3$, as for each $n \geq 3$ there are topologically knotted locally PL flat PL embeddings $f\colon S^{n-2} \to \mathbb{R}^n$ arbitrarily close to the identity (here $S^k = \partial I^{k+1}$). Similarly, for part (2) with $n = q+2$, note that by [28, 5.3] for each even $n \geq 8$ there is a topological embedding $f\colon S^1 \times S^{n-3} \to \mathbb{R}^n$ that can be approximated neither by PL embeddings nor by locally flat embeddings. Furthermore, by [38, 2.4] for each $n \geq 6$ there is a compact connected PL $(n-2)$-submanifold of $\mathbb{R}^n$ without boundary whose inclusion in $\mathbb{R}^n$ cannot be approximated by locally flat embeddings.

Rather than in embeddings between CAT manifolds without boundary, we are more generally interested in embeddings $f\colon Q \to M$ between CAT manifolds with boundary which are allowable in the sense that $f^{-1}\partial M$ is a fixed locally CAT flat CAT $(q-1)$-submanifold Q_0 of ∂Q (if $q \geq 1$).

As mentioned above, we establish our main CAT results, which generalize 1.1, with the aid of [25]. Here boundaries cause no problems. The main result of [25], a generalization of results in [41] and [42], deals with CAT approximation of homeomorphisms between locally CAT flat CAT manifold pairs of arbitrary codimension. These CAT approximations were constructed by piecing together local CAT approximations, provided by various known PL and DIFF results; the assumption $n \neq 4$ stems from these results. The piecing-together was performed with the help of a pairwise CAT deformation result of [23], which generalizes results in [40] and [41] and contains no restrictions on n and q. In fact, essentially the same result of [23] shows that (3) holds for $n = q = 4$, too.

Next we outline our CAT proofs in the situation of 1.1. In (1) and (2) we may assume f to be closed. Then in (1) there is a CAT structure $\mathcal{A}$ on M such that f is a locally CAT flat CAT embedding into $M_{\mathcal{A}}$. Now there is a small ambient isotopy $H = (h_t)_{t\in I}$ of M such that $h_1\colon M_{\mathcal{A}} \to M$ is a CAT homeomorphism; and then $h_1 f$ is a locally CAT flat CAT embedding. In (2) we may assume, by the analogous TOP result, that f is locally flat; if now H is as given by (1), then we can choose $g = h_1 f$. In (3) we may assume the existence of a homeomorphism $g\colon M \to M$ close to the identity with $gf_0 = f_1$. Then the CAT homeomorphism $g|\colon f_0Q \to f_1Q$ can be extended to a CAT homeomorphism $h\colon M \to M$ close to g. Now there is a small ambient CAT isotopy $H = (h_t)_{t\in I}$ of M with $h_1 = h$; and for this, $h_1 f_0 = f_1$.

The principal accomplishment of this paper is to establish the auxiliary TOP results on the existence and equivalence of locally flat approximations of embeddings starting from known results. Unfortunately, if $n = q$ or $n = q + 1$ with $n \geq 4$, then due to open problems (or these are open as far as the author knows) we have to make additional assumptions on Q_0 in these TOP results and consequently in the ensuing LIP and LQS results, too. For this reason, after the preliminary Section 2 and Section 3 on taming, we begin both Section 4 on equivalence and Section 5 on approximation by formulating simultaneously for all CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$ certain conjectures without these assumptions. Next we reduce these conjectures to the TOP case. Then we try to establish as much as possible of these TOP conjectures, allowing the case $n = 4 \neq q + 2$ also. Finally we state the main theorems for all three categories.

We remark that we close the statement of many lemmata with mentions of certain modifications without devoting any word to them in the proof; the modifications that would be needed in the proof are obvious.

Some of the results of [23,25] and of the present paper, which completes this series of investigations, were already announced in my talk at the Fourth International Conference on Topology and its Applications in Dubrovnik in October 1985.

2 Preliminaries

Notation and terminology 2.1. We use notation and terminology of [25]. We recall some of this and introduce some new with the consequence that for the TOP results no reference to [25] is needed.

For a metric space X, let $C_+(X) = \{\, \varepsilon \mid \varepsilon\colon X \to (0, \infty) \text{ continuous} \,\}$. For $A \subset X$ and $\varepsilon \in C_+(X)$, let $N(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon(x))$ be the open ε-neighborhood of A in X; then $N(A, \varepsilon) = N(\overline{A}, \varepsilon)$, and these sets form a neighborhood base of $\overline{A}$.

As in [25] we say that an (open) isotopy $F = (f_t)_{t \in I}$ of a metric space X to a metric space Y is an (open) LIP isotopy if F is LIP as an embedding $X \times I \to Y \times I$ or an (open) LQS isotopy if for all $x \in X$ and $t \in I$ there are a homeomorphism $\eta\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+$ and neighborhoods X' of x and I' of t such that the embedding $f_{t'}|X'$ is η-quasisymmetric for each $t' \in I'$. Before these conditions can be applied via charts in a manifold setting it might be necessary to first reduce the parameter interval I. For LQS isotopies this definition is stronger than that in [23], where it is only required that each f_t is LQS, but [25, Appendix A] improves [23] accordingly. The isotopy results of [22] use this weaker LQS condition and its LIP analogue. If $A \subset X$ and $f_t|A = f_0|A$ for each $t \in I$, then F is here said to be fixed on A, while $\operatorname{rel} A$ in [25]. If F is a homeomorphic isotopy of X onto X and $f_0 = \mathrm{id}$, then F is called an ambient isotopy of X. For ambient isotopies $F = (f_t)_{t \in I}$ and $G = (g_t)_{t \in I}$ of X we let $F \diamond G$ denote the ambient isotopy $H = (h_t)_{t \in I}$ of X defined by $h_t = g_{2t}$ if $t \leq \frac{1}{2}$ and $h_t = f_{2t-1} g_1$ if $t \geq \frac{1}{2}$. For CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ the operation $\diamond$ preserves ambient CAT isotopies of CAT manifolds.

In this paper n and q denote nonnegative integers with $q \leq n$. Let $\mathbb{R}^n_+ = \{\, x \in \mathbb{R}^n \mid x_n \geq 0 \,\}$ and $\mathbb{R}^n_{++} = \{\, x \in \mathbb{R}^n_+ \mid x_{n-1} \geq 0 \,\}$, identify $\mathbb{R}^q$ with $\{\, x \in \mathbb{R}^n \mid$

$x_i = 0$ if $i \geq q+1\,\}$, and define $\mathbb{R}^{n,q}_+ = \{\, x \in \mathbb{R}^n_+ \mid x_i = 0$ if $i \leq n-q\,\}$ and $\mathbb{R}^{n,q}_{++} = \{\, x \in \mathbb{R}^{n,q}_+ \mid x_{n-1} \geq 0\,\}$; then $\mathbb{R}^{n,1}_{++} = \mathbb{R}^{n,1}_+$. (Note that the definition $\mathbb{R}^{n,1}_{++} = \{\, x \in \mathbb{R}^{n,1}_+ \mid x_n \in I\,\}$ used in [22] is not equivalent to the present one.) We let an (n,q)-*model*, a model for a locally flat manifold pair at a point of the submanifold, mean one of the pairs $(\mathbb{R}^n, \mathbb{R}^q)$, $(\mathbb{R}^n, \mathbb{R}^q_+)$, $(\mathbb{R}^n_+, \mathbb{R}^{n,q}_+)$, and $(\mathbb{R}^n_+, \mathbb{R}^{n,q}_{++})$ if $q \geq 2$, one of the pairs $(\mathbb{R}^n, \mathbb{R}^1)$, $(\mathbb{R}^n, \mathbb{R}^1_+)$, and $(\mathbb{R}^n_+, \mathbb{R}^{n,1}_+)$ if $q = 1$, or one of the pairs $(\mathbb{R}^n, \mathbb{R}^0)$ and $(\mathbb{R}^n_+, \mathbb{R}^0)$ if $q = 0$. Let $I^n = [-1,1]^n$, $I^n_+ = I^n \cap \mathbb{R}^n_+$, and $I^n_{++} = I^n \cap \mathbb{R}^n_{++}$; if V is any of these intervals, denote $V(r) = rV$ for $r > 0$.

We recall from [25] and [22] the definitions of locally CAT flat CAT submanifolds and embeddings for CAT $\in \{\mathrm{TOP}, \mathrm{PL}, \mathrm{LIP}, \mathrm{LQS}\}$ (the prefix TOP is usually omitted). Let M be a CAT n-manifold (in the atlas sense, assumed to be a separable metrizable space endowed with a topologically compatible metric). A subspace Q of M is said to be a locally CAT flat CAT q-submanifold of M if for each point $x \in Q$ there are an open neighborhood U of x in M, an (n,q)-model (N,T), and a CAT homeomorphism $h\colon U \to N$ such that $h[U \cap Q] = T$ and $h(x) = 0$. In this case Q is a CAT q-manifold whose inclusion map to M is a CAT embedding (that is, Q is what is called a CAT q-submanifold of M). Moreover, if $q \geq 1$, then $Q \cap \partial M$ is a locally CAT flat CAT $(q-1)$-submanifold of both ∂Q and ∂M, and $\partial Q \setminus \mathrm{Int}(Q \cap \partial M)$ is a closed locally CAT flat CAT $(q-1)$-submanifold of both ∂Q and M. For $q = 0$ the definition means that Q is just a discrete subset of M.

Let M be a CAT n-manifold, Q a CAT q-manifold, and $f\colon Q \to M$ an embedding. If for each point $x \in Q$ there are an open neighborhood V of x, an (n,q)-model (N,T), a CAT homeomorphism $h\colon V \to T$ with $h(x) = 0$, and an open CAT embedding $g\colon N \to M$ such that $gh = f|V$ (in which case we may assume that $gN \cap fQ = fV$), then f is called a locally CAT flat CAT embedding. It follows trivially that f is a locally CAT flat CAT embedding if and only if fQ is a locally CAT flat CAT submanifold of M and f induces a CAT homeomorphism of Q onto fQ or, equivalently, is a CAT embedding into M. Note that (N,T) depends uniquely on x in these two definitions. In the latter definition the existence of g, at least near 0, is independent of the choice of (V,h), as can be seen from the following fact. Let (N,T) be an (n,q)-model, $V \subset T$ open, and $f\colon V \to T$ an open CAT embedding with $f^{-1}\partial N = V \cap \partial N$; then there are an open set $U \subset N$ with $U \cap T = V$ and an open CAT embedding $g\colon U \to N$ extending f. To prove this fact it is enough to consider the following four cases for $q \geq 1$.

Case 1: $(N,T) = (\mathbb{R}^n, \mathbb{R}^q)$ with $n = q+1$. First construct, by [42, 4.10] if CAT $=$ LQS, an open set $U_+ \subset \mathbb{R}^n_+$ with $U_+ \cap T = V$ and an open CAT embedding $g_+\colon U_+ \to \mathbb{R}^n_+$ extending f. Then use reflection in T.

Case 2: $(N,T) = (\mathbb{R}^n, \mathbb{R}^q_+)$ with $n = q$. Use reflection in ∂T.

Case 3: $(N,T) = (\mathbb{R}^n_+, \mathbb{R}^{n;q}_+)$ with $n = q+1$. Let $N_+ = \{\, x \in \mathbb{R}^n_+ \mid x_1 \geq 0 \,\}$. First construct by Cases 2 and 1 an open set $U_+ \subset N_+$ with $U_+ \cap T = V$ and an open CAT embedding $g_+ \colon U_+ \to N_+$ extending f with $g_+^{-1}\partial N = U_+ \cap \partial N$. Then use reflection in T.

Case 4: $(N,T) = (\mathbb{R}^n_+, \mathbb{R}^{n;q}_{++})$ with $n = q \geq 2$. Use reflection in $\overline{\partial T \smallsetminus \partial N}$.

Facts of local flatness 2.2. (a) If CAT $\in \{\mathrm{TOP}, \mathrm{PL}, \mathrm{LIP}, \mathrm{LQS}\}$, M is a CAT n-manifold, Q a CAT q-manifold, $f \colon Q \to M$ a locally CAT flat CAT embedding, $Q_0 = f^{-1}\partial M$, P a locally CAT flat CAT q-submanifold of Q, and $S = P \cap \overline{Q \smallsetminus P} \cap \partial Q_0$, then the CAT embedding $f|P$ is locally CAT flat provided that either $q \leq 1$ or $q = 2$ and $S = \emptyset$ or $q \geq 3$ and each point $x \in S$ has an open neighborhood U in ∂Q such that $(U, U \cap Q_0, U \cap P, x)$ is CAT homeomorphic to $(\mathbb{R}^{q-1}, \mathbb{R}^{q-1}_+, \mathbb{R}^{q-2}_+ \times \mathbb{R}^1, 0)$.

To prove this fact, we only consider the most demanding case where $q \geq 3$ and $f|P$ must be shown to be locally CAT flat at a point $x \in S$. We may assume that $M = \mathbb{R}^n_+$, that there is an open neighborhood V of 0 in $\mathbb{R}^n_+$ with $Q = V \cap \mathbb{R}^{n;q}_{++}$, $Q_0 = Q \cap \mathbb{R}^{n-1}$, and $x = 0$, and that there is an open CAT embedding $g \colon V \to \mathbb{R}^n_+$ extending f. Let $\mathbb{R}^{n;q}_{+++} = \{\, x \in \mathbb{R}^{n;q}_{++} \mid x_{n-2} \geq 0 \,\}$. Then by making V smaller we may assume, by [25, 2.5] and [24, 5.2] if CAT $=$ LQS, that there is an open CAT embedding $\varphi \colon Q \to \mathbb{R}^{n;q}_{++}$ for which $Q_0 = \varphi^{-1}\mathbb{R}^{n-1}$, $P = \varphi^{-1}\mathbb{R}^{n;q}_{+++}$, and $\varphi(0) = 0$. By making V smaller it is easy to extend φ to an open CAT embedding $\psi \colon V \to \mathbb{R}^n_+$ with $\psi^{-1}\mathbb{R}^{n;q}_{++} = Q$. Thus, with the aid of ψ we may assume that, in addition, $P = V \cap \mathbb{R}^{n;q}_{+++}$. As $(\mathbb{R}^n_+, \mathbb{R}^{n;q}_{+++}, 0)$ is PL homeomorphic to $(\mathbb{R}^n_+, \mathbb{R}^{n;q}_{++}, 0)$, we may finally assume, forgetting now the previously assumed expression to Q, that $P = V \cap \mathbb{R}^{n;q}_{++}$; then from $f|P = g|P$ it follows that $f|P$ is locally CAT flat at x.

In our applications we have a PL q-manifold Q with $q \geq 1$, a closed PL $(q-1)$-submanifold Q_0 of ∂Q, in which case Q_0 is locally PL flat in ∂Q, and a closed PL q-submanifold P of Q with $P \cap \partial Q \subset \partial P$ a PL $(q-1)$-manifold, in which case P is locally PL flat in Q, such that if $x \in S$, where S is defined as above, then $q \geq 3$ and there is an open neighborhood V of x in $P \cap \partial Q$ such that $(V, V \cap Q_0, V \cap \partial Q_0, x)$ is PL homeomorphic to $(\mathbb{R}^{q-2}_+ \times \mathbb{R}^1, \mathbb{R}^{q-1}_{++}, \mathbb{R}^{q-2}_+, 0)$, in which case there is an open neighborhood U of x in ∂Q such that $(U, U \cap Q_0, U \cap P, x)$ is PL homeomorphic to $(\mathbb{R}^{q-1}, \mathbb{R}^{q-1}_+, \mathbb{R}^{q-2}_+ \times \mathbb{R}^1, 0)$. Here the three conclusions are based on consideration of links; see [36, 3.14] and [19, 2.14 (p. 60)]. It thus follows that if $f \colon Q \to M$ is a locally CAT flat CAT embedding into a CAT manifold M with $f^{-1}\partial M = Q_0$, then $f|P$ is locally CAT flat. Note that the assumptions about the closed PL q-submanifold P of Q are equivalent to the assumption that $P \cap Q_0$ and $P \cap \overline{\partial Q \smallsetminus Q_0}$ are PL $(q-1)$-manifolds and $P \cap \partial Q_0$ a PL $(q-2)$-manifold.

(b) If $f \colon Q \to M$ is a locally CAT flat CAT embedding between positive-dimensional CAT manifolds with CAT as in (a), then trivially the CAT embedding $f|\overline{\partial Q \smallsetminus f^{-1}\partial M}$ is locally CAT flat.

(c) If $q \geq 1$ and $n \geq q+3$, if M is a PL n-manifold and Q a PL q-manifold, and if $f\colon Q \to M$ is a PL embedding for which $f^{-1}\partial M$ is a PL $(q-1)$-submanifold of ∂Q, then f is locally PL flat. See [36, 7.2] for the case $f^{-1}\partial M = \partial Q$. Even the case $\partial f^{-1}\partial M = \emptyset$ can be deduced from [36, 7.1] because every locally PL flat PL ball in a PL sphere is PL flat. The general case follows from [35, Corollary 1 of Theorem 3].

3 Taming

This section is devoted to the proof of the following taming result, which for $n = q+1$ and $Q_0 = \emptyset$ is [22, 5.10].

Theorem 3.1. *Let* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let* $0 \leq q \leq n$ *with* $n \neq 4$, *let* M *be a* CAT n*-manifold and* Q *a* CAT q*-manifold, let* Q_0 *be a closed locally* CAT *flat* CAT $(q-1)$*-submanifold of* ∂Q *if* $q \geq 1$ *or a subset of* Q *if* $q = 0$, *let* $A \subset Q$ *be closed, let* $f\colon Q \to M$ *be a locally flat topological embedding with* $f^{-1}\partial M = Q_0$ *such that* f *is a locally* CAT *flat* CAT *embedding near* A *and such that the restriction* $f|\colon Q_0 \to \partial M$ *is a locally* CAT *flat* CAT *embedding if* $n = 5$, *and let* $\varepsilon \in C_+(M)$. *Then there is an ambient* ε*-isotopy* $H = (h_t)_{t\in I}$ *of* M *fixed near* fA *and outside* $N(f[Q \smallsetminus A], \varepsilon)$ *such that* $h_1 f$ *is a locally* CAT *flat* CAT *embedding and such that* H *is fixed on* ∂M *if* $f|\colon Q_0 \to \partial M$ *is a locally* CAT *flat* CAT *embedding.*

PROOF. Say that (∂) holds whenever $f|\colon Q_0 \to \partial M$ is a locally CAT flat CAT embedding. Observe that if we choose an open submanifold M' of M containing fQ as a closed subset and define $\varepsilon' \in C_+(M')$ by $\varepsilon'(x) = \min(\varepsilon(x), d(x, M \smallsetminus M'))$, then we can extend an ambient ε'-isotopy of M' to an ambient ε-isotopy of M by the trivial ambient isotopy of $M \smallsetminus M'$. Hence we may assume f to be closed. Choose open neighborhoods U and V of A such that $f|U$ is a locally CAT flat CAT embedding and $\overline{V} \subset U$. Choose an open submanifold M_0 of M with $fU = M_0 \cap fQ$. Now fQ is a closed locally flat submanifold of M, and $M_0 \cap fQ$ is a locally CAT flat CAT submanifold of M_0; moreover, $fQ \cap \partial M = fQ_0$ is a locally CAT flat CAT submanifold of ∂M if (∂) holds and thus, in particular, if $n = 5$. Let $\mathcal{A}_1$ denote the CAT structure of M, and let $\mathcal{B}$ be the CAT structure on fQ making $f\colon Q \to (fQ, \mathcal{B})$ a CAT homeomorphism. Apply [25, 7.1] with the substitutions CAT $\mapsto$ CAT, $n \mapsto n$, $q \mapsto q$, $M \mapsto M$, $Q \mapsto fQ$ (then $Q_0 \mapsto fQ_0$), $M_0 \mapsto M_0$, $\mathcal{A}_0 \mapsto \mathcal{A}_1|M_0$, $\mathcal{B}_Q \mapsto \mathcal{B}$, and, if (∂) holds, $\mathcal{B}_{\partial M} \mapsto \mathcal{A}_1|\partial M$. Then we obtain a CAT structure $\mathcal{A}$ on M, topologically compatible with $\mathcal{A}_1$, such that $(fQ, \mathcal{B})$ is a locally CAT flat CAT submanifold of $(M, \mathcal{A})$, such that $\mathcal{A}|\partial M = \mathcal{A}_1|\partial M$ if (∂) holds, and such that $\mathcal{A}|M_0 = \mathcal{A}_1|M_0$. Now [25, 2.4] with the substitutions

CAT $\mapsto$ CAT, $n \mapsto n$, $M \mapsto (M, \mathcal{A})$, $M' \mapsto (M, \mathcal{A}_1)$, $A \mapsto f\overline{V}$, $B \mapsto f[Q \smallsetminus V]$, $B' \mapsto N = N(f[Q \smallsetminus V], \varepsilon)$, $f \mapsto \mathrm{id}$, and $\varepsilon \mapsto \varepsilon$ yields an ambient ε-isotopy $H = (h_t)_{t \in I}$ of M fixed near $f\overline{V}$, on $M \smallsetminus N$, and on ∂M if (∂) holds such that h_1 is CAT near fQ. Then $h_1 f$ is a locally CAT flat CAT embedding. Hence H is the desired isotopy. $\square$

4 Equivalence

In this section we try to establish as much as possible of the following conjecture. It generalizes the result [22, 4.1 and 4.2], where $n = q + 1 \neq 4$. Our main results are Theorems 4.24 and 4.25.

Equivalence conjecture 4.1. *Let* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$, *let* $0 \leq q \leq n$, *let* $n \neq q + 2$ *if* $q \geq 1$, *let* M *be a* CAT n*-manifold and* Q *a* CAT q*-manifold, let* Q_0 *be a closed locally* CAT *flat* CAT $(q-1)$*-submanifold of* ∂Q *if* $q \geq 1$ *or a subset of* Q *if* $q = 0$, *let* $A \subset Q$ *be closed,* $B \subset Q$ *a neighborhood of* A, *and* $U \subset Q$ *open, let* $f\colon Q \to M$ *be a topological embedding with* $f^{-1}\partial M = Q_0$, *and let* $\varepsilon \in C_+(M)$. *If* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$, *let* $f[A \smallsetminus U]$ *be closed in* M. *Then there is* $\delta \in C_+(Q)$ *with the following property*:

Let $f_0, f_1\colon Q \to M$ *be locally* CAT *flat* CAT *embeddings such that* $f_i|B$ *is* $(\delta|B)$*-close to* $f|B$ *and* $f_i^{-1}\partial M = Q_0$ *for* $i = 0, 1$ *and such that* $f_0|U = f_1|U$. *Then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t \in I}$ *of* M *fixed outside* $N(f[A \smallsetminus U], \varepsilon)$ *such that* $h_1 f_0 = f_1$ *on* $N(A, \delta)$. *Moreover, if* $f_0 = f_1$ *on* $B \cap Q_0$, *then, with the modification described in* $(*)$ *below,* H *can be chosen to be fixed on* ∂M.

$(*)$ *Suppose that* CAT $=$ LQS, *that* $q = 1$, *that* $n \geq 2$, *and that* $f_0 = f_1$ *on* $B \cap Q_0$. *Then the last claim that* H *can be chosen to be fixed on* ∂M *needs the following modification. Let* Σ_0 *be the set of all points* $x \in (A \smallsetminus U) \cap Q_0$ *for which the embedding defined near* $f_0(x)$ *in* $f_0 Q \cup \partial M$ *by the* LQS *embeddings* $f_1 f_0^{-1}$ *and* $\mathrm{id}_{\partial M}$ *is not* LQS *at* $f_0(x)$. *Let* Ω_0 *be a neighborhood of* $f_0\Sigma_0$ *in* ∂M. *Then* H *can be chosen such that* $h_1|\partial M \smallsetminus \Omega_0 = \mathrm{id}$. *Concerning the whole isotopy* H, *let* Σ *be a subset of* $(A \smallsetminus U) \cap Q_0$ *and* Ω *a neighborhood of* $f\Sigma$ *in* ∂M, *and allow* δ *to depend on* (Σ, Ω). *Now, if* $f_0 = f_1$ *on* $B \cap Q_0$ *with* $\Sigma_0 \subset \Sigma$, *then* H *can be chosen such that, in addition,* H *is fixed on* $\partial M \smallsetminus \Omega$.

Remarks 4.2. (1) Conjecture 4.1 implies its strengthened form where a neighborhood N^* of $f[A \smallsetminus U]$ in M and a function $\varepsilon^* \in C_+(N^*)$ are assumed instead of ε (and $N(f[A \smallsetminus U], \varepsilon)$) and where H is required to be fixed on $M \smallsetminus N^*$ and to induce an ε^*-isotopy of N^*. This is trivial in the LIP and LQS cases where $f[A \smallsetminus U]$ is assumed to be closed. In the TOP case, too, the stronger form can easily be deduced from the original one. Below the proof of 4.5 reducing the TOP case of 4.1 to another (equivalent) conjecture can also be modified to yield this strengthening; it suffices to observe that there we may assume that

N^* is a closed subset of M', that $N(f[A \smallsetminus U], \varepsilon') \subset N^*$, and that $\varepsilon'|N^* \le \varepsilon^*$.

(2) Consider the case $q = 0$. Then 4.1 (strengthened as in (1)) holds for all $n \ge 0$. Moreover, Q_0 is not needed, although $f_0^{-1}\partial M = f_1^{-1}\partial M$ must still be assumed; now H can again be chosen to be fixed on ∂M whenever $f_0 = f_1$ on $B \cap f_0^{-1}\partial M$. To see this, we may first assume by subsequent reductions that $A = B = Q$ and $U = \emptyset$, that fQ is closed in M, and finally that Q is a singleton; then the claim follows from the fact that for $M \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$ and two points $x, y \in B^n \cap M$ with either $\{x, y\} \cap \partial M = \emptyset$ or $\{x, y\} \subset \partial M$ there is an ambient LIP isotopy of M carrying x to y and being fixed on $M \smallsetminus B^n$ and, if $\{x, y\} \cap \partial M = \emptyset$, on ∂M.

(3) For $q = 1$, the special case of 4.1 where $f_0 = f_1$ on $B \cap Q_0$ implies the general case. In fact, at least if $B \cap Q_0 = A \cap Q_0$, B is closed, and $U \subset B$, as we may assume, it is easy to perform with the aid of (2) an auxiliary small ambient CAT isotopy of M reducing the general case to the special case.

(4) For all CAT $\in$ {LIP, LQS} and $1 \le q \le n$ the assumption in 4.1 that $f[A \smallsetminus U]$ is closed in M is essential. For example, for every $\delta \in C_+(\operatorname{Int} I^q)$ the δ-neighborhood of the inclusion map $f_0 = f\colon \operatorname{Int} I^q \to \mathbb{R}^n$ contains a PL homeomorphism f_1 onto $\operatorname{Int} I^q$ which is not quasisymmetric and thus, by the compactness of I^q, not of the form $h_1 f_0$ for any LQS homeomorphism $h_1\colon \mathbb{R}^n \to \mathbb{R}^n$. For $q = 0$ and $n = 1$, however, (2) can be established even without this assumption (the case $n \ge 2$ we leave open).

(5) For $q \ge 1$ the special case of 4.1 where $n = q$ and $Q_0 = \partial Q$ or, equivalently, where f is open can be established directly by deformation theory of [23]; see 4.6–4.7.

We first reduce a main part of the LIP and LQS cases of 4.1 to the TOP case of 4.1. In this reduction the assumptions concerning $n \in \{4, 5\}$ are essential, while the assumption $n \ne q + 2$ would not be.

Lemma 4.3. *The* LIP *and* LQS *cases of Equivalence conjecture* 4.1 *for particular n and q follow from the* TOP *case of* 4.1 *for the same n and q, provided that $n \ne 4$ and that we only consider pairs (f_0, f_1) with $f_0 = f_1$ on $B \cap Q_0$ if $n = 5$. The same concerns the special cases of these conjectures where $f_0 = f_1$ on $B \cap Q_0$. The special cases of* 4.1 *for* LIP *and* LQS *where $Q_0 = \emptyset$ or $\partial Q_0 = \emptyset$ or $Q_0 = \partial Q$ follow from the same respective special cases of* 4.1 *for* TOP.

PROOF. Let CAT $\in$ {LIP, LQS}. By replacing B by a smaller neighborhood of A we may assume B to be closed. For $q = 1$ we may also assume that $B \cap Q_0 = A \cap Q_0$. Choose an open submanifold M' of M containing fB as a closed subset. Let $N_0 = N(f[A \smallsetminus U], \varepsilon)$. Since $f[A \smallsetminus U]$ is closed in M, we may assume ε to be so small that $\overline{N}_0 \subset M'$. We show that there are an open neighborhood A' of A in Q with $\overline{A'} \subset \operatorname{int}_Q B$ and $\varepsilon' \in C_+(M)$ such that

$\overline{N}_4 \subset N_0$ for $N_4 = N(f[\overline{A}' \setminus U], \varepsilon')$. In fact, choose open sets V and W in Q such that $A \setminus U \subset V \subset \overline{V} \subset f^{-1}N_0 \cap \operatorname{int}_Q B$ and $A \setminus V \subset W \subset \overline{W} \subset U \cap \operatorname{int}_Q B$. Define $A' = V \cup W \supset A$. Since $f[\overline{A}' \setminus U] \subset f\overline{V} \subset N_0$, we first conclude that $f[\overline{A}' \setminus U]$ is closed in M and then that the desired ε' exists. Choose a closed neighborhood $A'' \subset A'$ of A in Q and open sets $N_1, N_2, N_3 \subset M$ such that $\overline{N}_i \subset N_{i-1}$ for $1 \leq i \leq 4$.

Consider $\eta \in C_+(M)$ with $\eta \leq \min(\varepsilon', \varepsilon/3)$; we fix η later. Let $\delta \in C_+(Q)$ with $N(A, \delta) \subset A''$ be the function yielded by the TOP case of 4.1 whenever M, Q, Q_0, B, U, and f are the same but A is replaced by $\overline{A}'$ and ε by η. We show this δ to satisfy the CAT case of 4.1. Thus, let f_0 and f_1 be as in the CAT case of 4.1. Then we obtain a homeomorphism $h\colon M \to M$ within η of id such that $h = \mathrm{id}$ outside N_4, that $hf_0 = f_1$ on A', and that $h|\partial M = \mathrm{id}$ whenever $f_0 = f_1$ on $B \cap Q_0$. By choosing δ sufficiently small we may assume $f_0[A \setminus U]$ to be closed in M and $f_0 B$ to be a closed subset of M'.

Choose an open set $M_0 \subset M'$ with $f_0 A' = f_0 B \cap M_0$; then $f_0 A'$ is closed in M_0. Hence, $M_1 = hM_0$ is open in M, and $f_1 A' = hf_0 A'$ is closed in M_1. Now $f_i A'$ is a locally CAT flat CAT q-submanifold of M_i for $i = 0, 1$, and the homeomorphism $f_0 A' \to f_1 A'$ induced by h is the CAT homeomorphism defined by $f_1 f_0^{-1}$; moreover, $h|\partial M = \mathrm{id}$ if $n = 5$. In the case of $(*)$, let Σ_0 and Ω_0 be as in $(*)$; then, since $f_0\Sigma_0$ is closed in ∂M and $f_0\Sigma_0 \subset M_0$, we may assume that Ω_0 is closed in ∂M and $\Omega_0 \subset M_0$. Define $\varepsilon_0 \in C_+(M_0)$ by $\varepsilon_0(x) = \min(\eta(x), d(x, M \setminus M_0))$. Apply [25, 6.3], together with [25, 6.4(3)] in the case of $(*)$, with the substitutions $M, B, B' \mapsto M_0$, $M' \mapsto M_1$, $Q \mapsto f_0 A'$, $Q' \mapsto f_1 A'$, $A \mapsto M_0 \setminus N_3$, $f \mapsto h|M_0$, $\varepsilon \mapsto \varepsilon_0$, and, in the case of $(*)$, $\Sigma \mapsto f_0\Sigma_0$ (note that $f_0 A' \cap \partial M \subset f_0[(A \setminus U) \cap Q_0] \cup f_0 U$) and $\Omega' \mapsto \Omega_0$. Then we get a homeomorphism $h'\colon M \to M$ within η of h such that $h' = h$ on $M \setminus M_0$, that h' induces a CAT homeomorphism $M_0 \to M_1$, that $h' f_0 = f_1$ on A', that $h' = \mathrm{id}$ outside N_3, and that, if $f_0 = f_1$ on $B \cap Q_0$, then $h'|\partial M = \mathrm{id}$, with the modification in the case of $(*)$ where we can only conclude that $h'|\partial M \setminus \Omega_0 = \mathrm{id}$ and that, hence, $h'|\partial M$ is CAT. Now note that $f_0 A'' \subset M_0$ is closed in M' and that $h' = \mathrm{id}$ on $M \setminus \overline{N}_3 \supset M \setminus M'$. Hence, by [25, 2.4] with the substitutions $M, M', B, B' \mapsto M'$, $A \mapsto f_0 A'' \cup (M' \setminus N_2)$, $f \mapsto h'|M'$, and $\varepsilon \mapsto \eta|M'$ there is a homeomorphism $h''\colon M \to M$ within η of h' such that $h''|M'$ is CAT, that $h'' = h'$ near $f_0 A''$, that $h'' = \mathrm{id}$ on $M \setminus \overline{N}_2 \supset M \setminus M'$, and that $h'' = h'$ on ∂M whenever $h'|\partial M$ is CAT; then h'' is CAT on M, h'' is within 3η, and thus within ε, of id, and $h'' f_0 = f_1$ on A''. In the case of $(*)$ we have $h''|\partial M \setminus \Omega_0 = \mathrm{id}$.

It suffices to construct the isotopy H from id to h''. For the case of $(*)$, let Σ and Ω be as in $(*)$; then, since $f\Sigma$ is closed in ∂M, there are closed neighborhoods Ω' of $f\Sigma$ and $\Omega'' \subset \Omega$ of Ω' in ∂M. By choosing δ small enough, we may and will assume that $\Omega_0 \subset \Omega'$ whenever $f_0 = f_1$ on $B \cap Q_0$ with $\Sigma_0 \subset \Sigma$. Now, if η is chosen small enough, then we can apply [25, 6.8], together with [25, 6.9] in the case of $(*)$, with the substitutions $M, M', B, B' \mapsto M$, $Q, Q' \mapsto \emptyset$, $f \mapsto \mathrm{id}$, $A \mapsto M \setminus N_0$, $A_i \mapsto M \setminus \overline{N_i}$ for $i = 1, 2$, $\varepsilon \mapsto \varepsilon$, $\delta \mapsto 3\eta$, $f^* \mapsto \mathrm{id}$, $f^{**} \mapsto h''$, and, in the case of $(*)$ (although this time $\Sigma \mapsto \emptyset$), $\Omega' \mapsto \Omega$, $W_1' \mapsto \partial M \setminus \Omega''$, and $W_2' \mapsto \partial M \setminus \Omega'$; we substitute $Z \mapsto \partial M$ only if $f_0 = f_1$ on $B \cap Q_0$ with $\Sigma_0 \subset \Sigma$. Then we obtain an ambient CAT ε-isotopy $H = (h_t)_{t \in I}$ of M fixed outside N_0 such that $h_1 = h''$, and hence $h_1 f_0 = f_1$ on $N(A, \delta)$, and such that H is fixed on ∂M whenever $f_0 = f_1$ on $B \cap Q_0$, with the modification in the case of $(*)$ where we can only require that if $\Sigma_0 \subset \Sigma$, then H is fixed on $\partial M \setminus \Omega$ with $h_1 | \partial M \setminus \Omega_0 = \mathrm{id}$. Note that in the case of $(*)$ the choices $\Sigma = (A \setminus U) \cap Q_0$ and $\Omega' = \Omega = \partial M$ yield an absolute δ. Thus H is the desired isotopy. $\square$

In 4.4 and 4.8 we present conjectures, which are simpler than 4.1 but which we show in Lemmata 4.5 and 4.10 and Remark 4.9(1) to be equivalent to it. With CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ this is needed for our theorems in the case $n = q \in \{4, 5\}$ only, but we include all possibilities of (n, q) for our discussion about 4.1. For simplicity, however, we remove here from 4.1 its last claim concerning the isotopy H on the boundary ∂M in the exceptional case where CAT = LQS and $q = 1 < n$. In the proofs the corresponding weakenings in this exceptional case are assumed to have been made tacitly.

We first reduce 4.1 for $q \geq 1$ to the following conjecture about a local situation. This statement is an analogue of [22, 4.11, 4.15, and 4.17].

Local conjecture 4.4. (a) *Let* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$, *let* $1 \leq q \leq n$ *with* $n \neq q + 2$, *let* G *be an open subset of* $\mathbb{R}^q_{++}$, *let* A, B, *and* U *be open subsets of* G *such that* $\overline{A} \cap G \subset U$, $\overline{B} \subset G$, *and* $\overline{B}$ *is compact, let* $G_0 = G \cap \mathbb{R}^{q-1}_+$, *let* $f \colon G \to \mathbb{R}^n_+$ *be an embedding with* $f^{-1}\mathbb{R}^{n-1} = G_0$, *and let* $\varepsilon > 0$. *Then there is* $\delta > 0$ *with the following property*: *Let* $f_0, f_1 \colon G \to \mathbb{R}^n_+$ *be locally* CAT *flat* CAT *embeddings* δ*-close to* f *such that* $f_i^{-1}\mathbb{R}^{n-1} = G_0$ *for* $i = 0, 1$ *and such that* $f_0|U = f_1|U$. *Then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t \in I}$ *of* $\mathbb{R}^n_+$ *fixed outside* $N(fB, \varepsilon)$ *such that* $h_1 f_0 = f_1$ *on* $A \cup B$. *Moreover, if* $f_0|G_0 = f_1|G_0$ *and for* CAT = LQS *the case* $q = 1 < n$ *is excluded, then* H *can be chosen to be fixed on* $\mathbb{R}^{n-1}$.

(b) *This is the modification of* (a) *for* $q = 1$ *where* $\mathbb{R}^n_+$ *is replaced by* $\mathbb{R}^n$ (*but* $\mathbb{R}^q_{++} = \mathbb{R}^1_+$ *is preserved*) *and where* G_0 *and all conditions involving* G_0 *are deleted.*

Lemma 4.5. *Equivalence conjecture* 4.1 *for particular* CAT, n, *and* $q \geq 1$ *follows from Local conjecture* 4.4 *for the same* CAT, n, *and* q, *provided that the claim of* 4.1 *concerning* H *on* ∂M *is removed if* CAT = LQS *and* $q = 1 < n$. *The special case of* 4.1 *where* $f_0 = f_1$ *on* $B \cap Q_0$ *follows from the special case of* 4.4 *where* $f_0|G_0 = f_1|G_0$. *The special cases of* 4.1 *where* (1) $Q_0 = \emptyset$ *or* (2) $Q_0 = \partial Q$ *or* (3) $\partial Q_0 = \emptyset$ *follow from the special cases of* 4.4 *where in* 4.4(a) *we have, respectively,* (1) $G_0 = \emptyset$ *or* (2) $G_0 = \partial G$ (*whence* G *is open* *in* $\mathbb{R}^q_+$) *or* (3) $G_0 = \emptyset$ *or* $G_0 = \partial G$ *and where* 4.4(b) *is not needed for* (2).

PROOF. We may assume B to be closed. By replacing U by $U \cap \operatorname{int}_Q B$ we may assume that $U \subset B$. Choose an open submanifold M' of M containing fB as a closed subset. Choose $\varepsilon' \in C_+(M')$ with $\varepsilon'(x) \le \min(\varepsilon(x), d(x, M \setminus M'))$ and with $\operatorname{cl}_M N(f[A \setminus U], \varepsilon') \subset M'$ if CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$. There is a cover $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}'$ of A by open subsets of Q with $\mathcal{V}_0 = \{U\}$ such that $\mathcal{V}$ is locally finite in Q and such that, if $V \in \mathcal{V}'$, then $V \cap (A \setminus U) \neq \emptyset$ and $\overline{V}$ is compact in B, there is an open CAT embedding $h_V \colon V \to \mathbb{R}^q_{++}$, with $h_V^{-1}\mathbb{R}^{q-1}_+ = V \cap Q_0$ whenever $q \ge 2$, and $f\overline{V}$ is contained in an open subset M_V of M' with $M_V \subset N(f[A \setminus U], \varepsilon')$ such that there is an open CAT embedding $h'_V \colon M_V \to T_V$, uniformly continuous along with its inverse, where $T_V = \mathbb{R}^n$ whenever $q = 1$ and $V \cap Q_0 = \emptyset$ and where $T_V = \mathbb{R}^n_+$ otherwise. By [31, 2.7] we may assume $\mathcal{V}'$ to have a disjoint decomposition $\mathcal{V}' = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{q+1}$ into disjoint families $\mathcal{V}_i$. We choose for each $V \in \mathcal{V}$ open sets $V = V^{-1} \supset V^0 \supset \cdots \supset V^{q+1}$ such that $\overline{V^i} \subset V^{i-1}$ and such that $A \subset \bigcup\{\, V^{q+1} \mid V \in \mathcal{V} \,\}$. Define open sets $D_j = \bigcup\{\, V^{j-i} \mid 0 \le i \le j,\ V \in \mathcal{V}_i \,\}$ for $0 \le j \le q+1$ and $D'_j = \bigcup\{\, V^{j-i+1} \mid 0 \le i \le j,\ V \in \mathcal{V}_i \,\}$ for $0 \le j \le q$. Then $D_0 = U^0$, $D_{q+1} \supset A$, and for $1 \le j \le q+1$ we have $\overline{D}'_{j-1} \subset D_{j-1}$ and $D_j = D'_{j-1} \cup \bigcup\{\, V^0 \mid V \in \mathcal{V}_j \,\}$.

Consider $j \in \{1, \ldots, q+1\}$ and a function $\varepsilon_j \in C_+(M')$, to be specified below. For each $V \in \mathcal{V}_j$ we choose $\varepsilon_V > 0$ such that $N_V = \overline{N(fV^0, \varepsilon_V)}$ is compact in M_V, that $\varepsilon_V \le \inf \varepsilon_j N_V$, that $N_V \cap f[B \setminus V] = \emptyset$, and that $\{\, N_V \mid V \in \mathcal{V}_j \,\}$ is a locally finite disjoint family in M'. We next construct $\alpha_j \in C_+(B)$ such that $\alpha_j(x) \le d(f(x), N_V)$ for all $V \in \mathcal{V}_j$ and $x \in B \setminus V$. For $y \in M'$ define a set

$$F_y = \bigcup\{\, f[B \setminus V] \mid V \in \mathcal{V}_j,\ y \in N_V \,\} \cup \bigcup\{\, N_V \mid V \in \mathcal{V}_j,\ y \notin N_V \,\}$$

closed in M' and disjoint from y. Then there are a locally finite cover $(K_\lambda)_{\lambda \in \Lambda}$ of M' by compact sets, a family $(y_\lambda)_{\lambda \in \Lambda}$ of points of M', and a function $\beta_j \in C_+(M')$ such that $\sup \beta_j K_\lambda \le d(K_\lambda, F_{y_\lambda})$ for each $\lambda \in \Lambda$. Define $\alpha_j = \beta_j(f|B)$. Consider $V \in \mathcal{V}_j$ and $x \in B \setminus V$. Pick $\lambda \in \Lambda$ with $f(x) \in K_\lambda$. Then $0 < \alpha_j(x) \le d(f(x), F_{y_\lambda})$, which implies that $f[B \setminus V] \not\subset F_{y_\lambda}$ and thus that $y_\lambda \notin N_V$; hence, $N_V \subset F_{y_\lambda}$, from which it follows that $\alpha_j(x) \le d(f(x), N_V)$.

Consider $V \in \mathcal{V}_j$, the set $G_V = h_V V$, and the embedding $f_V = h'_V f h_V^{-1} \colon G_V \to T_V$. If $T_V = \mathbb{R}^n_+$, then $f_V^{-1}\mathbb{R}^{n-1} = G_V \cap \mathbb{R}^{q-1}_+$. Choose a number $\varepsilon'_V > 0$ such that, if $h \colon h'_V M_V \to h'_V M_V$ is a homeomorphism ε'_V-close to id, then the homeomorphism ${h'_V}^{-1} h h'_V \colon M_V \to M_V$ is ε_V-close to id, and such that $N(h'_V f V^0, \varepsilon'_V) \subset h'_V N_V$ in T_V. Now apply 4.4(a) if $T_V = \mathbb{R}^n_+$ or 4.4(b) if $T_V = \mathbb{R}^n$ with the substitutions $G \mapsto G_V$, $A \mapsto h_V[D'_{j-1} \cap V]$, $B \mapsto h_V V^0$, $U \mapsto h_V[D_{j-1} \cap V]$, $f \mapsto f_V$, and $\varepsilon \mapsto \varepsilon'_V$; let $\delta'_V > 0$ be the number δ thus obtained. Choose a number $\delta_V > 0$ with $\delta_V \le d(f\overline{V}, M \setminus M_V)$ such that if $g \colon V \to M$ is an embedding δ_V-close to $f|V$, in which case $gV \subset M_V$, then the embedding $h'_V g h_V^{-1}\colon G_V \to T_V$ is δ'_V-close to f_V. Choose $\delta_j \in C_+(B)$ with $\delta_j \le \alpha_j$ such that $\max \delta_j \overline{V} \le \delta_V$ for each $V \in \mathcal{V}_j$.

We fix the functions $\varepsilon_1, \delta_1, \dots, \varepsilon_{q+1}, \delta_{q+1}$ so small that $h_{q+1} \cdots h_1$ is ε'-close to id and $h_j g_j$ is δ_{j+1}-close to $f|B$ for $1 \le j \le q$ whenever $h_j\colon M' \to M'$ is a homeomorphism ε_j-close to id and $g_j\colon B \to M'$ is an embedding δ_j-close to $f|B$. Choose $\delta \in C_+(Q)$ such that $\delta(x) \le \min(\delta_1(x), d(f(x), M \smallsetminus M'))$ for each $x \in B$ and such that $N(A, \delta) \subset D_{q+1}$.

We show that this δ satisfies 4.1. Thus, let f_0 and f_1 be as in 4.1. Then $f_i B \subset M'$ for $i = 0, 1$. Below we construct inductively for each $j \in \{1, \dots, q+1\}$ an ambient CAT ε_j-isotopy $H^j = (h^j_t)_{t \in I}$ of M' fixed outside $N(f[A \smallsetminus U], \varepsilon')$ such that $h^j_1 \cdots h^1_1 f_0 = f_1$ on D_j (true for $j = 0$ as $D_0 \subset U$) and such that H^j is fixed on $\partial M'$ whenever $f_0 = f_1$ on $B \cap Q_0$. The desired CAT isotopy $H = (h_t)_{t \in I}$ is then obtained by extending $H^{q+1} \diamond \cdots \diamond H^1$ by the trivial ambient isotopy of $M \smallsetminus M'$.

Thus, let $j \in \{1, \dots, q+1\}$, and suppose the isotopies $H^1, \dots, H^{j-1}$ already to be defined. Then the CAT embeddings $g_{j-1} = h^{j-1}_1 \cdots h^1_1 f_0 | B$ (where $g_0 = f_0|B$) and $f_1|B$ of B into M' are locally CAT flat on $\operatorname{int}_Q B$, both g_{j-1} and $f_1|B$ are δ_j-close to $f|B$, we have $g^{-1}_{j-1} \partial M' = B \cap Q_0$, and $g_{j-1} = f_1$ on D_{j-1}. Moreover, if $f_0 = f_1$ on $B \cap Q_0$, then $g_{j-1} = f_1$ on $B \cap Q_0$. Hence, for each $V \in \mathcal{V}_j$ there is, by the definition of δ_V, an ambient CAT ε_V-isotopy $H^V = (h^V_t)_{t \in I}$ of M_V fixed on $M_V \smallsetminus N_V$ such that $h^V_1 g_{j-1} = f_1$ on $D_j \cap V$ and such that, if $f_0 = f_1$ on $B \cap Q_0$, then H^V is fixed on ∂M_V. For $(x, t) \in M' \times I$, set $h^j_t(x) = h^V_t(x)$ whenever there is $V \in \mathcal{V}_j$ with $x \in N_V$ and $h^j_t(x) = x$ otherwise. Then $H^j = (h^j_t)_{t \in I}$ is an ambient CAT ε_j-isotopy of M' fixed outside $N(f[A \smallsetminus U], \varepsilon')$ such that H^j is fixed on $\partial M'$ whenever $f_0 = f_1$ on $B \cap Q_0$. We must still show that $h^j_1 g_{j-1}(x) = f_1(x)$ for each $x \in D_j$. If $x \in V \in \mathcal{V}_j$, then for each $W \in \mathcal{V}_j$ with $W \neq V$ we have $x \notin W$, whence $d(g_{j-1}(x), f(x)) < \delta_j(x) \le d(f(x), N_W)$ and thus $g_{j-1}(x) \notin N_W$; hence $h^j_1 g_{j-1}(x) = h^V_1 g_{j-1}(x) = f_1(x)$. If $x \notin \bigcup \mathcal{V}_j$, then $g_{j-1}(x) \notin \bigcup \{ N_V \mid V \in \mathcal{V}_j \}$ and $x \in D'_{j-1}$, whence $h^j_1 g_{j-1}(x) = g_{j-1}(x) = f_1(x)$. $\square$

In 4.7 we prove 4.1 for f open without resorting to 4.3, by establishing the following lemma directly.

Lemma 4.6. *For* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$ *and* $n = q \ge 1$, *the modification of* 4.4(a) *holds where* G *is an open subset of* $\mathbb{R}^n_+$ *and* $G_0 = \partial G$.

PROOF. Now f is open. Choose open subsets A_i for $i = 1, 2$ and B_i for $i = 1, \dots, 5$ of G such that if $A_0 = A$ and $B_0 = B$, then $\overline{A}_{i-1} \cap G \subset A_i$, $\overline{A}_2 \cap G \subset U$, $\overline{B}_{i-1} \subset B_i$, $\overline{B}_5 \subset G$, $\overline{B}_5$ is compact, and $f\overline{B}_2 \subset N(fB, \varepsilon)$. By [39, 1.7] or directly, there is $\delta_0 > 0$ such that if $f_0\colon G \to \mathbb{R}^n_+$ is an open embedding

δ_0-close to f, then (in the order we need these inclusions)

$$f\overline{B}_5 \subset f_0 G, \qquad f[\,\overline{A}_2 \cap \overline{B}_5] \subset f_0 U, \qquad f_0 \overline{B} \subset f B_1,$$
$$f\overline{B}_2 \subset f_0 B_3, \qquad f_0[\,\overline{A} \cap \overline{B}_3] \subset f[A_1 \cap B_4].$$

Consider $\delta \in (0, \delta_0)$ and open CAT embeddings $f_0, f_1\colon G \to \mathbb{R}^n_+$ within δ of f such that $f_0|U = f_1|U$. Then we can define an open CAT embedding $h = f_1 f_0^{-1}|\colon fB_5 \to \mathbb{R}^n_+$ with $d(h, \mathrm{id}) \leq d(f_1, f_0) \leq 2\delta$ and with $h = \mathrm{id}$ on $f[A_2 \cap B_5]$. Thus, if δ is small enough, an application of [23, 6.4] and its strengthening for the case CAT = LQS in [25, A.8] with $X = \mathbb{R}^n_+$ and with the substitutions $U \mapsto fB_5$, $B \mapsto f\overline{B}_1$, $B' \mapsto f\overline{B}_2$, $(Y_\gamma, C_\gamma, C'_\gamma)_{\gamma\in\Gamma} \mapsto ((\mathbb{R}^n_+, f[A_1 \cap B_4], f[A_2 \cap B_5]), (\mathbb{R}^n_+, fB_5, fB_5))$, and $\mathcal{Q} \mapsto \{\, g \in H(\mathbb{R}^n_+) \mid d(g, \mathrm{id}; f\overline{B}_2) < \varepsilon \,\}$ yields an ambient CAT ε-isotopy $H = (h_t)_{t\in I}$ of $\mathbb{R}^n_+$ fixed outside $f\overline{B}_2$ and thus also outside $N(fB, \varepsilon)$, on $f[A_1 \cap B_4]$, and, if $h|\partial fB_5 = \mathrm{id}$ and thus if $f_0|\partial G = f_1|\partial G$, on $\mathbb{R}^{n-1}$, too, such that $h_1 = h$ on $f\overline{B}_1$. For CAT = TOP a suitable corollary of [16, 5.1 and Note] can alternatively be used. Then $h_1 f_0|B = hf_0|B = f_1|B$. Finally, from $f_0 A \cap f\overline{B}_2 \subset f_0 A \cap f_0 B_3 \subset f[A_1 \cap B_4]$ it follows that H is fixed on $f_0 A$ implying that $h_1 f_0|A = f_0|A = f_1|A$. ☐

Lemma 4.7. *Conjecture* 4.1 *holds for* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$ *and* $n = q \geq 1$ *whenever* $Q_0 = \partial Q$.

PROOF. This follows from 4.6 by 4.5(2). ☐

In the general case we further reduce Local conjecture 4.4 to the following conjecture, which is a set of handle lemmata analogous to [22, 4.9, 4.10, 4.14, and 4.16]. For our theorems we do not directly need its part (d), and part (c) we only need for $q = 1$, but we need them indirectly, through 4.9(1), for the case $(n, q) = (5, 2)$, too. Also our discussion about the open cases of 4.1 is based on this conjecture in its entirety.

Handle conjecture 4.8. *Let* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$, *let* $1 \leq q \leq n$ *with* $n \neq q + 2$, *let* k *and* m *be nonnegative integers with* $q = k + m$, *and let* $\varepsilon > 0$.

(a) *Let* $f\colon I^q \to \mathbb{R}^n$ *be an embedding. Then there is* $\delta > 0$ *with the following property*: *Let* $f_0, f_1\colon I^q \to \mathbb{R}^n$ *be locally* CAT *flat* CAT *embeddings such that* $d(f_i, f) < \delta$ *for* $i = 0, 1$ *and* $f_0 = f_1$ *on* $(I^k \smallsetminus I^k(\frac{1}{2})) \times I^m$. *Then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t\in I}$ *of* $\mathbb{R}^n$ *fixed outside* $N(fI^q(\frac{1}{2}), \varepsilon)$ *such that* $h_1 f_0 = f_1$ *on* $I^k \times I^m(\frac{1}{2})$.

(b) *Let* $m \geq 1$, *and let* $f\colon I^q_+ \to \mathbb{R}^n$ *be an embedding. Then there is* $\delta > 0$ *with the following property*: *Let* $f_0, f_1\colon I^q_+ \to \mathbb{R}^n$ *be locally* CAT *flat* CAT *embeddings such that* $d(f_i, f) < \delta$ *for* $i = 0, 1$ *and* $f_0 = f_1$ *on* $(I^k \smallsetminus I^k(\frac{1}{2})) \times I^m_+$.

Then there is an ambient CAT ε*-isotopy* $H = (h_t)_{t\in I}$ *of* $\mathbb{R}^n$ *fixed outside* $N(fI^q_+(\frac{1}{2}),\varepsilon)$ *such that* $h_1f_0 = f_1$ *on* $I^k \times I^m_+(\frac{1}{2})$.

(c) *Let* $m \geq 1$, *and let* $f\colon I^q_+ \to \mathbb{R}^n_+$ *be an embedding with* $f^{-1}\mathbb{R}^{n-1} = I^{q-1}$. *Then there is* $\delta > 0$ *with the following property*: *Let* $f_0, f_1\colon I^q_+ \to \mathbb{R}^n_+$ *be locally* CAT *flat* CAT *embeddings such that* $f_i^{-1}\mathbb{R}^{n-1} = I^{q-1}$ *and* $d(f_i, f) < \delta$ *for* $i = 0, 1$ *and such that* $f_0 = f_1$ *on* $(I^k \smallsetminus I^k(\frac{1}{2})) \times I^m_+$. *Then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t\in I}$ *of* $\mathbb{R}^n_+$ *fixed outside* $N(fI^q_+(\frac{1}{2}),\varepsilon)$ *such that* $h_1f_0 = f_1$ *on* $I^k \times I^m_+(\frac{1}{2})$ *and such that* H *is fixed on* $\mathbb{R}^{n-1}$ *if* $f_0 = f_1$ *on* I^{q-1} *and for* CAT = LQS *the case* $q = 1 < n$ *is excluded.*

(d) *Let* $m \geq 2$, *and let* $f\colon I^q_{++} \to \mathbb{R}^n_+$ *be an embedding with* $f^{-1}\mathbb{R}^{n-1} = I^{q-1}_+$. *Then there is* $\delta > 0$ *with the following property*: *Let* $f_0, f_1\colon I^q_{++} \to \mathbb{R}^n_+$ *be locally* CAT *flat* CAT *embeddings such that* $f_i^{-1}\mathbb{R}^{n-1} = I^{q-1}_+$ *and* $d(f_i, f) < \delta$ *for* $i = 0, 1$ *and such that* $f_0 = f_1$ *on* $(I^k \smallsetminus I^k(\frac{1}{2})) \times I^m_{++}$. *Then there is an ambient* CAT ε*-isotopy* $H = (h_t)_{t\in I}$ *of* $\mathbb{R}^n_+$ *fixed outside* $N(fI^q_{++}(\frac{1}{2}),\varepsilon)$ *such that* $h_1f_0 = f_1$ *on* $I^k \times I^m_{++}(\frac{1}{2})$ *and such that* H *is fixed on* $\mathbb{R}^{n-1}$ *if* $f_0 = f_1$ *on* I^{q-1}_+.

Remarks 4.9. (1) Conjecture 4.8 is a corollary of the special case $A = B = Q$ of 4.1. In fact, for Q take the domain of f intersected by $I^k \times I^m(\frac{1}{2})$, and note that it can be shown by 2.2(a) that $f_i|Q$ is locally CAT flat for $i = 0, 1$. From 4.10 below and 4.5 it then follows that each of Conjectures 4.8 and 4.4 is equivalent to the weakening of 4.1 discussed before 4.4.

(2) In 4.8 for CAT = TOP and for fixed n, q, k, and m with $m \geq 1$, part (b) can be deduced from part (a) in the same way as [22, 4.10] is deduced from [22, 4.9]. In [22] only replace $n-1$ and $n-2$, respectively, by q and $q-1$, and $I^1(s)$ and sx_n in the definitions of B_s and ψ_s, respectively, by $I^{n-q}(s)$ and $(sx_{q+1}, \ldots, sx_n)$. I thank David B. Gauld for an explicit construction of Γ_1 as a diffeomorphism (which suffices) in this proof.

Lemma 4.10. *Local conjecture* 4.4 *for particular* n *and* q *follows from Handle conjecture* 4.8 *for the same* n *and* q. *The special case of* 4.4 *where* $f_0|G_0 = f_1|G_0$ *follows from the special case of* 4.8 *where* $f_0 = f_1$ *on* $f^{-1}\mathbb{R}^{n-1}$ *in* (c)–(d). *The special cases of* 4.4 *where* $\partial G_0 = \emptyset$ *or* $G_0 = \emptyset$, *respectively, in* 4.4(a) *follow from parts* (a)–(c) *or parts* (a)–(b), *respectively, of* 4.8.

PROOF. First consider 4.4(a). Let $G_1 = \partial G \smallsetminus \operatorname{Int} G_0$ and $G_2 = \partial G_0 = \partial G_1$. Choose a rectilinear triangulation K of G containing for $i = 0, 1, 2$ a subcomplex K_i triangulating G_i. Let K' be the first barycentric subdivision of K and K'' the second one. For a simplex $\sigma \in K$, let $\hat\sigma$ be the barycenter of σ, $H_\sigma = \bigcup\{\, \tau \mid \hat\sigma \in \tau \in K'' \,\}$, and $i(H_\sigma) = \dim\sigma$. Then $\mathcal{H} = \{\, H_\sigma \mid \sigma \in K \,\}$ is a handle decomposition of G as in [19, pp. 224–227], [36, p. 82], and [37, pp. 31–33]. In fact, if $\sigma \in K$, $P_\sigma = \bigcup\{\, H_\tau \mid \tau \in K, \dim\tau < \dim\sigma \,\}$, $\widetilde{H}_\sigma = \operatorname{cl}\bigl(\bigcup\{\, \tau \mid \hat\sigma \in \tau \in K' \,\} \smallsetminus P_\sigma\bigr)$, $k = i(H_\sigma)$, and $m = q - k$ (thus $m \geq 1$ whenever

$\sigma \in K_0 \cup K_1$, and $m \geq 2$ whenever $\sigma \in K_2$), then there is a PL homeomorphism of $(\widetilde{H}_\sigma, H_\sigma, \widetilde{H}_\sigma \cap G_0, \widetilde{H}_\sigma \cap G_1, \widetilde{H}_\sigma \cap G_2)$ onto $I^k \times (2I^m_\sigma, I^m_\sigma, 2I^{m-1}_{0\sigma}, 2I^{m-1}_{1\sigma}, 2I^{m-2}_{2\sigma})$ which maps $\widetilde{H}_\sigma \cap P_\sigma$ onto $\partial I^k \times 2I^m_\sigma$, where I^j_σ with $j = m$ (or also with $j = q$ to imply $I^q_\sigma = I^k \times I^m_\sigma$) is defined as I^j for $\sigma \in K \setminus (K_0 \cup K_1)$, as I^j_+ for $\sigma \in (K_0 \cup K_1) \setminus K_2$, and as I^j_{++} for $\sigma \in K_2$, where $I^{m-1}_{i\sigma}$ with $i = 0, 1$ is defined as $\emptyset$ for $\sigma \in K \setminus K_i$, as I^{m-1} for $\sigma \in K_i \setminus K_2$, and as $\{0\}^i \times I^{m-1}_+$ for $\sigma \in K_2$, and where $I^{m-2}_{2\sigma}$ is defined as $\emptyset$ for $\sigma \in K \setminus K_2$ and as I^{m-2} for $\sigma \in K_2$. (See [36, p. 52] for regular neighborhoods used in the construction.) We also define $J^m_\sigma = (-1,1)^m \cap I^m_\sigma$. If V is any of these intervals, denote $V(r) = rV$ for $r > 0$. By setting $H_\sigma \leq H_\tau$ if $H_\sigma \cap H_\tau \neq \emptyset$ and $i(H_\sigma) \leq i(H_\tau)$ (that is, if σ is a face of τ) we obtain a partial order $\leq$ for $\mathcal{H}$. For $H \in \mathcal{H}$ denote $\widetilde{H} = \widetilde{H}_\sigma$ if $\sigma \in K$ and $H = H_\sigma$; then $\widetilde{H} \subset \bigcup\{\, H' \mid H \leq H' \in \mathcal{H} \,\}$. If $g\colon G \to \mathbb{R}^n_+$ is a locally CAT flat CAT embedding with $g^{-1}\mathbb{R}^{n-1} = G_0$, then $g|\widetilde{H}$ is locally CAT flat for each $H \in \mathcal{H}$ by 2.2(a).

Let $A' = \overline{A} \cap G$ and $N = N(fB, \varepsilon)$. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the sets of the handles $H \in \mathcal{H}$ for which there is $H' \in \mathcal{H}$ with $H \leq H'$ and with $H' \cap A' \neq \emptyset$ or $H' \cap \overline{B} \neq \emptyset$, respectively; the latter set is finite. Then $X \in \{A, B\}$, $H \in \mathcal{H}$, $H' \in \mathcal{H}_X$, and $H \leq H'$ imply $H \in \mathcal{H}_X$. It follows that $\widetilde{H} \cap A' = \emptyset$ if $H \in \mathcal{H} \setminus \mathcal{H}_A$. We may assume that K is so fine that $\bigcup \mathcal{H}_A \subset U$ and $\bigcup\{\, \widetilde{H} \mid H \in \mathcal{H}_B \,\} \subset f^{-1}N$. Write bijectively $\mathcal{H}_B \setminus \mathcal{H}_A = \{H_1, \ldots, H_s\}$ with $i(H_j) \leq i(H_{j+1})$. Let $\mathcal{H}_j = (\mathcal{H}_A \cap \mathcal{H}_B) \cup \{H_1, \ldots, H_j\}$ and $P_j = \bigcup \mathcal{H}_j$ for $0 \leq j \leq s$. Then $P_s \supset B$. For each $H \in \mathcal{H}$ meeting H_j we have $i(H) < i(H_j)$ if and only if $H \in \mathcal{H}_{j-1}$. We have $\widetilde{H}_j \subset f^{-1}N \setminus A'$. For $1 \leq j \leq s$ pick $\sigma_j \in K$ with $H_j = H_{\sigma_j}$.

We set $U_0 = U \supset A' \cup P_0$. For each $j \in \{1, \ldots, s\}$ we inductively first choose a PL homeomorphism $\mu_j\colon I^q_{\sigma_j} \to \widetilde{H}_j$ and then define an open neighborhood U_j of $A' \cup P_j$ in G such that, by setting $k = i(H_j)$ and $m = q - k$, we have

$$\begin{aligned}
\mu_j[I^k \times I^m_{\sigma_j}(\tfrac{1}{3})] &= H_j, \\
\mu_j^{-1} G_i &= I^k \times I^{m-1}_{i\sigma_j} \quad \text{for } i = 0, 1, \\
\mu_j^{-1} G_2 &= I^k \times I^{m-2}_{2\sigma_j}, \\
\mu_j^{-1} P_{j-1} &= \partial I^k \times I^m_{\sigma_j}, \\
\mu_j[(I^k \setminus I^k(\tfrac{1}{2})) \times I^m_{\sigma_j}] &\subset U_{j-1}, \\
U_j &= (U_{j-1} \setminus \mu_j[I^k(\tfrac{2}{3}) \times I^m_{\sigma_j}]) \cup \mu_j[I^k \times J^m_{\sigma_j}(\tfrac{1}{2})].
\end{aligned}$$

Choose numbers $0 < \delta_j \leq \varepsilon_j \leq \varepsilon/s$ for $1 \leq j \leq s$ such that $\varepsilon_j \leq \delta_{j+1}/2$ for $j < s$, that

$$N_j = N(f\mu_j I^q_{\sigma_j}(\tfrac{1}{2}), \varepsilon_j) \subset N \setminus N(f[G \setminus \mu_j[I^k(\tfrac{2}{3}) \times I^m_{\sigma_j}]], \delta_j)$$

where k and m are as above, that $N_j \subset \operatorname{Int} \mathbb{R}^n_+$ if $\mu_j^{-1} G_0 = \emptyset$, and that δ_j is the number given by 4.8 with these same n, q, k, and m and with the substitutions $f \mapsto f\mu_j$ and $\varepsilon \mapsto \varepsilon_j$; more precisely, by 4.8(a) if $\sigma_j \in K \smallsetminus (K_0 \cup K_1)$, by 4.8(b) if $\sigma_j \in K_1 \smallsetminus K_2$, by 4.8(c) if $\sigma_j \in K_0 \smallsetminus K_2$, and by 4.8(d) if $\sigma_j \in K_2$. Then $\delta = \delta_1$ satisfies 4.4(a). In fact, let f_0 and f_1 be as in 4.4(a). Then it follows inductively that for each $j \in \{1, \ldots, s\}$ there is an ambient CAT ε_j-isotopy $H^j = (h^j_t)_{t \in I}$ of $\mathbb{R}^n_+$ fixed outside N_j such that $H^j_1 \cdots H^1_1 f_0 = f_1$ on U_j and such that H^j is fixed on $\mathbb{R}^{n-1}$ if $f_0 = f_1$ on G_0. Hence $H = H^s \diamond \cdots \diamond H^1$ is the desired CAT isotopy.

The proof of 4.4(b) is similar, based on 4.8(a) and 4.8(b). $\square$

For CAT = TOP we now reduce the case $q = n$ of 4.1 to the case $q = n - 1$.

Lemma 4.11. *The* TOP *case of Handle conjecture* 4.8 *for a particular n and $q = n$ follows from the special case $Q_0 = \partial Q$ of the* TOP *case of Equivalence conjecture* 4.1 *for the same n and $q = n - 1$. The special case of* 4.8 *where $f_0 = f_1$ on $f^{-1}\mathbb{R}^{n-1}$ in* (c)–(d) *follows from the special case of* 4.1 *where $f_0 = f_1$ on $B \cap Q_0$. Parts* (a)–(b) *of* 4.8 *follow from the special case $\partial Q = \emptyset$ of* 4.1.

PROOF. To prove parts (a)–(d) of 4.8 simultaneously, let M denote $\mathbb{R}^n$ in (a)–(b) and $\mathbb{R}^n_+$ in (c)–(d), and for $i \in \{n, m\}$ let $\hat{I}^i$ denote I^i in (a), I^i_+ in (b)–(c), and I^i_{++} in (d), with $\hat{I}^i(r) = r\hat{I}^i$ for $r > 0$; then $\hat{I}^n = I^k \times \hat{I}^m$. Let $L = f^{-1}\partial M$; then $L = \hat{I}^n \cap \mathbb{R}^{n-1}$ in (c)–(d). Choose $\varepsilon_0 \in (0, \frac{1}{2})$ such that for the PL n-ball $Q = I^k(\frac{1}{2} + \varepsilon_0) \times \hat{I}^m(\frac{1}{2}) \supset \hat{I}^n(\frac{1}{2})$ and $N_0 = N(fQ, \varepsilon_0) \subset M$ we have $\overline{N_0} \subset N(f\hat{I}^n(\frac{1}{2}), \varepsilon)$. Let $A = (I^k \smallsetminus I^k(\frac{1}{2} + \varepsilon_0)) \times \hat{I}^m(\frac{1}{2})$, and choose $\varepsilon_1 > 0$ with $2\varepsilon_1 \leq d(fA, f\hat{I}^n(\frac{1}{2}))$. Define a PL $(n-1)$-sphere $Q' = \partial Q$ in (a)–(b) and a PL $(n-1)$-ball $Q' = \overline{\partial Q \smallsetminus \mathbb{R}^{n-1}} = \partial Q \smallsetminus \operatorname{Int}(Q \cap \mathbb{R}^{n-1})$ in (c)–(d), and let $Q'_0 = \emptyset$ in (a)–(b) and $Q'_0 = Q' \cap \mathbb{R}^{n-1}$ in (c)–(d); then $Q' = \overline{\partial Q \smallsetminus L}$ and $Q'_0 = \partial Q' = Q' \cap L$. Let $U = (I^k \smallsetminus I^k(\frac{1}{2})) \times \hat{I}^m(\frac{1}{2})$. Consider $\eta > 0$ with $\eta \leq \min(\varepsilon_0, \varepsilon_1)$, to be fixed later. We apply the TOP case of 4.1 with the substitutions $n \mapsto n$, $q \mapsto n-1$, $M \mapsto M$, $Q, A, B \mapsto Q'$, $Q_0 \mapsto Q'_0$, $U \mapsto Q' \cap U$, $f \mapsto f|Q'$, and $\varepsilon \mapsto \eta$; let $\delta \in (0, \eta]$ be the number (Q' is compact) thus obtained. We show that if η is small enough, then δ satisfies 4.8.

Thus, let f_0 and f_1 be as in 4.8. Then for $i = 0, 1$ the embedding $f_i|Q$ is locally flat by 2.2(a), whence the embedding $f'_i = f_i|\colon Q' \to M$ with $f'^{-1}_i \partial M = Q'_0$ is locally flat by 2.2(b). Further, $d(f'_i, f|Q') < \delta$ and $f'_0 = f'_1$ on $Q' \cap U$. Hence, as $Q' \smallsetminus U = Q' \cap \hat{I}^n(\frac{1}{2})$, we obtain a homeomorphism $\alpha\colon M \to M$ such that $d(\alpha, \mathrm{id}) < \eta$, that $\alpha = \mathrm{id}$ outside $N(f[Q' \smallsetminus U], \eta) \subset N_0 \cap N(f\hat{I}^n(\frac{1}{2}), \varepsilon_1)$, that $\alpha f'_0 = f'_1$, and that $\alpha|\partial M = \mathrm{id}$ if $f'_0 = f'_1$ on Q'_0 and, thus, if $f_0|L = f_1|L$. Then

$\alpha f_0[\partial Q] = f_1[\partial Q]$ and thus $\alpha f_0 Q = f_1 Q$ by the Jordan–Brouwer separation theorem (applied twice in (c)–(d)). As $\mathrm{bd}_M f_i Q = f_i Q'$, the homeomorphism $f_1(\alpha f_0)^{-1}\colon \alpha f_0 Q \to f_1 Q$ can be extended by the identity to a homeomorphism $\beta\colon M \to M$. Then $\beta\alpha f_0 = f_1$ on Q, and $d(\beta, \mathrm{id}) \le d(f_1, \alpha f_0) \le d(\alpha, \mathrm{id}) + d(f_0, f_1) < \eta + 2\delta \le 3\eta$. Since $f_1 Q \subset N_0$, we have $\beta | M \smallsetminus N_0 = \mathrm{id}$. Moreover, if $f_0|L = f_1|L$, then $\alpha f_0 = f_0 = f_1$ on $Q \cap L$, and hence $\beta|\partial M = \mathrm{id}$. The homeomorphism $h = \beta\alpha\colon M \to M$ has thus the properties that $d(h, \mathrm{id}) \le d(\beta, \mathrm{id}) + d(\alpha, \mathrm{id}) < 4\eta$, that $h|M \smallsetminus N_0 = \mathrm{id}$, that $hf_0 = f_1$ on Q, and that $h|\partial M = \mathrm{id}$ if $f_0|L = f_1|L$. Since $A = U \smallsetminus Q$, since $\beta = \mathrm{id}$ on $M \smallsetminus \alpha f_0 Q$, and since $d(f_0 A, f\hat{I}^n(\frac{1}{2})) \ge \varepsilon_1$, we have $hf_0 = \beta\alpha f_0 = \alpha f_0 = f_0 = f_1$ on A. Hence, $hf_0 = f_1$ on $I^k \times \hat{I}^m(\frac{1}{2}) = Q \cup A$. Now, if η is small enough, then by the Chernavskiĭ–Edwards–Kirby deformation theory [16, 5.1 and Note] there is an ambient ε-isotopy $H = (h_t)_{t\in I}$ of M fixed outside $N(f\hat{I}^n(\frac{1}{2}), \varepsilon)$ such that $h_1 = h$ and such that H is fixed on ∂M if $h|\partial M = \mathrm{id}$. □

Corollary 4.12. *In the* TOP *case,* 4.1 *for a particular* $n \ge 1$ *and* $q = n$ *follows from the special case* $Q_0 = \partial Q$ *of* 4.1 *for the same* n *and* $q = n - 1$. *The same concerns the special cases of these conjectures where* $f_0 = f_1$ *on* $B \cap Q_0$. *The special case* $Q_0 = \emptyset$ *of* 4.1 *for* $q = n$ *follows from the special case* $\partial Q = \emptyset$ *of* 4.1 *for* $q = n - 1$.

PROOF. See 4.5, 4.10, and 4.11. □

Lemma 4.13. *Conjecture* 4.1 *holds for* CAT = TOP, $n = 2$, *and* $q = 1$.

PROOF. With the claim of 4.1 concerning H on ∂M removed if $Q_0 \ne \emptyset$, this version of 4.1 is proved by means of the Schoenflies theorem in [22, 4.2] (where require $N \cap \partial Q = \emptyset$ if $Q_0 = \emptyset$) through 4.8, which is contained in [22, 4.9, 4.10, and 4.14]. To get the removed claim, it thus suffices to observe that in [22, 4.13] the ambient isotopy of Q associated with the ε-push h of (Q, fI) can be chosen to be $\mathrm{rel}\, \partial Q$ if $f_1(0) = f_2(0)$; this yields 4.8(c) in full. □

Lemma 4.14. *Conjecture* 4.1 *holds for* CAT = TOP *and* $n = q \in \{1, 2\}$.

PROOF. This follows from 4.2(2) and 4.13 by 4.12. □

Lemma 4.15. *Conjecture* 4.1 *holds for* CAT = TOP, $n = 3$, *and* $q \in \{2, 3\}$.

PROOF. Here we do not wish to apply 4.12. We prove 4.1 in the form mentioned in 4.2(1). For $q = 2 = n - 1$ the version of 4.1 where the claim of 4.1 concerning H on ∂M is removed if $Q_0 \ne \emptyset$ is proved in [22, 4.2].

Special case: $A = B = Q$. As easily seen, f may be assumed to be closed. In [22, 4.5] the respective weaker form is deduced from a result of Craggs [10, 8.1]. We modify this proof and use the notation of [22], this time assuming that M is a q-manifold with $q \in \{2, 3\}$.

Thus, consider the proof of [22, 4.5]. If $q = 3$, then $L = f^{-1}\partial Q$ is a closed locally flat 2-submanifold of ∂M and thus locally tame in the sense of [4, p. 146], and hence we may assume by [4, Theorem 8] that L is a PL $(q-1)$-submanifold of ∂M in this case, too. We construct the locally PL flat PL q-submanifolds M_1 and M_2 of M as regular neighborhoods such that $M_2 \cap L$ and $M_2 \cap \overline{\partial M \setminus L}$ are PL $(q-1)$-manifolds and $M_2 \cap \partial L$ a PL $(q-2)$-manifold. By [10, 8.1] we can choose ν and G such that G is $\operatorname{rel} \partial Q$ if $g_1 = g_2$ on L. The embedding $f_1|M_2 = f_2|M_2$ is locally flat for each q by 2.2(a). If $q = 3$, but we do not suppose $f_1|L = f_2|L$, then the proof can be finished as in the case $q = 2$; it only suffices to observe that the modification of [22, 4.4] where M is a PL 3-manifold with $f^{-1}\partial Q \subset \partial M$ a PL 2-submanifold holds with the same proof.

Suppose now that $f_1 = f_2$ on L. We show that H can then be chosen to be $\operatorname{rel} \partial Q$. It suffices to show that G^1 and G^2 can be chosen such that $G^1_t = G^2_t$ on ∂Q for each $t \in I$. In fact, then $g_1 = g_2$ on L implying that G is $\operatorname{rel} \partial Q$, in which case such is H, too. Thus, consider the construction of G^1 and G^2 in [22, 4.4] with the substitutions $M \mapsto M$, $Q \mapsto Q$, $f \mapsto f'_i$, $A \mapsto Q \setminus N_1$, and $\varepsilon \mapsto \eta|N_1$ for $i = 1, 2$; endow accordingly the objects constructed in the proof with the superscript i. Since $f'_1 = f'_2$ on L, we can choose N^i, K^i_0, and $K'^i = K^i_1|\partial Q$ to be the same for both i. The proof of [4, Theorem 4] reveals that the PL homeomorphisms g^1 and g^2 can then be chosen to coincide on ∂Q. Now H^1 and H^2 can be chosen to coincide on ∂Q. If not already from Kister's isotopy results used in [22], this follows from the Chernavskiĭ–Edwards–Kirby deformation theory: see [23, 3.2(6) and 7.7] or alternatively after the construction of H^1 choose by [16, 6.2 and Note after 5.1] an ambient isotopy H^0 of Q $\operatorname{rel} A \cup \partial Q$ which is small on $Q \setminus A$ and for which $H^0_1 = g^2(g^1)^{-1}$, and then define $H^2_t = H^0_t H^1_t$.

General case. As for the weaker form in [22], this is obtained by 4.9(1), 4.10, and 4.5 (see 4.2(1)) from the special case above. We provide here a more direct alternative.

Let N^* and ε^* be as in 4.2(1). Then $B^* = U \cup f^{-1}N^*$ is a neighborhood of A in Q with $f[B^* \setminus U] \subset N^*$. The construction of M_2 above shows that there is a closed neighborhood P of A in Q such that $P \subset B \cap B^*$, such that P is a q-manifold with $P \cap Q_0$ a locally flat $(q-1)$-submanifold of ∂P, and such that $g|P$ is locally flat for each locally flat embedding $g\colon Q \to M$ with $g^{-1}\partial M = Q_0$. Let $\delta' \in C_+(P)$ be the function given by the special case with the substitutions $Q \mapsto P$, $Q_0 \mapsto P \cap Q_0$, $U \mapsto U \cap P$, and $f \mapsto f|P$. Choose

$\delta \in C_+(Q)$ such that $\delta|P \le \delta'$ and $N(A,\delta) \subset P$; then δ satisfies 4.1. $\square$

The following result is related to the case $q = 1$ of a result of Gluck [18, 1.2] in the range $n \ge 2q + 2$.
Lemma 4.16. *Conjecture* 4.1 *holds for* CAT = TOP, $q = 1$, *and* $n \ge 4$.

PROOF. By 4.2(3), 4.5, and 4.10 it suffices to prove 4.8 assuming $f_0(0) = f_1(0)$ in part (c). We establish part (c) with $\delta = \frac{1}{3}\varepsilon$, and then comment on parts (a) and (b). Thus, suppose that $\delta_0 = \max(d(f_0,f), d(f_1,f)) < \frac{1}{3}\varepsilon$, and choose $\varepsilon_0 > 0$ with $\varepsilon_0 < \frac{1}{5}(\varepsilon - 3\delta_0)$. Let $x_0 = f_0(0) = f_1(0) \in \mathbb{R}^{n-1}$. Write $J = [\,0,1)$, $I^* = I^1_+(\frac{1}{2})$, and $N_0 = N(fI^*, \varepsilon)$.

We first show that we may assume that there is $t^* \in (0, \frac{1}{2})$, depending on (f_0, f_1), such that $f_0(t) = f_1(t) = (x_0, t)$ for each $t \in [\,0, t^*\,]$. By [37, Theorem 1.7.7] (where assume $\partial N \cap M = \partial N \cap \overline{M}$), for $i = 0,1$ there is an embedding $c_i\colon \mathbb{R}^{n-1} \times J \to \mathbb{R}^n_+$ with $c_i(x,0) = x$ for each $x \in \mathbb{R}^{n-1}$ such that $c_i^{-1} f_i I^1_+ = \{x_0\} \times J$. Then setting $\alpha_i(t) = f_i^{-1} c_i(x_0, t)$ yields a homeomorphism α_i of J onto $J_i = t_i J$ for some $t_i \in (0,1\,]$. Define open embeddings $c'_i, c''_i\colon \mathbb{R}^{n-1} \times J_i \to \mathbb{R}^n_+$ by $c'_i(x,t) = c_i(x, \alpha_i^{-1}(t))$ and $c''_i = \mathrm{id}$; then $c'_i(x_0,t) = f_i(t)$ for $t \in J_i$. Choose a continuous function $\eta\colon \mathbb{R}^n_+ \to \mathbb{R}^1_+$ with $\eta_i(x_0) > 0$ and $\eta_i \le \varepsilon_0$ such that $\eta_i(x) \le d(x, \mathbb{R}^n_+ \smallsetminus N_0)$ for each $x \in \mathbb{R}^n_+$. Now [20, A.1 on p. 40] with the substitutions CAT $\mapsto$ TOP, $(W,M) \mapsto (\mathbb{R}^n_+, \mathbb{R}^{n-1})$, $U \mapsto \mathbb{R}^{n-1} \times J_i$, $f \mapsto c''_i$, $g \mapsto c'_i$, $C \mapsto \emptyset$, $D \mapsto \{x_0\}$, and $\varepsilon \mapsto \eta_i$ yields an ambient isotopy $H^i = (h^i_t)_{t\in I}$ of $\mathbb{R}^n_+$ fixed on $\mathbb{R}^{n-1}$ such that $h^i_1 c'_i = c''_i$ near $(x_0, 0)$ and $|h^i_t(x) - x| \le \eta_i(x)$ for all $x \in \mathbb{R}^n_+$ and $t \in I$. Then there is $t^* \in (0, \min(t_0, t_1, \frac{1}{2}))$ such that $h^i_1(f_i(t)) = (x_0, t)$ for $t \in [\,0, t^*\,]$ and $i = 0,1$. Since $d(h^i_1 f_i, f) \le \varepsilon_0 + \delta_0 < \frac{1}{3}(\varepsilon - 2\varepsilon_0)$, we may replace f_i by $h^i_1 f_i$ and thus arrive at the desired situation.

We next show that we may assume that, in addition, $f_i|I^*$ is PL for $i = 0,1$. We may assume that $\mathrm{cl}\, N(f_i[\,t^*, \frac{1}{2}\,], \varepsilon_0) \subset N_0 \cap \mathrm{Int}\, \mathbb{R}^n_+$ for $i = 0,1$. Then an application of [37, Corollary 3.6.1] with the substitutions $k \mapsto q = 1$, $n \mapsto n$ $(\ge 2q+2)$, $M^k \mapsto (0, \frac{1}{2}\,]$, $R \mapsto (0, t^*\,]$, $M^n \mapsto \mathrm{Int}\, \mathbb{R}^n_+$, $f \mapsto f_i|(0, \frac{1}{2}\,]$, and $\varepsilon \mapsto .9\varepsilon_0$ yields an ambient ε_0-isotopy $H^i = (h^i_t)_{t\in I}$ of $\mathbb{R}^n_+$ fixed on $(\mathbb{R}^n_+ \smallsetminus N_0) \cup \mathbb{R}^{n-1} \cup f_i[\,0, t^*\,]$ such that $h^i_1 f_i|I^*$ is PL. Now we may replace f_i by $h^i_1 f_i$ for $i = 0,1$ to reach the desired situation.

To complete the proof we need the modification of [37, Theorem 3.5.1] where $E^n = \mathbb{R}^n$ is replaced by $E^n_+ = \mathbb{R}^n_+$, where $fL \cup gL \subset \mathrm{Int}\, \mathbb{R}^n_+$, and where the resulting ambient isotopies of $\mathbb{R}^n_+$ are fixed on $\mathbb{R}^{n-1}$; the proof of this modification only requires some obvious, small modifications in the original proof. Since $d(f_0, f_1) < 2\delta = \frac{2}{3}\varepsilon$ and $N(f_0 I^*, 2\delta) \subset N_0$, we can indeed apply this result with the substitutions $k \mapsto q = 1$, $n \mapsto n$ $(\ge 2q+2)$, $P \mapsto I^*$, $L \mapsto [\,t^*, \frac{1}{2}\,]$, $f \mapsto f_0|I^*$, $g \mapsto f_1|I^*$, and $\varepsilon \mapsto 2\delta$ to find an ambient ε-isotopy

$H = (h_t)_{t\in I}$ of $\mathbb{R}^n_+$ fixed on $(\mathbb{R}^n_+ \smallsetminus N_0) \cup \mathbb{R}^{n-1} \cup f_0[\,0, t^*\,]$ such that $h_1 f_0 = f_1$ on I^*.

Parts (a) with $k = 0$ and (b) are easier (and the same). In (a) with $k = 1$ choose $a \in (\frac{1}{2}, 1)$ such that $N(fR, \frac{1}{3}\varepsilon) \subset N_0 = N(fI^1(\frac{1}{2}), \varepsilon)$ for $R = I^1(a) \smallsetminus \operatorname{Int} I^1(\frac{1}{2})$ and let $b = d(fI^1(\frac{1}{2}), f[I^1 \smallsetminus I^1(a)])$; then the conjecture can be established with $\delta = \min(\frac{1}{3}\varepsilon, \frac{1}{4}b)$. In fact, since $f_0R \subset N_0$ and $N(f_iI^1(\frac{1}{2}), 2\delta) \subset N_0 \smallsetminus f_i[I^1 \smallsetminus I^1(a)]$, we can make, retaining the assumption that $f_0 = f_1$ on $I^1 \smallsetminus \operatorname{Int} I^1(\frac{1}{2})$, first $f_0|R$ and then $f_i|I^1(a)$ for $i = 0, 1$ PL, and in this situation construct the desired isotopy. $\square$

Codimension three case 4.17. Consider 4.1 with CAT = TOP for $q \geq 1$ and $n \geq \max(q + 3, 5)$. We now prove this statement. By 4.5 it suffices to establish 4.4 for these same q and n. We deduce 4.4 in 4.19 from an equivalence result for locally tame embeddings due to Edwards [14, 8.3(1)]. The formulation of this result in regard to the boundaries of manifolds is sufficiently general and explicit to take into account locally flat embeddings in the sense we have adopted. In [14] the domain X is an arbitrary polyhedron with the inverse image Y of the boundary of the domain topological manifold Q a subpolyhedron, but for us it would be enough to assume, as we indicate in the proof of 4.19, that X is a PL manifold, Y a codimension zero PL submanifold of ∂X, and, moreover, Q a PL manifold. However, even then we must possibly choose the subpolyhedron Z of X in [14] to be the union of Y and a codimension zero PL submanifold of X and thus a genuine polyhedron. The statement of [14, 8.3(1)] should apparently contain the same assumption corresponding to our assumption $n \neq 4$ ($\dim Q \neq 4$ in [14]) as the characterization [14, 8.2] of tameness contains. See also 4.18. An ingredient of [14, 8.3(1)] is the PL equivalence result [14, 6.1], which generalizes [8, Theorem 1] and [30, Theorem 2]. By 4.2(3), 4.16, and 4.26(2) we could assume in 4.1 the coincidence of f_0 and f_1 on $B \cap Q_0$ whenever $n = 5$; perhaps this assumption (whose counterpart in [14, 8.3(1)], and correspondingly in [14, 6.1], too, would be that $Y \subset Z$ whenever $\dim \partial Q = 4$) in addition to $n \neq 4$ would be needed for the application of [14] (namely, for the proposed proof of [14, 6.1] by engulfing methods).

Possibly an earlier result of Miller [30, Theorem 4], which is a simplified version of 4.1 with Q a manifold, admits a stronger formulation which would have been just suitable for us, but we do not try to establish such as we already have [14] at our disposal. Matveev [29, Theorem 4 and Remark], based on a result of Chernavskiĭ [29, Lemma 3], has proved a version of 4.1 in the case of manifolds without boundary. In fact, 4.8(a) follows from the slightly stronger handle lemma [29, Theorem 1]. By 4.9(2) and 4.10 this implies 4.4(a) with $G_0 = \emptyset$ and 4.4(b) (it is easy to see that now $\mathbb{R}^q_{++}$ could be replaced by $\mathbb{R}^q_+$ and $\mathbb{R}^n_+$ by $\mathbb{R}^n$), which results suffice in 4.5 to prove 4.1 with $Q_0 = \emptyset$. It seems that [29, Theorem 2] is formulated in a too strong way, with just the special case $\partial G = \emptyset$ of 4.4 being a correct formulation of it.

Remark 4.18. For [14, 8.3] it is needed that in [14, 8.2] it suffices to assume in the case $\dim \partial Q = 4$ that, at least if there in (3) the isotopy is not required to be PL on $Q \times [0,1)$, the embedding $f|Y$ is merely locally tame in the sense of [14, p. 176], not necessarily PL; cf.[8, Corollary 2] and [37, Theorem 5.5.1]. Indeed, we reduce this modification of [14, 8.2] to the original version in the following.

Thus, suppose that $\dim \partial Q = 4$ and that $f_Y = f|\colon Y \to \partial Q$ is locally tame. We may assume f to be closed. Consider $\eta \in C_+(Q)$ and $\delta \in C_+(\partial Q)$. By [37, Theorem 3.6.1] with the substitutions $k \mapsto 1$, $n \mapsto 4$, $P^k \mapsto Y$, $R \mapsto Y \cap Z$, $M^n \mapsto \partial Q$, $f \mapsto f_Y$, $\widetilde{P}^k \mapsto Y$, $h \mapsto \mathrm{id}$, and $\varepsilon \mapsto \delta$ there is an ambient δ-isotopy $G = (g_t)_{t \in I}$ of ∂Q fixed outside $N(f[Y \smallsetminus Z], \delta)$ and on $fZ \cap \partial Q = f[Y \cap Z]$ such that $g_1 f_Y$ is PL. For δ small enough, G can be extended to an ambient η-isotopy $G' = (g'_t)_{t \in I}$ of Q fixed outside $N(f[X \smallsetminus Z], \eta)$ and on fZ. For η small enough, we may replace (f, ε) by $(g'_1 f, \eta)$ and thus assume that $f|Y$ is PL.

Lemma 4.19. *Local conjecture* 4.4 *holds for* $n \geq \max(q+3, 5)$.

PROOF. We first prove 4.4(a). Choose a compact polyhedron X in G and $\varepsilon_0 \in (0, \varepsilon)$ such that $B \subset X$ and $N(fX, \varepsilon_0) \subset N(fB, \varepsilon)$, and let $Y = X \cap G_0$. Choose a compact polyhedral neighborhood Z_0 of $\overline{A} \cap X$ in X with $Z_0 \subset U$. (By replacing X by a second derived neighborhood of X in a sufficiently fine rectilinear triangulation of (G, G_0, X) we could assume that X is a PL q-manifold and Y a PL $(q-1)$-submanifold of ∂X; similarly we could then take Z_0 to be a PL q-submanifold of X.) As $\overline{X \smallsetminus Z_0} \cap \overline{A} = \emptyset$, there is $\varepsilon_1 \in (0, \varepsilon_0)$ with $N(f[X \smallsetminus Z_0], \varepsilon_1) \cap N(fA, \varepsilon_1) = \emptyset$. If $f_0|G_0 = f_1|G_0$, let $Z = Z_0 \cup Y$; otherwise, let $Z = Z_0$. Now let $\delta \in (0, \varepsilon_1)$ be the number given by [14, 8.3(1)] with the substitutions $X \mapsto X$, $Y \mapsto Y$, $Q \mapsto \mathbb{R}^n_+$, $f \mapsto f|X$, $Z \mapsto Z$, and $\varepsilon \mapsto \varepsilon_1$. We show this δ to satisfy 4.4(a).

Thus, let f_0 and f_1 be as in 4.4(a). We first verify that for $i = 0, 1$ the embedding $g_i = f_i|X$ with $g_i^{-1}\mathbb{R}^{n-1} = Y$ is locally tame in the sense of [14]. For this purpose we may identify $\mathbb{R}^q_{++}$ with $\mathbb{R}^{n,q}_{++}$ through the homeomorphism $x \mapsto (0, x)$; then $G_0 = G \cap \mathbb{R}^{n-1}$. If $x \in X$, then there are an open neighborhood V of x in $\mathbb{R}^n_+$ and an extension of $g_i|V \cap X$ to an open embedding $\overline{g}_i\colon V \to \mathbb{R}^n_+$; now $g_i|\colon V \cap X \to \overline{g}_i V$ is a PL embedding whenever $\overline{g}_i V$ has the PL manifold structure which makes $\overline{g}_i\colon V \to \overline{g}_i V$ a PL homeomorphism; hence, g_i is locally tame. We further have $d(g_i, f|X) < \delta$ and $g_0|Z = g_1|Z$. Thus, by the choice of δ there is an ambient ε-isotopy $H = (h_t)_{t \in I}$ of $\mathbb{R}^n_+$ fixed on $g_0 Z$ and outside $N(f[X \smallsetminus Z], \varepsilon_1)$ such that $h_1 g_0 = g_1$. Then H is fixed outside $N(fB, \varepsilon)$. As $\delta < \varepsilon_1$ and thus $f_0 A \subset N(fA, \varepsilon_1)$, it follows that H is fixed on $f_0 A$. Hence, $h_1 f_0 = f_1$ on $A \cup B$. If $f_0|G_0 = f_1|G_0$, then $Y \subset Z$, and thus H can be chosen to be fixed on $\mathbb{R}^{n-1}$.

To obtain of the above proof a proof for 4.4(b), replace $\mathbb{R}^n_+$ by $\mathbb{R}^n$ and delete G_0, Y, and all conditions involving them (but for [14] use the substitution $Y \mapsto \emptyset$). $\square$

Corollary 4.20. *Conjecture* 4.1 *holds for* CAT = TOP *and* $n \geq \max(q+3, 5)$.

PROOF. This follows from 4.19 by 4.5. $\square$

Lemma 4.21. *Conjecture* 4.1 *holds for* CAT = TOP *and* $n = q + 1 \geq 5$ *whenever* $Q_0 = \emptyset$.

PROOF. This is proved in [22, 4.1]. The result is due to Price and Seebeck [32]: the absolute case $A = B = Q$ and $U = \emptyset$ is [32, Theorem 3], and 4.8(a), which suffices by 4.9(2), 4.10, and 4.5, is [32, Theorem 2]. $\square$

Lemma 4.22. *Conjecture* 4.1 *holds for* CAT = TOP *and* $n = q \geq 5$ *whenever* $\partial Q_0 = \emptyset$.

PROOF. The special case $Q_0 = \emptyset$ follows from 4.21 by 4.12. By 4.9(1) and 4.10 this implies that 4.4(a) holds in the case $G_0 = \emptyset$. Recall from 4.6 that 4.4(a) holds in the case $G_0 = \partial G$, too. The lemma now follows by 4.5(3). $\square$

Lemma 4.23. *For* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$*, fixed* n *and* $q \geq 1$*, and* $R \in \{Q_0, \partial Q_0, \partial Q \setminus \operatorname{Int} Q_0\}$*, the special case* $A \cap R \subset U$ *of* 4.1 *follows from the special case* $R = \emptyset$ *of* 4.1*. The same concerns the special cases of these conjectures where* $f_0 = f_1$ *on* $B \cap Q_0$.

PROOF. For $q \geq 1$, note that $Q_0 = \partial Q$ if and only if $Q_1 = \partial Q \setminus \operatorname{Int} Q_0$ is empty. We may assume that B is closed and $B \cap R \subset U$. Choose open neighborhoods U_1 of $B \cap R$ and U_0 of $\overline{U_1}$ in Q with $\overline{U_0} \subset U$. Since $f[B \cap \overline{U_0}]$ and $f[A \setminus U]$ are disjoint closed subsets of fQ, there is an open neighborhood N^* of $f[A \setminus U]$ in M with $N^* \subset N(f[A \setminus U], \varepsilon)$ such that $f[B \cap \overline{U_0}] \cap \overline{N^*} = \emptyset$ and such that $N^* \subset \operatorname{Int} M$ if $R = Q_0$. Let $\delta^* \in C_+(Q \setminus R)$ be the function obtained when applying the special case $R = \emptyset$ of 4.1 with the substitutions $M \mapsto M$, $Q \mapsto Q \setminus R$, $Q_0 \mapsto Q_0 \setminus R$, $A \mapsto A \setminus U_0$, $B \mapsto B \setminus U_1$, $U \mapsto U \setminus R$, $f \mapsto f|Q \setminus R$, and $\varepsilon \mapsto \varepsilon$; here $(A \setminus U_0) \setminus (U \setminus R) = A \setminus U$. By 4.2(1) we may assume that δ^* works even when $N(f[A \setminus U], \varepsilon)$ is replaced by N^*. Choose $\delta \in C_+(Q)$ such that $\delta \leq \delta^*$ on $B \setminus U_1$, that $\delta(x) \leq d(f(x), \overline{N^*})$ if $x \in B \cap \overline{U_0}$, and that $N(A, \delta) \subset N(A \setminus U_0, \delta^*) \cup (B \cap U_0)$. We show this δ to satisfy 4.1.

Thus, let f_0 and f_1 be as in 4.1. Then there is an ambient CAT ε-isotopy $H = (h_t)_{t\in I}$ of M fixed on $M \setminus N^*$, and thus on ∂M if $R = Q_0$, such that $h_1 f_0 = f_1$ on $N(A \setminus U_0, \delta^*)$ and such that H is fixed on ∂M if $R \in \{\partial Q_0, Q_1\}$ and $f_0 = f_1$ on $B \cap Q_0$, with the modification $(*)$ of 4.1, where $Q_0 \subset Q \setminus R$. If $x \in B \cap U_0$, then $f_0(x) \notin N^*$, and so $h_1 f_0(x) = f_0(x) = f_1(x)$. Hence $h_1 f_0 = f_1$ on $N(A, \delta)$. □

Theorem 4.24. *Equivalence conjecture* 4.1 *holds for* CAT = TOP *provided that $(n, q) \neq (4, 3)$, that $A \cap Q_0 \subset U$ if $n = q+1 \geq 5$, that $A \cap (\partial Q \setminus \operatorname{Int} Q_0) \subset U$ if $n = q = 4$, and that $A \cap \partial Q_0 \subset U$ if $n = q \geq 5$.*

PROOF. See 4.2(2), 4.7, 4.13–4.16, and 4.20–4.23. □

Theorem 4.25. *Equivalence conjecture* 4.1 *holds for* CAT $\in$ {LIP, LQS} *provided that $(n, q) \neq (4, 1)$, that in the case $(n, q) = (5, 2)$ only pairs (f_0, f_1) with $f_0 = f_1$ on $B \cap Q_0$ are considered, that $(n, q) \neq (4, 3)$, that $A \cap Q_0 \subset U$ if $n = q+1 \geq 5$, that $A \cap (\partial Q \setminus \operatorname{Int} Q_0) \subset U$ if $n = q = 4$, and that $A \cap \partial Q_0 \subset U$ if $n = q \geq 5$.*

PROOF. Assuming $n \neq 4$ and that $f_0 = f_1$ on $B \cap Q_0$ if $n = 5$ this follows from 4.24 by 4.3 and 4.23. If $n = q + 1 = 5$, then we may assume that $B \cap Q_0 \subset U$, and hence the assumption on the pairs (f_0, f_1) can trivially be removed. In the case $n = q = 5$ we first conclude that 4.1 holds if $Q_0 = \emptyset$, then from this by 4.9(1) and 4.10 that 4.4(a) holds if $G_0 = \emptyset$, and finally by this, 4.6, and 4.5(3) that 4.1 holds if $\partial Q_0 = \emptyset$, with the assumption on the pairs (f_0, f_1) removed. The rest of the theorem follows from 4.2(2), 4.2(3), 4.7, and 4.23. □

Remarks 4.26. (1) For $n = q + 1$ the case $Q_0 = \emptyset$ of 4.25 is proved in [22, 4.1], but with a weaker definition of CAT isotopies, and a weak form of the case $Q_0 \neq \emptyset$ and $n \leq 3$ is proved in [22, 4.2].

(2) Conjecture 4.1 for CAT $\in$ {TOP, LIP} and a pair (n, q) with $q \geq 1$ can be reduced to its special case where $f_0 = f_1$ on $B \cap Q_0$ whenever the special case $Q_0 = \emptyset$ of 4.1 holds for the same CAT and with $(n-1, q-1)$ in place of (n, q). Here we may assume by 4.9(1) that $A = B = Q$ and hence that, in addition, f is closed; now the assertion follows by applying the assumed result to the embedding $f|\colon Q_0 \to \partial M$ and by extending the resulting small ambient CAT isotopy of ∂M to one of M with the aid of a narrow CAT collar of ∂M in M (see [27, 7.5 and 3.3] for LIP collars). Thus, the only real difference between the LIP case of 4.25 and 4.24 that remains is in the dimensions $(n, q) = (4, 1)$.

(3) By (2) and the proof of 4.24, the TOP case of Conjecture 4.1 would follow in full for all $n \geq 4$ and $q \in \{n, n-1\}$ from the weaker conjecture that parts

(c)–(d) of 4.8 hold with $f_0 = f_1$ on $f^{-1}\mathbb{R}^{n-1}$ for all $n \geq 4$ and $q = n-1$ and that parts (a)–(b) hold for $(n,q) = (4,3)$. By 4.3 the latter conjecture with $n \geq 5$ would imply the previous one for all CAT with $n \geq 5$ and with $f_0 = f_1$ on $B \cap Q_0$ if $n = 5$. We note that the extension and uniqueness theorems for boundary collars in [3] cannot be improved in any elementary way (causing thus no dimensional restrictions) to the (ε, δ)-versions which would be needed to reduce this last conjecture to 4.21; in fact, the validity of these versions would give a contradiction with the failure of 4.8 for $n = q+2$.

5 Approximation

In this section we try to establish as much as possible of the following two conjectures. For $n = q+1 \neq 4$ and $Q_0 = \emptyset$ they (the latter in the absolute case $A = \emptyset$) are the theorems [22, 5.1] and [22, 5.3], respectively. Our main results are Theorems 5.17–5.20.

Approximation conjecture 5.1. *Let* CAT $\in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$*, let* $0 \leq q \leq n$*, let* $n \neq q+2$ *if* $q \geq 2$*, let* $n \neq 4$ *if* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$*, let* M *be a* CAT *n-manifold and* Q *a* CAT *q-manifold, let* Q_0 *be a closed locally* CAT *flat* CAT *$(q-1)$-submanifold of* ∂Q *if* $q \geq 1$ *or a subset of* Q *if* $q = 0$*, let* $A \subset Q$ *be closed, let* $f\colon Q \to M$ *be a topological embedding with* $f^{-1}\partial M = Q_0$ *such that* f *is a locally* CAT *flat* CAT *embedding near* A*, and let* $\varepsilon \in C_+(Q)$*. If* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ *and* $n = 5$*, let the restriction* $f|\colon Q_0 \to \partial M$ *be a locally* CAT *flat* CAT *embedding. Then there is a locally* CAT *flat* CAT *embedding* $f^*\colon Q \to M$ *within* ε *of* f *such that* $f^{*-1}\partial M = Q_0$*, such that* $f^* = f$ *near* A*, and such that* $f^* = f$ *on* Q_0 *whenever* $f|\colon Q_0 \to \partial M$ *is a locally* CAT *flat* CAT *embedding.*

In 5.1 the assumptions on $n = 4$ and $n = 5$ are essential at least if $n = q$. In fact, from [13, Theorem 2] it follows by [41] that there is, cf. [13, Corollary (p. 183)], a homeomorphism $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ which cannot be approximated by LQS homeomorphisms and which thus cannot be approximated by LIP homeomorphisms, either; the same then concerns the homeomorphism $f \times \mathrm{id}\colon \mathbb{R}^5_+ = \mathbb{R}^4 \times \mathbb{R}^1_+ \to \mathbb{R}^5_+$ extending f, too.

The following conjecture on extending partial approximations combines Conjectures 4.1 and 5.1.

Extension conjecture 5.2. *Let* CAT*,* n*,* q*,* M*,* Q*,* Q_0*,* A*,* f*, and* ε *be as in* 5.1 *with* $(n,q) \neq (3,1)$*, let* $B \subset Q$ *be closed, let* $U \subset Q$ *be an open neighborhood of* B*, and let* $V \subset U$ *be an open neighborhood of* $A \cap U$*. Then there is* $\delta \in C_+(Q)$ *with the following property: If* $f_0\colon U \to M$ *is a locally* CAT *flat* CAT *embedding* $(\delta|U)$*-close to* $f|U$ *with* $f_0^{-1}\partial M = U \cap Q_0$*, if* $f_0 = f$ *on* V*, and if* $f_0 = f$ *on* $U \cap Q_0$ *whenever* CAT $\in \{\mathrm{LIP}, \mathrm{LQS}\}$ *and* $n = 5$*, then there is a locally* CAT *flat* CAT *embedding* $f^*\colon Q \to M$ *within* ε *of* f *such* *that $f^{*-1}\partial M = Q_0$, such that $f^* = f$ near A, such that $f^* = f_0$ near B, and such that $f^* = f$ on Q_0 whenever $f|\colon Q_0 \to \partial M$ is a locally* CAT *flat* CAT *embedding and $f_0 = f$ on $U \cap Q_0$.*

Lemma 5.3. *Extension conjecture* 5.2 *for particular* CAT*, n, and q follows from Approximation conjecture* 5.1 *and Equivalence conjecture* 4.1 *for the same* CAT*, n, and q. Here the absolute case $A = \emptyset$ of* 5.1 *suffices for the absolute case $A = \emptyset$ of* 5.2*. If $R \in \{Q_0, \partial Q_0, \partial Q \setminus \operatorname{Int} Q_0\}$, then the special cases $R = \emptyset$ and $R \subset A$ of* 5.2 *follow from the special cases $R = \emptyset$ or $R \subset A$ of* 5.1 *and $R = \emptyset$ or $A \cap R \subset U$ of* 4.1*, respectively.*

PROOF. By replacing M by an open submanifold containing fQ as a closed subset we may assume that f is closed. Choose open neighborhoods W_0 and W of A such that $f|W_0$ is a locally CAT flat CAT embedding and $\overline{W} \subset W_0$. We may assume that $V \subset W$. Choose a closed neighborhood B_0 of B with $B_0 \subset U$ and an open neighborhood B_1 of B with $\overline{B}_1 \subset \operatorname{int}_Q B_0$. In the case $R \subset A$ we have $B \cap (R \setminus V) = \emptyset$ and may thus assume that $\overline{B}_1 \cap R \subset V$. Consider $\eta \in C_+(M)$ and $\delta' \in C_+(Q)$, to be fixed later. Since $f[A \setminus B_1]$ and $f[\,\overline{B}_1 \setminus V]$ are disjoint closed sets in M, we may assume that for $N = N(f[\,\overline{B}_1 \setminus V], \eta)$ we have $f[A \setminus B_1] \cap \overline{N} = \emptyset$. Let $\delta_0 \in C_+(U)$ be the function given by 4.1 with the substitutions CAT $\mapsto$ CAT, $n \mapsto n$, $q \mapsto q$, $M \mapsto M$, $Q \mapsto U$, $Q_0 \mapsto U \cap Q_0$, $A \mapsto \overline{B}_1$, $B \mapsto B_0$, $U \mapsto V$, $f \mapsto f|U$, and $\varepsilon \mapsto \eta$. In the LQS case with $q = 1$ and $n \geq 2$, to eliminate here the need of the modification $(*)$ in 4.1, choose η to be so small that $f[Q_0 \setminus \overline{B}_1] \cap N = \emptyset$. Extend $\delta_0|B_0$ to $\delta_0' \in C_+(Q)$, and let $\delta = \min(\delta_0', \delta')$. We wish to show that δ satisfies 5.2. Thus, let f_0 be as in 5.2. By 5.1 choose a locally CAT flat CAT embedding $f'\colon Q \to M$ within δ of f such that $f'^{-1}\partial M = Q_0$, such that $f' = f$ on W, and such that $f' = f$ on Q_0 whenever $f|\colon Q_0 \to \partial M$ is a locally CAT flat CAT embedding. Then $f' = f_0$ on V, and $f' = f_0$ on $U \cap Q_0$ whenever CAT $\in \{$LIP, LQS$\}$ and $n = 5$. Hence, there is a CAT homeomorphism $h\colon M \to M$ within η of id such that $h = \mathrm{id}$ on $M \setminus N$, such that $hf' = f_0$ on $\overline{B}_1$ and thus $hf' = f$ on $B_1 \cap V$, and such that $h|\partial M = \mathrm{id}$ whenever $f_0 = f$ on $U \cap Q_0$ and $f' = f$ on Q_0, except possibly when CAT $=$ LQS, $q = 1$, and $n \geq 2$, in which case we still have $h = \mathrm{id}$ on $f'[Q_0 \setminus \overline{B}_1]$ implying, as $hf' = f_0 = f = f'$ on $Q_0 \cap \overline{B}_1$, that $h|f'Q_0 = \mathrm{id}$. On the open neighborhood $W_1 = W \setminus f^{-1}\overline{N}$ of $A \setminus B_1$ in Q we have $hf' = hf = f$, and thus $hf' = f$ on the open neighborhood $(B_1 \cap V) \cup W_1$ of A. Now, for η and δ' small enough, the embedding $f^* = hf'$ is ε-close to f and hence the desired one. $\square$

Remark 5.4. Consider 5.1 with $(n, q) \neq (3, 1)$. Then 5.2 is a generalization of 5.1 as seen by taking either $B = U = \emptyset$ or $A = V = \emptyset$ and $f_0 = f|U$ for a suitable U. Assuming 4.1, from the latter alternative it follows by 5.3 that 5.1 can be reduced to its special case $A = \emptyset$.

We now reduce the LIP and LQS cases of 5.1 to the TOP case of 5.1 with the aid of the taming theorem 3.1.

Lemma 5.5. *The* LIP *and* LQS *cases of Approximation conjecture* 5.1 *for particular n and q follow from the* TOP *case of* 5.1 *for the same n and q. The same concerns the special cases of these conjectures where $f|\colon Q_0 \to \partial M$ is a locally* CAT *flat* CAT *embedding for the respective category* CAT*. The special cases of* 5.1 *for* LIP *and* LQS *where $Q_0 = \emptyset$ or $\partial Q_0 = \emptyset$ or $Q_0 = \partial Q$ follow from the same respective special cases of* 5.1 *for* TOP.

PROOF. Let CAT $\in$ {LIP, LQS}. Say that (∂) holds if $f|\colon Q_0 \to \partial M$ is a locally CAT flat CAT embedding. By the TOP case of 5.1 there is a locally flat embedding $f_1\colon Q \to M$ within $\varepsilon/2$ of f such that $f_1^{-1}\partial M = Q_0$, such that $f_1 = f$ near A, and such that $f_1 = f$ on Q_0 if (∂) holds. Thus, by replacing (f, ε) by $(f_1, \varepsilon/2)$ we may assume f to be locally flat. We may assume f to be closed; then there is $\eta \in C_+(M)$ with $\eta f = \varepsilon$. Now by 3.1 there is a homeomorphism $h\colon M \to M$ within η of id such that hf is a locally CAT flat CAT embedding, such that $h = \mathrm{id}$ near fA, and such that $h|\partial M = \mathrm{id}$ if (∂) holds. Then $f^* = hf$ is the desired embedding. $\square$

Trivial cases 5.6. If $q = 0$, then 5.1 holds trivially with $f^* = f$ without any restrictions on n. If $1 \leq q \leq n \leq 2$, then the TOP case of 5.1 holds, by the Schoenflies theorem in the case $n = 2$, with $f^* = f$. If $n = q$ and $Q_0 = \partial Q$, then f is an open embedding, and in the TOP case we can thus again choose $f^* = f$. We now try to establish the rest of the TOP case of 5.1. We first prove three general lemmata.

Lemma 5.7. *The* TOP *case of* 5.1 *for $q \geq 1$ follows from the following two special cases of it*:

(a) *M is an open subset of $\mathbb{R}^n_+$ and Q an open subset of $\mathbb{R}^q_{++}$ with $Q_0 = Q \cap \mathbb{R}^{q-1}_+$.*

(b) (*for $q = 1$*) *M is an open subset of $\mathbb{R}^n$ and Q an open subset of $\mathbb{R}^1_+$ (in which case $Q_0 = \emptyset$).*

The same concerns the special cases of these conjectures where fQ_0 is locally flat in ∂M. The special cases of 5.1 *where* (1) *$Q_0 = \emptyset$ or* (2) *$Q_0 = \partial Q$ or* (3) *$\partial Q_0 = \emptyset$ follow from the special cases of* 5.1 *where M is an open subset of $\mathbb{R}^n_+$ and Q an open subset of $\mathbb{R}^q_+$ and where, respectively,* (1) *$Q_0 = \emptyset$ or* (2) *$Q_0 = \partial Q$ or* (3) *$Q_0 = \emptyset$ or $Q_0 = \partial Q$.*

PROOF. Choose an open neighborhood $U \subset Q$ of A such that $f|U$ is locally flat. There is a locally finite open cover $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}'$ of Q with $\mathcal{V}_0 = \{U\}$ such that, if $V \in \mathcal{V}'$, then $\overline{V}$ is compact in $Q \setminus A$, there is an open embedding $h_V\colon V \to \mathbb{R}^q_{++}$, with $h_V^{-1}\mathbb{R}^{q-1}_+ = V \cap Q_0$ whenever $q \geq 2$, and $f\overline{V}$ is contained in an open subset M_V of M such that there is an open embedding $h'_V\colon M_V \to T_V$ where $T_V = \mathbb{R}^n$ whenever $q = 1$ and $V \cap Q_0 = \emptyset$ and where $T_V = \mathbb{R}^n_+$ otherwise. By [31, 2.7] we may assume $\mathcal{V}'$ to have a disjoint decomposition $\mathcal{V}' = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{q+1}$ into disjoint families $\mathcal{V}_i$. We may assume that $\sup \varepsilon \overline{V} \leq d(f\overline{V}, M \smallsetminus M_V)$ if $V \in \mathcal{V}'$. We choose for each $V \in \mathcal{V}$ open sets $V = V^0 \supset \cdots \supset V^{q+1}$ such that $\overline{V}^i \subset V^{i-1}$ and such that $Q = \bigcup\{\, V^{q+1} \mid V \in \mathcal{V}\,\}$. Define open sets $D_j = \bigcup\{\, V^{j-i} \mid 0 \leq i \leq j,\, V \in \mathcal{V}_i \,\}$ for $0 \leq j \leq q+1$ and $D'_j = \bigcup\{\, V^{j-i+1} \mid 0 \leq i \leq j,\, V \in \mathcal{V}_i \,\}$ for $0 \leq j \leq q$. Then $D_0 = U$, $D_{q+1} = Q$, and for $1 \leq j \leq q+1$ we have $\overline{D}'_{j-1} \subset D_{j-1}$ and $D_j = D'_{j-1} \cup \bigcup \mathcal{V}_j$.

We construct embeddings $f_j\colon Q \to M$ for $0 \leq j \leq q+1$ with $f_0 = f$ such that $f_j^{-1}\partial M = Q_0$, that $f_j|D_j$ is locally flat, and, if $j \geq 1$, that $f_j = f_{j-1}$ on $D'_{j-1} \cup (M \smallsetminus \bigcup \mathcal{V}_j)$, that f_j is within $\varepsilon/(q+1)$ of f_{j-1}, and that $f_j = f_{j-1}$ on Q_0 whenever $f_{j-1}Q_0$ is locally flat in ∂M. Then $f^* = f_{q+1}$ is the desired locally flat embedding as $f^* = f$ on the open neighborhood(s) U^{q+1} (and $M \smallsetminus \bigcup\{\, \overline{V} \mid V \in \mathcal{V}' \,\}$) of A. Thus, let $1 \leq j \leq q+1$ be such that f_{j-1} is already defined. If $V \in \mathcal{V}_j$, then f_{j-1} defines an embedding $f_V\colon V \to M_V$ with $f_V^{-1}\partial M_V = V \cap Q_0$ such that $f_V|D_{j-1} \cap V$ is locally flat. Hence, the existence of f_j follows from Cases (a) and (b). □

Lemma 5.8. *Conjecture* 5.1 *for* CAT = TOP *and particular n and $q \geq 1$ follows with its last claim removed from its special case $Q_0 = \partial Q$ for the same n and q. The special case $Q_0 = \emptyset$ follows from the special case $\partial Q = \emptyset$.*

PROOF. Let $Q_1 = \partial Q \smallsetminus \operatorname{Int} Q_0$. By 5.7 we may assume that Q is an open subset of $\mathbb{R}^q_{++}$ and $Q_0 = Q \cap \mathbb{R}^{q-1}_+$ if $q \geq 2$ or $Q_0 = \emptyset$ if $q = 1$. Choose an open neighborhood A' of A in Q with f locally flat near $\overline{A}'$. By taking a second derived neighborhood of the simplicial neighborhood of $A \cap Q_1$ in a sufficiently fine rectilinear triangulation of $(Q_1, \partial Q_1)$ we find a neighborhood $N \subset A'$ of $A \cap Q_1$ in Q_1 such that N is a closed PL $(q-1)$-submanifold of Q_1 and $N_0 = N \cap \partial Q_1$ a PL $(q-2)$-submanifold of ∂N. Let $N_1 = \partial N \smallsetminus \operatorname{Int} N_0$. Then by [37, Theorem 1.7.7] (consider the union of Q and its image under the orthogonal reflection in $\mathbb{R}^{q-1}$) there are closed embeddings $c_0\colon N_1 \times [-1,2\,] \to Q_1$ and $c\colon Q_1 \times I \to Q$ such that $c_0(x,0) = x$ if $x \in N_1$ and $c(x,0) = x$ if $x \in Q_1$, such that $c_0^{-1}N = N_1 \times [-1,\, 0]$, such that $c_0^{-1}\partial Q_1 = \partial N_1 \times [-1,2\,]$ and $c^{-1}Q_0 = \partial Q_1 \times I$, such that $d(fC_x) < \min \frac{1}{2}\varepsilon C_x$ if $x \in Q_1$ and $C_x = c[\{x\} \times I]$, and such that the closed set $C = c[(Q_1 \smallsetminus (N \smallsetminus N_1)) \times I]$ does not meet A; now $c_0[N_1 \times (-1,2)]$ is open in Q_1 and $c[Q_1 \times [\,0,1)]$ open in Q. Construct a closed embedding $h\colon Q \to Q$ such that if either $y = c_0(x,s)$ for some $(x,s) \in N_1 \times I$ or $y \in Q_1 \smallsetminus (N \cup c_0[N_1 \times I])$ and $s = 1$, then $h(c(y,t)) = c(y, \frac{1}{2}(s+t))$ for each $t \in [\,0,s\,]$, and such that $h = \mathrm{id}$ elsewhere. Then $h[Q \smallsetminus N] \subset Q \smallsetminus Q_1$, $h|N = \mathrm{id}$, $h^{-1}Q_0 = Q_0$, and it is easy to see that hQ is locally flat in Q and that each point x of the set $S = hQ \cap \overline{Q \smallsetminus hQ} \cap \partial Q_0$ has an open neighborhood U in ∂Q such that $(U, U \cap Q_0, U \cap hQ, x)$ is homeomorphic to $(\mathbb{R}^{q-1}, \mathbb{R}^{q-1}_+, \mathbb{R}^{q-2}_+ \times \mathbb{R}^1, 0)$

if $q \geq 3$ while $S = \emptyset$ if $q \leq 2$. Since $(Q \setminus Q_1) \cap Q_0 = \operatorname{Int} Q_0 = \partial(Q \setminus Q_1)$, we can apply the special case to find an embedding $g\colon Q \to M$ such that $g|Q \setminus Q_1$ is locally flat, such that $g^{-1}\partial M = Q_0$, such that $d(g(h(x)), f(h(x))) < \frac{1}{2}\varepsilon(x)$ for each $x \in Q \setminus N$, and such that $g = f$ on $A' \cup Q_1$. Then the embedding $f^* = gh\colon Q \to M$ is ε-close to f and satisfies $f^{*-1}\partial M = Q_0$, and $f^* = f$ on the open neighborhood $A' \setminus C$ of A. The embeddings $g|h[Q \setminus N]$ and $g|A' \cap hQ$ are locally flat by 2.2(a). Hence, $g|hQ$ and thus f^*, too, are locally flat. $\square$

Lemma 5.9. *With* CAT = TOP*, consider* $1 \leq q \leq n$ *for which the following two conjectures hold*: 5.1 *with its last claim removed, and* 4.1 *with* $Q_0 = \emptyset$ *and with* $(n-1, q-1)$ *in place of* (n, q)*. Then* 5.1 *holds in the case where the embedding* $f_0 = f|\colon Q_0 \to \partial M$ *is locally flat.*

PROOF. We may assume f to be closed. Choose open neighborhoods U, V, and W of A with $f|U$ locally flat, $\overline{V} \subset U$, and $\overline{W} \subset V$. Consider $\eta \in C_+(M)$ and $\eta_0 \in C_+(\partial M)$, to be fixed later, such that if $N_0 = N(f[Q_0 \setminus V], \eta_0) \subset \partial M$, then $\overline{N_0} \cap f\overline{W} = \emptyset$. We can apply 4.1 with the substitutions $n \mapsto n-1$, $q \mapsto q-1$, $M \mapsto \partial M$, $Q \mapsto Q_0$, $Q_0 \mapsto \emptyset$, $A, B \mapsto Q_0$, $U \mapsto V \cap Q_0$, $f \mapsto f_0$, and $\varepsilon \mapsto \eta_0$; let $\delta_0 \in C_+(Q_0)$ be the function thus obtained. Consider $\delta \in C_+(Q)$, to be fixed later, with $\delta|Q_0 \leq \delta_0$. By 5.1 choose a locally flat embedding $g\colon Q \to M$ within δ of f such that $g^{-1}\partial M = Q_0$ and $g|V = f|V$. Then the definition of δ_0 applied to the locally flat embeddings $g_0 = g|\colon Q_0 \to \partial M$ and f_0 gives an ambient η_0-isotopy $H = (h_t)_{t\in I}$ of ∂M fixed outside N_0 such that $h_1 g_0 = f_0$. If η_0 is small enough, then we can extend h_1 with the aid of a narrow collar of ∂M in M to a homeomorphism $h\colon M \to M$ within η of id such that $h|fW = \mathrm{id}$. Now $f^* = hg\colon Q \to M$ is a locally flat embedding with $f^{*-1}\partial M = Q_0$ and with $f^* = f$ on $W \cup Q_0$. If η and δ are small enough, then f^* is ε-close to f and thus the desired embedding. $\square$

Lemma 5.10. *Conjecture* 5.1 *holds for* CAT = TOP*,* $n = 3$*, and* $1 \leq q \leq 3$*.*

PROOF. Note that now the embedding $f|\colon Q_0 \to \partial M$ is always locally flat. We may assume f to be closed. Choose an open neighborhood U of A such that $f|U$ is locally flat. We may assume M to be a PL manifold. Then fU is locally tame in M in the sense of [4]. Thus by applying [4, Theorem 9] to an open submanifold of M where fU is closed we can find a homeomorphism $h\colon M \to M$ with hfU a polyhedron. Hence we may assume fU to be a polyhedron. Choose a closed neighborhood A' of A with $A' \subset U$. We may assume Q to be a PL manifold. If $q \leq 2$, then Q_0 is a PL submanifold of Q, whereas if $q = 3$, then Q_0 is locally tame in Q, and thus again we may assume Q_0 to be a PL submanifold.

Suppose $q = 1$. We may assume that A' and $Q' = \overline{Q \setminus A'}$ are PL 1-submanifolds of Q. We may assume ε to be so small that all continuous $(\varepsilon|Q')$-approximations of $f|Q'$ are proper. Then applying [19, 4.8 (p. 102)] with the substitutions $P \mapsto Q'$, $P_0 \mapsto \partial Q'$, $Q \mapsto M$, $f \mapsto f|Q'$, $\{R_1, \ldots, R_N\} \mapsto \{fA', f[\partial Q']\}$, and $\varepsilon \mapsto \varepsilon|Q'$ yields a closed embedding $f^*\colon Q \to M$ within ε of f such that $f^* = f$ outside $\operatorname{Int} Q'$, such that $f^*|Q'$ is PL, and such that $f^{*-1}\partial M = Q_0$. Then f^*Q is a PL submanifold of M and f^* thus locally flat. As $f^* = f$ on $A' \cup \partial Q$, we have proved 5.1 completely in this case.

Suppose $q \in \{2, 3\}$. We first establish 5.1 with the last claim removed by using results of Bing [5]. If $q = 2$, apply [5, Theorem 5] with the substitutions $M \mapsto M$, $P \mapsto fQ$, $C \mapsto Q$, $g \mapsto f$, $f \mapsto \varepsilon f^{-1}$, and $K \mapsto fA'$ to obtain an embedding $h\colon fQ \to M$ within εf^{-1} of id for which hfQ is a polyhedron, $h|fA' = \mathrm{id}$, and $h^{-1}\partial M = fQ \cap \partial M = fQ_0$. (We remark that for the proof of [5, Theorem 5] we must apparently add the assumption that $C_0 = g^{-1}[P \cap \partial M]$ is a subpolyhedron of C, and that in the proof we must then choose the triangulation of C so as to induce a triangulation of C_0; then in the theorem we can choose h and $P' = hP$ such that $h[P \cap \partial M] = P' \cap \partial M$.) If $q = 3$, apply similarly [5, Theorem 10]. (The relativity part of this result is mentioned in the last two lines of its proof; note also that for us it is not necessary to construct h_1 as a PL embedding but only as an embedding mapping our Q onto a polyhedron. Regarding $g^{-1}[P \cap \partial M]$ and $h[P \cap \partial M]$, we must make a similar remark as above.) Now $f^* = hf\colon Q \to M$ is an embedding ε-close to f such that $f^{*-1}\partial M = Q_0$ and $f^*|A' = f|A'$. Then f^*Q is a PL q-submanifold of M, and $f^*Q \cap \partial M = f^*Q_0$ is a PL $(q-1)$-submanifold of ∂f^*Q as it is both a polyhedron and a topological $(q-1)$-submanifold. Hence f^*Q is a locally PL flat PL submanifold of M (consider links), and f^* thus the desired embedding. However, this proof leaves open the last claim of 5.1 that f^* can always be chosen to satisfy $f^*|Q_0 = f|Q_0$. In fact, it seems that the proofs of the applied results of [5] do not provide the needed condition $h|fQ_0 = \mathrm{id}$ even under the additional assumption that fQ_0 is a polyhedron, and this assumption may always be made, with a smaller neighborhood U of A'. (To see the possibility of this arrangement, choose a neighborhood P of fA' in fU such that P is a closed polyhedron in M; then the closed set $K = P \cup fQ_0$ is locally tame in M because fQ_0 is locally tame and because the neighborhood $fU \cap K$ of $P \cap fQ_0$ in K is a polyhedron as the union of the closed subpolyhedra P and $fU \cap \partial M$ of fU. Hence, by [4, Theorem 9] we may assume that K is a polyhedron implying that such is $fQ_0 = K \cap \partial M$, too.)

We obtain 5.1 for $q \in \{2, 3\}$ in full from 5.9 by the just established special case of 5.1 and 4.13–4.14. $\square$

Lemma 5.11. *Conjecture* 5.1 *holds for* CAT = TOP, $q = 1$, *and* $n \geq 4$.

PROOF. The following two modifications to the proof of the case $n = 3$ in 5.10 suffice. First, we may assume that M and Q are as in 5.7. Second, the methods of the proof of 4.16 allow us to assume that $f|U$ is even a PL embedding. $\square$

Codimension three case 5.12. Consider 5.1 for CAT = TOP in the case $n \geq q+3$ with $n \geq 5$. We deduce 5.1 in 5.13 from a locally tame approximation result of Edwards [14, 8.3(2)]. This result of [14] generalizes earlier locally flat approximation results due to Miller [30, Theorem 5] and Matveev [29, Theorem 3 and Remark] and later ones due to Cannon, Bryant, and Lacher [21, Section 7] and Quinn [33, 3.5.3]. As compared with these results, the result of [14] has a formulation in regard to the boundaries of manifolds which is sufficiently general and explicit to take into account locally flat embeddings in the sense we have adopted. Concerning X, Y, Q, and Z in [14] and in our application, we can repeat what we said in 4.17. The statement of [14, 8.3(2)] should apparently contain the same assumption corresponding to our assumption $n \neq 4$ ($\dim Q \neq 4$ in [14]) as the characterization [14, 8.2] of tameness contains. An ingredient of [14, 8.3(2)] is the PL approximation result [14, 8.1], which generalizes [30, Theorem 3], [6, Theorem 1], and [7, 3.1] (and which for $\dim Q \leq 5$ also follows from [19, 4.8 (p. 102)]). In 5.13 we also provide an alternative proof for 5.1 based on [15], allowing $n = 4$ if $Q_0 = \partial Q$.
Lemma 5.13. *Conjecture* 5.1 *holds for* CAT = TOP *and* $n \geq \max(q+3, 5)$.

PROOF. We may assume that $q \geq 1$. It suffices to consider the cases (a) and (b) of 5.7.

Case (a). We may assume f to be closed. Choose a closed neighborhood Z_0 of A in Q such that Z_0 is a polyhedron and such that f is locally flat near Z_0. (We could choose Z_0 to be a locally PL flat PL q-submanifold of Q such that $Z_0 \cap Q_0$ and $Z_0 \cap \overline{\partial Q \smallsetminus Q_0}$ are PL $(q-1)$-manifolds and $Z_0 \cap \partial Q_0$ a PL $(q-2)$-manifold.) If fQ_0 is locally flat in ∂M, let $Z = Z_0 \cup Q_0$; otherwise, let $Z = Z_0$. Then Z is a closed subpolyhedron of Q.

We show that the embedding $f|Z$ is locally tame in the sense of [14]. For this purpose we may identify $\mathbb{R}^q_{++}$ with $\mathbb{R}^{n;q}_{++}$ through the homeomorphism $x \mapsto (0, x)$; then $Q_0 = Q \cap \mathbb{R}^{n-1}$. It suffices to construct for each $x \in Z$ an open neighborhood U of x in $\mathbb{R}^n_+$ and an open embedding $g\colon U \to M$ extending $f|U \cap Z$, for then $f|\colon U \cap Z \to gU$ is a PL embedding whenever gU has the PL manifold structure which makes $g\colon U \to gU$ a PL homeomorphism. If $x \in Z_0$, then the existence of (U, g) follows from the local flatness of f near x. Consider $x \in Z \smallsetminus Z_0$. By the local flatness of fQ_0 in ∂M there are an open neighborhood U_0 of x in $\mathbb{R}^{n-1}$, an embedding $g_0\colon U_0 \to \partial M$ extending $f|U_0 \cap Q_0$, and $r > 0$ such that for the open neighborhood $U = U_0 \times [\,0, r)$ of x in $\mathbb{R}^n_+$ and the open embedding $g = g_0 \times (\mathrm{id}|[\,0, r))\colon U \to \mathbb{R}^n_+$ extending g_0

we have $U \cap Z_0 = \emptyset$, whence $U \cap Z = U_0 \cap Q_0$, and $gU \subset M$. Hence, (U, g) satisfies the requirements.

Now we can apply [14, 8.3(2)] with the substitutions $X \mapsto Q$, $Y \mapsto Q_0$, $Q \mapsto M$, $f \mapsto f$, $\varepsilon \mapsto \varepsilon$, and $Z \mapsto Z$ to obtain an embedding $f^*\colon Q \to M$ within ε of f such that f^* is locally tame in the sense of [14], that $f^{*-1}\partial M = Q_0$, and that $f^*|Z = f|Z$. Then $f^* = f$ near A, and $f^* = f$ on Q_0 whenever fQ_0 is locally flat in ∂M. Finally, by the local tameness of f^*, for each $x \in Q$ there are open neighborhoods $U \subset Q$ of x and $V \subset M$ of f^*U such that the embedding $f^*|\colon U \to V$ is PL with respect to some PL manifold structure on V; hence, f^* is locally flat by 2.2(c).

Case (b). The proof is now similar to that in Case (a); only replace $\mathbb{R}^n_+$ by $\mathbb{R}^n$ throughout.

An alternative proof. Allow $n = 4$. There is a closed neighborhood B of A such that fB and $f[B \cap Q_0]$ are 1-LCC in M or in ∂M, respectively, in the sense of [15]. Moreover, if fQ_0 is locally flat in ∂M, then fQ_0 and $f[B \cup Q_0]$ are 1-LCC in ∂M or in M, respectively. Hence, by [15, Approximation Theorem (pp. 95–96)] there is a 1-LCC embedding $f^*\colon Q \to M$ within ε of f such that $f^{*-1}\partial M = Q_0$, that f^*Q_0 is 1-LCC in ∂M, that $f^* = f$ near A, and that $f^* = f$ on Q_0 if fQ_0 is locally flat in ∂M.

It suffices to show that f^* is locally flat. If $n \geq 6$ or if $n = 5$ and fQ_0 is locally flat in ∂M, this follows from [8, Corollary 2]. But, at least if $Q_0 = \partial Q$, no restrictions concerning $n = 4$ and $n = 5$ are needed. In fact, now [33, 3.4.1] and its extension in [34] after [34, 2.5.1] first imply that f^*Q_0 is locally flat in ∂M if $n \geq 5$ (for $n = 4$ this is trivial) and then that f^* is locally flat for each $n \geq 4$. □

Lemma 5.14. *Conjecture* 5.1 *holds for* CAT = TOP *and* $n = q + 1 \geq 4$ *whenever* $Q_0 = \emptyset$.

PROOF. Suppose first $n \geq 5$. Ancel and Cannon [2] proved the absolute case $A = \emptyset$ of the lemma; an argument as for 5.8 dispenses with their original assumption $\partial Q = \emptyset$. The general case [22, 5.1] follows from this absolute case by 4.21 and 5.3; see 5.4. A new proof of the Ancel–Cannon theorem due to Cannon and, independently, Bryant and Lacher [9, p. 263] is reported in [21, Section 7]. Ancel [1, 2.2] has proved the special case $\partial Q = \emptyset$; his result also covers the dimension $n = 4$. Ferry [17], too, has proved the Ancel–Cannon theorem, with Q compact. The general case for all $n \geq 4$ follows from Ancel's result by 5.8. □

Lemma 5.15. *Conjecture* 5.1 *holds for* CAT = TOP *and* $n = q \geq 1$ *whenever in the case* $n = 5$ *either its last claim is removed or* $\partial Q_0 = \emptyset$.

PROOF. As seen in 5.6, the case $Q_0 = \partial Q$ is trivial. The special case $\partial Q = \emptyset$ of this and 5.8 imply the case $Q_0 = \emptyset$. The case $\partial Q_0 = \emptyset$ follows from these two cases by 5.7(3). From the case $Q_0 = \partial Q$ it follows by 5.8 that 5.1 holds whenever its last claim is removed. If $n \neq 5$, then 5.1 follows in full from this result by 4.24 and 5.9. $\square$

Lemma 5.16. *For* $\mathrm{CAT} \in \{\mathrm{TOP}, \mathrm{LIP}, \mathrm{LQS}\}$*, fixed* n *and* $q \geq 1$*, and* $R \in \{Q_0, \partial Q_0, \partial Q \setminus \operatorname{Int} Q_0\}$*, the special case* $R \subset A$ *of* 5.1 *follows from the special case* $R = \emptyset$ *of* 5.1.

PROOF. Choose an open neighborhood A' of A with f a locally CAT flat CAT embedding near $\overline{A}'$. Define $\varepsilon' \in C_+(Q \setminus R)$ by setting $\varepsilon'(x) = \min(\varepsilon(x), \frac{1}{2}d(f(x), fR))$. Let f_0^* be the embedding obtained when applying the special case $R = \emptyset$ of 5.1 with the substitutions $M \mapsto M$, $Q \mapsto Q \setminus R$, $Q_0 \mapsto Q_0 \setminus R$, $A \mapsto \overline{A}' \setminus R$, $f \mapsto f|Q \setminus R$, and $\varepsilon \mapsto \varepsilon'$. Then setting $f^* = f_0^* \cup (f|A')$ yields an embedding as $d(f^*(x), f^*(y)) > \frac{1}{2}d(f(x), f(y))$ for $x \in Q \setminus R$ and $y \in R$, and clearly f^* has the desired properties. $\square$

Theorem 5.17. *Approximation conjecture* 5.1 *holds for* $\mathrm{CAT} = \mathrm{TOP}$ *provided that* $Q_0 \subset A$ *if* $n = q + 1 \geq 4$ *and that in the case* $n = q = 5$ *either its last claim concerning* $f^*|Q_0$ *is removed or* $\partial Q_0 \subset A$.

PROOF. See 5.6, 5.10–5.11, and 5.13–5.16. $\square$

Theorem 5.18. *Approximation conjecture* 5.1 *holds for* $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$ *provided that* $Q_0 \subset A$ *if* $n = q + 1 \geq 5$ *and that* $\partial Q_0 \subset A$ *if* $n = q = 5$.

PROOF. This follows from 5.17 by 5.5 and 5.16. $\square$

Theorem 5.19. *Extension conjecture* 5.2 *holds for* $\mathrm{CAT} = \mathrm{TOP}$ *provided that* $(n, q) \neq (4, 3)$*, that* $Q_0 \subset A$ *if* $n = q + 1 \geq 5$*, that* $\partial Q \setminus \operatorname{Int} Q_0 \subset A$ *if* $n = q = 4$*, and that* $\partial Q_0 \subset A$ *if* $n = q \geq 5$.

PROOF. This follows from 5.17 and 4.24 by 5.3. $\square$

Theorem 5.20. *Extension conjecture* 5.2 *holds for* $\mathrm{CAT} \in \{\mathrm{LIP}, \mathrm{LQS}\}$ *provided that* $Q_0 \subset A$ *if* $n = q + 1 \geq 5$ *and that* $\partial Q_0 \subset A$ *if* $n = q \geq 5$.

PROOF. This follows from 5.18 and 4.25 by 5.3. $\square$

Remarks 5.21. (1) It follows from 5.8, 5.9, and 4.24 that 5.1 for CAT = TOP and $n = q + 1$ can be reduced to the case $Q_0 = \partial Q$ if $n = 4$ or $n \geq 6$.

(2) Results of Dancis [11,12] show that a proper continuous map of a q-manifold to an n-manifold can be approximated, taking boundaries into account, by locally flat embeddings if $n \geq \max(2q + 1, q + 3)$.

References

[1] F.D. Ancel, Resolving wild embeddings of codimension-one manifolds in manifolds of dimensions greater than 3, Topology Appl. 24 (1986) 13–40.

[2] F.D. Ancel and J.W. Cannon, The locally flat approximation of cell-like embedding relations, Ann. of Math. (2) 109 (1979) 61–86.

[3] M.A. Armstrong, Collars and concordances of topological manifolds, Comment. Math. Helv. 45 (1970) 119–128.

[4] R.H. Bing, Locally tame sets are tame, Ann. of Math. (2) 59 (1954) 145–158.

[5] R.H. Bing, An alternative proof that 3-manifolds can be triangulated, Ann. of Math. (2) 69 (1959) 37–65.

[6] J.L. Bryant, Approximating embeddings of polyhedra in codimension three, Trans. Amer. Math. Soc. 170 (1972) 85–95.

[7] J.L. Bryant, Triangulation and general position of PL diagrams, Topology Appl. 34 (1990) 211-233.

[8] J.L. Bryant and C.L. Seebeck III, Locally nice embeddings in codimension three, Quart. J. Math. Oxford Ser. (2) 21 (1970) 265–272.

[9] J.W. Cannon, J.L. Bryant, and R.C. Lacher, The structure of generalized manifolds having nonmanifold set of trivial dimension, in: J.C. Cantrell, ed., Geometric Topology, Proc. Georgia Topology Conf. (Athens, Ga., 1977) (Academic Press, New York, 1979) 261–300.

[10] R. Craggs, Small ambient isotopies of a 3-manifold which transform one embedding of a polyhedron into another, Fund. Math. 68 (1970) 225–256.

[11] J. Dancis, General position maps for topological manifolds in the 2/3rds range, Trans. Amer. Math. Soc. 216 (1976) 249–266.

[12] J. Dancis, General position maps for topological manifolds, in: J.C. Cantrell, ed., Geometric Topology, Proc. Georgia Topology Conf. (Athens, Ga., 1977) (Academic Press, New York, 1979) 305–321.

[13] S.K. Donaldson and D.P. Sullivan, Quasiconformal 4-manifolds, Acta Math. 163 (1989) 181–252.

[14] R.D. Edwards, The equivalence of close piecewise linear embeddings, General Topology Appl. 5 (1975) 147–180.

[15] R.D. Edwards, Approximating codimension $\geq$ 3 σ-compacta with locally homotopically unknotted embeddings, Topology Appl. 24 (1986) 95–122.

[16] R.D. Edwards and R.C. Kirby, Deformations of spaces of imbeddings, Ann. of Math. (2) 93 (1971) 63–88.

[17] S.C. Ferry, On the Ancel-Cannon theorem, Topology Proc. 17 (1992) 41–58.

[18] H. Gluck, Embeddings in the trivial range, Ann. of Math. (2) 81 (1965) 195–210.

[19] J.F.P. Hudson, Piecewise Linear Topology (Benjamin, New York, 1969).

[20] R.C. Kirby and L.C. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Ann. of Math. Stud. 88 (Princeton Univ. Press, Princeton, 1977).

[21] R.C. Lacher, Resolutions of generalized manifolds, in: K. Borsuk and A. Kirkor, eds., Proc. International Conf. on Geometric Topology (Warsaw 1978) (PWN, Warsaw, 1980) 277–292.

[22] J. Luukkainen, Topologically, quasiconformally or Lipschitz locally flat embeddings in codimension one, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983) 107–138.

[23] J. Luukkainen, Respectful deformation of bi-Lipschitz and quasisymmetric embeddings, Ann. Acad. Sci. Fenn. Ser. A I Math. 13 (1988) 137–177.

[24] J. Luukkainen, Respectful quasiconformal extension from dimension $n-1$ to n, Ark. Mat. 31 (1993) 355–376.

[25] J. Luukkainen, Lipschitz and quasiconformal approximation of homeomorphism pairs, Manuscript.

[26] J. Luukkainen and P. Tukia, Quasisymmetric and Lipschitz approximation of embeddings, Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981) 343–367.

[27] J. Luukkainen and J. Väisälä, Elements of Lipschitz topology, Ann. Acad. Sci. Fenn. Ser. A I Math. 3 (1977) 85–122.

[28] Y. Matsumoto, Wild embeddings of piecewise linear manifolds in codimension two, in: J.C. Cantrell, ed., Geometric Topology, Proc. Georgia Topology Conf. (Athens, Ga., 1977) (Academic Press, New York, 1979) 393–428.

[29] S.V. Matveev, Approximation of embeddings of manifolds by locally flat embeddings in codimension greater than 2, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 27:6 (1972) 53–57 (in Russian); English transl. in Moscow Univ. Math. Bull. 27:5–6 ((1972) 1973) 112–116.

[30] R.T. Miller, Approximating codimension 3 embeddings, Ann. of Math. (2) 95 (1972) 406–416.

[31] J.R. Munkres, Elementary Differential Topology, Ann. of Math. Stud. 54 (Princeton Univ. Press, Princeton, 1963).

[32] T.M. Price and C.L. Seebeck III, Somewhere locally flat codimension one manifolds with 1-ULC complements are locally flat, Trans. Amer. Math. Soc. 193 (1974) 111–122.

[33] F. Quinn, Ends of maps. I, Ann. of Math. (2) 110 (1979) 275–331.

[34] F. Quinn, Ends of maps. III: Dimensions 4 and 5, J. Differential Geom. 17 (1982) 503–521.

[35] C.P. Rourke, Improper embeddings of p.l. spheres and balls, Topology 6 (1967) 297–330.

[36] C.P. Rourke and B.J. Sanderson, Introduction to Piecewise-linear Topology (Springer, Berlin, 1972 (revised printing 1982)).

[37] T.B. Rushing, Topological Embeddings (Academic Press, New York, 1973).

[38] M.A. Shtan′ko, Nonexistence of locally flat approximation in codimension two, Studies in topology III, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 83 (1979) 93–100, 104 (Russian. English summary).

[39] L.C. Siebenmann, Deformation of homeomorphisms on stratified sets, Comment. Math. Helv. 47 (1972) 123–163.

[40] L. Siebenmann and D. Sullivan, On complexes that are Lipschitz manifolds, in: J.C. Cantrell, ed., Geometric Topology, Proc. Georgia Topology Conf. (Athens, Ga., 1977) (Academic Press, New York, 1979) 503–525.

[41] D. Sullivan, Hyperbolic geometry and homeomorphisms, in: J.C. Cantrell, ed., Geometric Topology, Proc. Georgia Topology Conf. (Athens, Ga., 1977) (Academic Press, New York, 1979) 543–555.

[42] P. Tukia and J. Väisälä, Lipschitz and quasiconformal approximation and extension, Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981) 303–342.