AN ISOMETRIC APPROACH TO STOCHASTIC INTEGRALS

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Abstract. The possibility to extend the classical Ito's construction of stochastic integrals is studied. This construction can be applied to fractional Brownian motions with Hurst index $H \in (0, 1/2)$. A change of variables formula for fractional Brownain motions in terms of the stochastic integrals is given.

1. Introduction, notation, definitions

1. (*Introduction*) Let x be a continuous stochastic process adapted to a filtration $\mathbf{F}$. We assume that $x_0 = 0$ and x is defined on some complete probability space (Ω, F, P). Adapted means that x_t is F_t measurable for each t; write this shortly as $x_t \in F_t$, where $\mathbf{F} = (F_t)_{t\geq 0}$.

Let $\pi^{(n)} \doteq \{t_i : 0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of the interval $[0, T]$. Given a partition $\pi^{(n)}$ and a stochastic process x, define Δx_i by $\Delta x_i \doteq x_{t_i} - x_{t_{i-1}}$ for $i \leq n$.

Take the integrand f to be a simple predictable process with respect to $\mathbf{F}$: $f_t = \sum_i \alpha_i 1_{(t_{i-1},t_i]}(t)$, where the random variables $\alpha_i \in F_{t_{i-1}}$ and $t_i \in \pi^{(n)}$; denote the class of simple predictable processes by $f \in L^s$. With such an $f \in L^s$ and any (continuous) process x, define stochastic integral of f with respect to x by

$$(f, x) \doteq \sum_i \alpha_i \Delta x_i.$$

Assume now that $\text{mesh}(\pi^{(n)}) \to 0$, as $n \to \infty$, where $\text{mesh}(\pi) \doteq \max_i |t_i - t_{i-1}|$, $t_i \in \pi$ and f is a limit of $f^{(n)} \in L^s$. Define now the stochastic integral (f, x) as the limit of the sequence $(f^{(n)}, x)$. Of course, the above procedure depends of the properties of the paths of the process x: if its paths are of bounded variation then the stochastic integral is usually the standard Lebesgue-Stieltjes-integral, if the paths are of unbounded variation, then a more careful definition is needed. For more details of this procedure in the case, when x is a semimartingale we refer to [10].

We study an isometric approach to define stochastic integrals. Our approach is a modification of the classical approach to define an Ito integral in the case when the process x is a standard Brownian motion. In our case the isometry is not a natural one and this will cause some peculiarities for the defined integrals. Nevertheless, we can give an Ito formula for fractional Brownian motions, even in the case when the process has infinite energy. When the process x is a fractional Brownian motion, some of the peculiar properties of the integral can be studied.

2. (*The isometry*) Denote by E^s the space of random variables $\xi = (f, x)$, where now the process $f \in L^s$ is deterministic: the coefficients $\alpha_i \in \mathbb{R}$ are constants. Introduce the following pre-norm on E^s:

$$||(f, x)||_{E^s} \doteq ||f||_{L^2}, \tag{1.1}$$

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where $||f||_{L^2}$ is the usual L^2 norm on $[0,T]$:

$$||f||_{L^2} \doteq \left(\int_0^T f_s^2 ds\right)^{\frac{1}{2}}.$$

Since f is a step function we can rewrite (1.1) as

$$||\sum_{i\le n} \alpha_i \Delta x_i||_{E^s} = \left(\sum_{i\le n} \alpha_i^2 (t_i - t_{i-1})\right)^{1/2}.$$

Let $\mathcal{I}$ be the mapping $\mathcal{I}(f) = (f,x)$, i.e. $\mathcal{I}$ is a surjection from L^s to E^s. It is easy to check that the following is a necessary and a sufficient condition for $\mathcal{I}$ to be a bijection and for $||\cdot||_{E^s}$ to be a norm on E^s:

(A) $\sum_{i\le n} \alpha_i \Delta x_i = 0$ if and only if $\alpha_i = 0$, $i = 0, \ldots, n$.

Note that without the condition **(A)** it may happen that $(f,x) = 0$ but $||(f,x)||_{E^s} > 0$. Our construction of the stochastic integral goes as follows. Assume that the process x satisfies the property (**A**). Then $\mathcal{I}$ is an isometry $L^s \mapsto E^s$. Since the space L^2 is complete we can use the isometry $\mathcal{I}$ to define integrals: define $\int_0^T f dx_s \doteq \lim_n \mathcal{I}(f^{(n)})$, when $f^{(n)} \xrightarrow{L^2} f$, $f^{(n)} \in L^s$.

3. (*Notation and definitions*) If the process f is deterministic, then the integral $\int_0^T f_s dx_s$ is called Wiener integral (of f respect to x). If f is stochastic, then this integral is called stochastic integral.

We denote $P-\lim$ for the limit in probability and if $(X, ||\cdot||_X)$ is a complete metric space, then the $X-\lim$ stands for the limit in the space X with respect the norm $||\cdot||_X$.

Denote by L^2 the space of measurable functions f with the property $||f||_{L^2} < \infty$ and $L^2(P\otimes Leb)$ is the space of predictable process f with the property $E\int_0^T f_s^2 ds < \infty$ and Leb is the Lebesgue measure. Finally $L^2(P)$ is the space of square integrable random variables.

4. (*The structure of this paper*)

In the next section we define Wiener integrals with respect to x. Then we define stochastic integrals with respect to x and we end this section by defining higher order integrals with respect to x. The third section deals with fractional Brownian motions. We show that with our approach we can define first and higher order integrals with respect to fractional Brownian motions. In this section compare our integrals with various other integrals suggested in the litterature and study some of the properties of our integrals.

The reason for defining higher order stochastic integrals is motivated by the following fact: if x is a fractional Brownian motion, then

$$\sum_{t_i\in\pi^{(n)}} |\Delta x_i|^p \xrightarrow{P} \infty$$

when $p < \frac{1}{H}$ and

$$\sum_{t_i\in\pi^{(n)}} |\Delta x_i|^p \xrightarrow{P} 0,$$

when $p > \frac{1}{H}$, as $\mathrm{mesh}(\pi^{(n)}) \to 0$ (see Section 3 for more information on the Hurst index H). We use the higher order integrals in the last section and use these to give an Ito formula for fractional Brownian motions when $H < 1/2$.

2. First and higher order integrals with respect to x

We start first with Wiener integrals. Then we will define the stochastic integrals and end with higher order integrals.

1. (*Basic setup for Wiener integrals*) Let $x = (x_t)_{t\in[0,T]}$ be a square integrable process with $Ex_t = 0$, $x_0 = 0$, and write $R(t,s)$ for the covariance function, $R(t,s) \doteq E(x_t x_s)$.

Consider the quadratic forms

$$B_n \doteq E\left((f,x)\right)^2,$$

where $f \in L^s$ has deterministic coefficients α_i, $1 \le i \le n$.

Then the condition **(A)** is equivalent to the following:

(B) The quadratic form B_n is positive definite for each $n \ge 1$.

We can write B_n in terms of the covariance function R:

$$\begin{aligned} B_n &= \sum_{i\le n} \Big[\alpha_i^2 \left(R(t_i,t_i) - 2R(t_{i-1},t_i) + R(t_{i-1},t_{i-1})\right)\Big] \\ &+ 2\sum_{i\ne j, i,j\le n} \alpha_i\alpha_j \Big[R(t_i,t_j) - R(t_{i-1},t_j) - R(t_i,t_{j-1}) + R(t_{i-1},t_{j-1})\Big] \end{aligned} \tag{2.1}$$

Put

$$\delta_{ii} \doteq R(t_i,t_i) - 2R(t_{i-1},t_i) + R(t_{i-1},t_{i-1})$$

and

$$\delta_{ij} \doteq R(t_i,t_j) - R(t_{i-1},t_j) - R(t_i,t_{j-1}) + R(t_{i-1},t_{j-1}).$$

Then the condition **(B)** is equivalent to the property that the matrix $(\delta_{ij})_{i,j\le n}$ is positive definite for each $n \ge 1$.

2. (*Definition of the Wiener integral with respect to x*) Assume that the condition **(A)** is valid for the process x and assume that $f \in L^2$. Then there exists $f^{(n)} \in L^s$ with deterministic coefficients such that $||f^{(n)} - f||_{L^2} \to 0$ as $n \to \infty$. Moreover, the sequence $(f^{(n)}, x)$ is a Cauchy sequence in the space $(E^s, ||\cdot||_{E^s})$. Complete E^s with respect the norm $||\cdot||_{E^s}$ and denote this Banach space by $\bar{E}$. Define now the integral $\int_0^T f_s dx_s$ as the limit of $(f^{(n)}, x)$ in the space $\bar{E}$. We say that $\int_0^T f_s dx_s$ is the generalized Wiener integral with respect to process x. Note that L^s in dense in L^2 and hence also E^s in dense in $\bar{E}$, by using the isometry $\mathcal{I}$.

We clarify the connection with random variables and Wiener integrals defined above. Let $\zeta^{(n)}$ be a sequence of random variables of the form

$$\zeta^{(n)} \doteq (f^{(n)}, x),$$

with some $f^{(n)} \in L^s$. Assume now that $\zeta = P - \lim_n \zeta^{(n)}$ and $||f - f^{(n)}||_{L^2} \to 0$, $n \to \infty$. We show later that it may happen that $P\{|\zeta| < \infty\} < 1$ or even $P\{|\zeta| < \infty\} = 0$. But even in the above situation the limit

$$\int_0^T f_s dx_s = \bar{E} - \lim_n (f^{(n)}, x)$$

defines the generalized Wiener integral. In this kind of situation we say that the random variable ζ is one of the representatives of $\int_0^T f_s dx_s$ in the space of random variables and $\int_0^T f_s dx_s$ is one of the representatives of the random variable ζ in the space $\bar{E}$; denote this by $\zeta \leftrightarrow \int_0^T f_s dx_s$.

3. (*Examples of Wiener integrals*) It is easy to check that if x is a process with non-correlated increments and with the property

$$Ex_t^2 > Ex_s^2, \tag{2.2}$$

where $s < t$, then the condition (2.1) is satisfied. Note first that the condition (2.2) is equivalent to the condition $E(x_t - x_s)^2 > 0$ for $s < t$. Since the process x has non-correlated increments, we have that

$$E\left(\sum_{i\le n}\alpha_i\Delta x_i\right)^2 = \sum_{i\le n}\alpha_i^2 E\left(\Delta x_i\right)^2 = 0$$

if and only if $\alpha_i = 0$, $i \le n$. Note that if x is a square integrable martingale and $Ex_t^2 > Ex_s^2$, $s < t$, then **(B)** is satisfied.

Similarly, if x is a stationary process with so called orthogonal vector measure $\varphi(d\lambda)$ such that the spectral measure $F(d\lambda) \doteq E|\varphi(d\lambda)|^2$ is equivalent to the Lebesgue measure, then the condition **(B)** is satisfied.

Example 2.1. ***(Standard Brownian motion)*** *Assume that x is the standard Brownian motion w. Then*

$$||\sum_{i\le n}\alpha_i\Delta w_i||_{E^s} = \left(\sum_{i\le n}\alpha_i^2\Delta t_i\right)^{1/2} = \left(E\left(\sum_{i\le n}\alpha_i\Delta w_i\right)^2\right)^{1/2}.$$

It follows that the generalized Wiener integral is isometrically equivalent to the standard Wiener integral, which can be defined as a limit of $(f^{(n)}, w)$ in the space $L^2(P)$ (see also subsection 3.3 for the $L^2(P)$-integrals).

One can easily extend the Example 2.1 for continuous square integrable martingales x with the angle bracket $< x >_t = \int_0^t a_s ds$, where $0 < K^{-1} \le Ea_s \le K$ for all $s \in [0,T]$.

4. (*Basic setup for stochastic integrals*) Let $x \in \mathbf{F}$ be a process and introduce the space G^s of random variables ξ:

$$\xi = \sum_{i\le n}\alpha_i\Delta x_i,$$

where $\alpha_i \in F_{t_{i-1}}$ and $\alpha_i \in L^2(P)$, $i = 1,\dots,n$, $n \ge 1$. Then $\xi \in \mathbf{F}$. Define a surjection $\mathcal{J}$ from $L^s \to G^s$ by

$$\mathcal{J}(f) \doteq (f,x) = \sum_{i\le n}\alpha_i\Delta x_i.$$

Introduce the following pre-norm on G^s:

$$||(f,x)||_{G^s} \doteq \left(E\sum_{i\le n}\left[\alpha_i^2(t_i - t_{i-1})\right]\right)^{1/2}. \tag{2.3}$$

It is easy to check that the condition

(A) $(f,x) = 0$ almost surely if and only if $\alpha_i = 0$ almost surely for $1 \le i \le n$

is a necessary and a sufficient condition for $\mathcal{J}$ to be a bijection and $||\cdot||_{G^s}$ to be a norm.

5. (*Definition of the generalized integral*) Let x be a square integrable process, which satisfies **(A)**. Let now f be a predictable process with $E\int_0^T f_s^2 ds < \infty$. Then there exists processes $f^{(n)} \in L^s$

such that

$$E\left(\int_0^T \left(f_s - f_s^{(n)}\right)^2 ds\right) \to 0$$

as $n \to \infty$. Complete the space G^s with respect to the norm $||\cdot||_{G^s}$. The integral $\int_0^T f_s dx_s \doteq \mathcal{J}(f)$ is defined using the extension of the isometry $\mathcal{J}$ on the completed Banach space $\bar{G}$. The sequence $f^{(n)}$ is a Cauchy sequence with respect the norm $||\cdot||_{\bar{G}}$ and the integral $\int_0^T f_s dx_s$ is the limit of the elementary integrals $(f^{(n)}, x)$ in the space $(\bar{G}, ||\cdot||_{\bar{G}})$.

We say that the integral $\int_0^T f_s dx_s$ defined for predictable $f \in L^2(P \otimes Leb)$ is the first order stochastic integral with respect to the process x. Later we will use the notation $\int_0^T f_s dx_s^{(1)}$ for this integral.

If $\zeta^{(n)}$ be a sequence of random variables of the form

$$\zeta^{(n)} \doteq (f^{(n)}, x),$$

with some $f^{(n)} \in L^s$ and assume that $\zeta = P - \lim_n \zeta^{(n)}$ and $||f - f^{(n)}||_{L^2(P\otimes Leb)} \to 0$, $n \to \infty$. Hence also

$$\int_0^T f_s dx_s = \bar{G} - \lim_n (f^{(n)}, x).$$

It may happen that $P\{|\zeta| < \infty\} < 1$ or even $P\{|\zeta| < \infty\} = 0$. Again the random variable ζ is one of the representatives of the integral $\int_0^T f_s dx_s$ in the space of random variables and $\int_0^T f_s dx_s$ is one of the representatives of the random variable ζ in the space $\bar{G}$: $\zeta \leftrightarrow \int_0^T f_s dx_s$.

6. (*Basic setup for higher order stochastic integrals*) Let $(x, \mathbf{F})$ be again a stochastic process defined on (Ω, F, P). Introduce the space of $G^{s,k}$ of the random variables ξ:

$$\xi \doteq \sum_{i \le n} \alpha_i \left(\Delta x_i\right)^k,$$

where $k > 1$, $\alpha_i \in F_{t_{i-1}}$, $\alpha_i \in L^2(P)$, $i = 1, \ldots, n$. With $f \in L^s$ define a surjection $\mathcal{J}^k$ from L^s to $G^{s,k}$ by putting

$$\mathcal{J}^k(f) \doteq (f, x^{(k)}) \doteq \sum_{i \le n} \alpha_i \left(\Delta x_i\right)^k.$$

Introduce the following pre-norm on $G^{s,k}$:

$$||(f, x^{(k)})||_{G^{s,k}} \doteq \left(E \sum_{i \le n} \left(\alpha_i^2 \left(t_i - t_{i-1}\right)\right)\right)^{1/2} = ||f||_{L^2(P\otimes Leb)}.$$

Let π_1 and π_2 be two partitions of $[0, T]$, $\pi = \pi_1 \cup \pi_2$, $f \in L^s$ defined on π_1 and $g \in L^s$ defined on π_2, $(f, x^{(k)}) = \sum_i \alpha_i (\Delta x_i)^k$ and $(g, x^{(k)}) = \sum_j \beta_j (\Delta x_j)^k$. Write $\pi = \{t_{i,j}\}$ and define the sum $(f, x^{(k)}) + (g, x^{(k)})$ by

$$(f, x^{(k)}) + (g, x^{(k)}) \doteq \sum_{t_{i,j} \in \pi} \left(\alpha_i + \beta_j\right) \left(\Delta x_{i,j}\right)^k.$$

With this convention,

$$||(f, x^{(k)}) + (g, x^{(k)})||_{G^{s,k}} \le ||(f, x^{(k)})||_{G^{s,k}} + ||(g, x^{(k)})||_{G^{s,k}}. \tag{2.4}$$

Again it is easy to check that the condition

(AA) $(f, x^{(k)}) = 0$ almost surely if and only if $\alpha_i = 0$ for $i = 1, \ldots, n$, when $f \in L^s$, $f = \sum_i \alpha_i 1_{(t_{i-1}, t_i]}(\cdot)$

is a necessary and a sufficient condition for $\mathcal{J}^k$ to be a bijection, for $G^{s,k}$ to be a linear space and for $||\cdot||_{G^{s,k}}$ to be a norm.

7. (*Definition of the higher order integral with respect to x*) If f is a predictable process for $L^2(P\otimes Leb)$, take $f^{(n)}\in L^s$ such that $||f-f^{(n)}||_{L^2(P\otimes Leb)}\to 0$. Assume that the property **(AA)** holds for the process x with some $k>1$. Define the integral $\int_0^T f_s dx_s^{(k)} \doteq \mathcal{J}^k(f)$ as the limit of $(f^{(n)},x^{(k)})$ in the completed Banach space $(\bar{G}^k, ||\cdot||_{\bar{G}^k})$, where $\bar{G}^k$ is the completion of $G^{s,k}$ with respect to norm $||\cdot||_{G^{s,k}}$. We say that the so defined integral $\int_0^T f_s dx_s^{(k)}$ is the k^{th} order stochastic integral of f with respect to the process x.

Assume now that the property of **(AA)** holds for all $k\le N$. Define the Banach space G^N by

$$G^N \doteq \bar{G}^1\times\bar{G}^2\times\cdots\times\bar{G}^N$$

and define the norm in G^N by

$$||\cdot||_{G^N} \doteq \sum_{k=1}^N ||\cdot||_{\bar{G}^k}.$$

In view of (2.4) $||\cdot||_{G^N}$ satisfies the triangle inequality and hence it is really a norm.

The elements $g\in\bar{G}^N$ have the form

$$g=\sum_{k=1}^N\int_0^T f_k(s)dx_s^{(k)},$$

where f_k is a predictable process from $L^2(P\otimes Leb)$. Note also that there is a bijection between such a g from $\bar{G}^N$ and $(f_1,\dots,f_N)\in\bigotimes_{k=1}^N L^2(P\otimes Leb)$ equipped with the norm $\sum_{k=1}^N ||f_k||_{L^2(P\otimes Leb)}$.

8. (*Examples of higher order integrals*) The following examples clarify the definition of the generalized integrals of higher order. We assume that the process x below satisfies the property **(AA)** for each $k\le N$.

Example 2.2. ***(Processes with bounded variation)*** *Assume that the process x is a continuous process with bounded variation and consider the random variables $x_T^{n,N}$, where*

$$x_T^{n,N}\doteq\sum_{l\le N}\sum_{k\le n}(\Delta x_k)^l.$$

When $\mathrm{mesh}(\pi^{(n)})\to 0$ *we have that $x_T^{n,N}\xrightarrow{P} x_T$ and the right-hand side converges in the space $\bar{G}^N$ towards the element*

$$\sum_{l=1}^N\int_0^T dx_s^{(l)}.$$

Here: $x_T\leftrightarrow\int_0^T dx_s^{(1)}$ and $0\leftrightarrow\sum_{l=2}^N\int_0^T dx_s^{(l)}$.

Example 2.3. ***(Standard Brownian motion)*** *Assume that x is a standard Brownian motion, $x=w$. Define again the random variable $x_T^{n,N}$ by*

$$x_T^{n,N}\doteq\sum_{l\le N}\sum_{k\le n}(\Delta x_k)^l.$$

Now, when $\mathrm{mesh}(\pi^{(n)})\to 0$, *$x_T^{(n)}\xrightarrow{P} x_T+T$, and we have $x_T\leftrightarrow\int_0^T dx_s^{(1)}$, $T\leftrightarrow\int_0^T dx_s^{(2)}$ and $0\leftrightarrow\sum_{l=3}^N\int_0^T dx_s^{(l)}$.*

3. Integration with respect to fractional Brownian motions

For the rest of the paper we assume that the process x is a fractional Brownian motion.

1. (*Fractional Brownian motions*) The process x^H is a fractional Brownian motion with Hurst index $H \in (0,1)$, if it is a continuous Gaussian process with stationary increments, $x_0^H = 0$, $Ex_t^H = 0$, and covariance function

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

If $H > \frac{1}{2}$ then we can also write the covariance function in the form

$$R(t,s) = H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2} du dv.$$

The process x^H is self-similar with index H: the law of $(x^H_{at})_{t\leq T}$ is the same as the law of $(a^H x_t^H)_{t\leq T}$ for $a>0$.

Let w be a Brownian motion defined on some probability space (Ω, F, P). The following representation result was obtained in [8] (for related work see also [2])

$$x_t^H = \int_0^t z^H(t,s) dw_s, \tag{3.1}$$

where

$$z^H(t,s) \doteq c_H\Big[\left(\frac{t}{s}\right)^{H-1/2}(t-s)^{H-1/2} - (H-1/2)s^{1/2-H}\int_s^t u^{H-3/2}(u-s)^{H-1/2}du\Big]$$

and the integral in (3.1) is defined 'ω by ω'.

2. (*Fractional Brownian motion and the property* **(A)**)

Theorem 3.1. *The property* **(A)** *holds for fractional Brownian motion* x^H, $H\in(0,1)$.

Proof Assume that $\sum_{i\leq n}\alpha_i \Delta x_i^H = 0$ almost surely. Assume that n_0 is the largest index for which $P\{\alpha_{n_0} \neq 0\} > 0$. By (3.1) we have

$$\Delta x_{n_0} = \int_{t_{n_0-1}}^{t_{n_0}} z^H(t_{n_0}, s)dw_s + \int_0^{t_{n_0-1}} (z^H(t_{n_0},s) - z^H(t_{n_0-1},s))dw_s \doteq w^z_{n_0} + i_{n_0-1}.$$

For the term i_{n_0-1} we have $i_{n_0-1} \in F_{t_{n_0-1}}$. Put $B \doteq \{\omega : |\alpha_i| \leq c, i\leq n_0\}$. Then $B \in F_{t_{n_0-1}}$ and

$$\sum_{i\leq n_0} 1_B \alpha_i \Delta x_i = \sum_{i\leq n} 1_B\alpha_i\Delta x_i = 0.$$

Hence we can conclude the following

$$0 = E\left(\sum_{i\leq n_0} 1_B\alpha_i\Delta x_i\right)^2 = E\Big(\left(\sum_{i\leq n_0-1} 1_B\alpha_i\Delta x_i\right) + \alpha_{n_0}1_B i_{n_0-1} + \alpha_{n_0}w^z_{n_0}\Big)^2. \tag{3.2}$$

The right-hand side of (3.2) is equal to

$$E\Big(\sum_{i\leq n_0-1}(\alpha_i\Delta x_i 1_B) + \alpha_{n_0}1_B i_{n_0-1}\Big)^2 + E\Big(\alpha_{n_0}^2 1_B\int_{t_{n_0-1}}^{t_{n_0}} \left(z^H(t_{n_0},s)\right)^2 ds\Big).$$

Hence, from (3.2), since

$$\int_{t_{n_0}-1}^{t_{n_0}} \left(z^H\left(t_{n_0}, s\right)\right)^2 ds > 0,$$

we have that $\alpha_{n_0} 1_B = 0$ almost surely for any $c > 0$ and so $\alpha_{n_0} = 0$ almost surely. This shows that the condition **(A)** is fulfilled. Hence $\alpha_i = 0$ for all $i \leq n$. □

3. (*Discussion and examples; Wiener integrals*) We can clarify some of the properties of the defined integrals in the case of Wiener integrals defined with respect to fractional Brownian motions.

Example 3.1. ***(Some counterexamples)*** *Assume that x is again fractional Brownian motion with index H. Take*

$$f_s^{(n)} \doteq n^\gamma 1_{(\frac{T}{2}-\frac{1}{2n}, \frac{T}{2}+\frac{1}{2n}]}(s),$$

Then $||f^{(n)}||_{L^2}^2 = n^{2\gamma-1}$. If $H > 1/2$, $\frac{1}{2} < \gamma < H$, then $||f^{(n)}||_{L^2} \to \infty$ and the generalized integral does not exist, but $E\left(\left(f^{(n)}, x^H\right)\right)^2 = n^{2\gamma-2H} \to 0$, and the limit exists in $L^2(P)$. If $H < 1/2$ and $H < \gamma < 1/2$, then $E\left(\left(f^{(n)}, x^H\right)\right)^2 \to \infty$, but $||f^{(n)}||_{L^2} \to 0$. Hence the integral exits in $\bar{E}$ and it is $= 0$, but the limit does not exist in $L^2(P)$. Note also that here we have that $|(f^{(n)}, x^H)| \xrightarrow{P} \infty$.

Recall that if x is a centered Gaussian process with covariance function $R(\cdot,\cdot)$ and f is a function such that

$$\int_0^T \int_0^T f(u)f(v)d_uR(u,v)d_vR(u,v) < \infty$$

one can define the integral $\int_0^T f(s)dx_s$ as a limit in $L^2(P)$ and one has

$$E\left(\int_0^T f(s)dx_s\right) = \int_0^T \int_0^T f(u)f(v)d_uR(u,v)d_vR(u,v)$$

(see e.g. [5] for more details on $L^2(P)$-integrals). Below we compare these $L^2(P)$-integrals with the isometric construction of Wiener integrals.

Example 3.2. *($L^2(P)$**-integrals and Wiener integrals,** $H > 1/2$) If x^H is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$, then for $f \in L^{\frac{1}{H}}$ we have the following estimate for the $L^2(P)$-integral*

$$E(\int_0^T f_s dx_s)^p \leq c_H ||f||_{L^{\frac{1}{H}}}^p, \tag{3.3}$$

where $p > 0$ (see [7]). Hence, if $\int_0^T f(s)dx_s = \bar{E} - \lim(f^{(n)}, x)$, then also $\int_0^T f(s)dx_s$ exists as a $L^2(P)$-integral.

>From Example 2.1 we have that the classical $L^2(P)$-integrals and the isometric Wiener integrals can be defined for all $f \in L^2$.

Before we can continue, we need the following, which is opposite to (3.3).

Theorem 3.2. *Let $f \in L_T^s$ and x is a fractional Brownian motion with Hurst index $H < 1/2$. Then*

$$E|\int_0^T f_s dx_s|^2 \geq C(T,H)||f||_{L^2}^2. \tag{3.4}$$

Proof

(i) We can assume that the coefficients α_i of f satisfy $\alpha_i \geq 0$. Indeed, if $f \in L^s$ and $(f,x) = \sum_i (\alpha_i \Delta x_i)$, then

$$E\left((f,x)\right)^2 = \sum_i \left(\alpha_i^2 E(\Delta x_i)^2\right) + \sum_{i \neq k} \left(\alpha_i \alpha_k E\left(\Delta x_i \Delta x_k\right)\right). \tag{3.5}$$

But $E\left(\Delta x_i \Delta x_k\right) < 0$ and hence $\alpha_i \alpha_k E\left(\Delta x_i \Delta x_k\right) \geq |\alpha_i||\alpha_k| E\left(\Delta x_i \Delta x_k\right)$. Use this in (3.5) to obtain the inequality

$$E\left((f,x)\right)^2 \geq E\left(\sum_i |\alpha_i| \Delta x_i\right)^2. \tag{3.6}$$

This proves **(i)**.

Next we use the approach of Dasgupta [1]. Denote by $\mathcal{D}(\mathbb{R})$ the space of functions ϕ with the two properties: $\phi \in C^\infty(\mathbb{R})$ and ϕ has compact support. Then, using integration by parts:

$$\int_{-\infty}^{\infty} \phi(s) dx_s = -\int_{-\infty}^{\infty} \phi'(s) x_s ds.$$

(ii) We show that if $\phi^{(n)}(t) \to f(t)$ for all $t \in [0,T]$, $\phi^{(n)} \in \mathcal{D}(\mathbb{R})$, then $\int_{-\infty}^{\infty} \phi(s) dx_s \overset{L^2(P)}{\longrightarrow} (f,x)$.

To prove **(ii)** it is sufficient to consider the case $f \in L_T^s$, $f_u = a1_{(s,t]}(u)$, $s < t \leq T$ and $a > 0$. Take $\phi^{(n)} \in \mathcal{D}(\mathbb{R})$ such that $supp(\phi^{(n)}) \subset [s - \frac{1}{n}, t + \frac{1}{n}]$ and $\phi^{(n)} = a$ on $[s + \frac{1}{n}, t - \frac{1}{n}]$. Then

$$|(f,x) - \int_{-\infty}^{\infty} \phi^{(n)}(u) dx_u| \leq a\omega^{(n)}(x;s) + a\omega^{(n)}(x;t), \tag{3.7}$$

where

$$\omega^{(n)}(x;u) \doteq \sup_{v \in [u - \frac{1}{n}, u + \frac{1}{n}]} |x_v - x_{u - \frac{1}{n}}|.$$

By [9, Theorem 1.1]

$$\omega^{(n)}(x;u) \overset{L^2(P)}{\longrightarrow} 0.$$

Use this in (3.7) to prove **(ii)**.

(iii) Next we show that

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 |\lambda|^{1-2H} d\lambda \leq c_H E(f,x)^2, \tag{3.8}$$

and $\hat{f}$ is the Fourier transform of f.

Let $\phi \in \mathcal{D}(\mathbb{R})$ and x is a fractional Brownian motion with Hurst index $H < 1/2$. The from [1, p. 14] we have the following

$$E|\int_{-\infty}^{\infty} \phi'(t) x_t dt|^2 = c_H \int_{-\infty}^{\infty} |\hat{\phi}(t)|^2 |\lambda|^{1-2H} d\lambda, \tag{3.9}$$

where c_H is a constant.

Since for any $\lambda \in \mathbb{R}$ $\hat{f}(\lambda) = \lim_n \hat{\phi}^{(n)}(\lambda)$, we have, using Fatou lemma, **(ii)** and the relation (3.9),

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)||\lambda|^{1-2H} d\lambda \leq \liminf_n \int_{-\infty}^{\infty} |\hat{\phi}^{(n)}(\lambda)|^2 \lambda = E(f,x)^2.$$

This proves the step **(iii)**.

(iv) The proof of (3.4).

To prove (3.4) we estimate the left-hand side of (3.8) downwards. Start with the estimate

$$\int_{-\infty}^{\infty}|\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda \geq \epsilon^{1-2H}\int_{|\lambda|>\epsilon}|\hat{f}(\lambda)|^2d\lambda+\int_{|\lambda|\leq\epsilon}|\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda, \tag{3.10}$$

where the constant $\epsilon>0$ is not yet fixed.

Put $\epsilon(\lambda)\doteq|\lambda|^{\frac{1}{2}-H}1_{[-\epsilon,\epsilon]}(\lambda)$. Since $H\in(0,\frac{1}{2})$, we have that $\epsilon(\cdot)\in L^1(\mathbb{R})$ for any $\epsilon>0$. Moreover

$$\hat{\epsilon}(t)\doteq\int_{-\infty}^{\infty}e^{it\lambda}\epsilon(\lambda)d\lambda=\int_{-\epsilon}^{\epsilon}\cos(t\lambda)|\lambda|^{\frac{1}{2}-H}d\lambda.$$

Next we show that $\hat{\epsilon}\in L^1(\mathbb{R})$. Note that

$$\int_0^{\infty}|\hat{\epsilon}(t)|dt=\int_0^{\infty}t^{H-\frac{3}{2}}|\int_0^{\epsilon t}\cos(z)z^{\frac{1}{2}-H}dz|dt.$$

This integral is finite and hence $\epsilon(\cdot)$ is the Fourier transform of $\hat{\epsilon}(\cdot)$. Use the Parceval's identity to obtain

$$\int_{|\lambda|<\epsilon}|\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda=\int_{-\infty}^{\infty}|\int_{-\infty}^{\infty}f(s)\left(\int_{-\epsilon}^{\epsilon}\cos\left((t-s)\lambda\right)|\lambda|^{\frac{1}{2}-H}d\lambda\right)ds|^2dt. \tag{3.11}$$

Estimate the right hand side of (3.11) from below by

$$\int_{-1}^{1}|\int_0^T f(s)\left(\int_{-\epsilon}^{\epsilon}\cos\left((t-s)\lambda\right)|\lambda|^{\frac{1}{2}-H}d\lambda\right)ds|^2dt. \tag{3.12}$$

Fix ϵ by taking in (3.12) such an ϵ that $\epsilon(T+1)\leq\frac{\pi}{3}$. Then $\cos((t-s)\lambda)\geq\frac{1}{2}$ and the left hand of the inequality (3.11) can be estimated from below, using the estimate (3.12) and the chosen ϵ by the expression

$$\frac{1}{4}\int_1^1|\int_0^T f(s)ds|^2dt\times\left(\int_{-\epsilon}^{\epsilon}|\lambda|^{\frac{1}{2}-H}d\lambda\right)^2=\frac{2\epsilon^{3-2H}}{(\frac{3}{2}-H)^2}|\int_0^T f(s)ds|^2.$$

The nonnegative function f has its support on $[0,T]$, and hence we have $\hat{f}(0)=\int_{\infty}^{\infty}f_sds=\int_0^T f_sds$ and so

$$\frac{2\epsilon^{3-2H}}{(\frac{3}{2}-H)^2}|\int_0^T f(s)ds|^2=\frac{2\epsilon^{3-2H}}{(\frac{3}{2}-H)^2}|\hat{f}(0)|^2;$$

but since f is nonnegative, we also have the estimate $|\hat{f}(0)|\geq|\hat{f}(\lambda)|$. Therefore, from the above estimates we obtain

$$\int_{|\lambda|<\epsilon}|\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda\geq\frac{\epsilon^{2-2H}}{(\frac{3}{2}-H)^2}\int_{-\epsilon}^{\epsilon}|\hat{f}(\lambda)|^2d\lambda. \tag{3.13}$$

Take $C=\min\{\epsilon^{1-2H},\frac{\epsilon^{2-2H}}{(\frac{3}{2}-H)^2}\}$ and use (3.13) in (3.10) to obtain

$$\int_{-\infty}^{\infty}|\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda\geq C\int_{\infty}^{\infty}|\hat{f}(\lambda)|^2d\lambda=C||f||^2_{L^2_T}.$$

This finishes the proof of **(iv)** and we are done. □

Example 3.3. *(L^2-integrals and Wiener integrals, $H<1/2$) Assume now that $L^2(P)$-integral exists for some $f\in L^2$ with respect to fractional Brownian motion x with $H<\frac{1}{2}$. Then there exists a sequence $f^{(n)}\in L^s$ with deterministic coefficients such that $(f^{(n)},x)$ is a Cauchy* *sequence in $L^2(P)$. By (3.4) $f^{(n)}$ is a Cauchy sequence also in L^2 and the isometric Wiener integral exists.*

4. (*Discussion and examples; stochastic integrals*) We compare our approach to other recent approaches to define stochastic integrals with respect to fractional Brownian motions. We start with a simple example, whcih shows some of the peculiar properties of stochastic integrals in our sense.

Example 3.4. *(* ***Fractional Brownian motions,*** *$H < \frac{1}{2}$) Assume that $f = x^H$, where x^H is a fractional Brownian motion with Hurst index $H \in (0, 1/2)$. Now $f \in L^2(P \otimes Leb)$ and hence the generalized integral $\int_0^T x_s^H dx_s^H$ exists in $\bar{G}$. If $\pi^{(n)}$ is a partition of $[0,T]$ and*

$$f_s^{(n)} = \sum_{k \leq n} \left(x_{t_{k-1}}^H 1_{(t_{k-1}, t_k]}(s) \right),$$

then

$$E(f^{(n)}, x^H) = \frac{T^{2H}}{2} - \frac{1}{2} \sum_{t_k \in \pi^{(n)}} (\Delta t_k)^{2H} \to -\infty$$

as $\mathrm{mesh}(\pi^{(n)}) \to 0$. *Hence the generalized integral exists as a limit in $\bar{G}$, but even the expectations diverge in $L^1(P)$.*

For the fractional Brownian motion x^H there already exists several different definitions of stochastic integrals in the case of $H > \frac{1}{2}$. We discuss our approach in connection to some of these.

Remark 3.1. *(* ***The approach of Lin****) Assume that x is fractional Brownian motion with Hurst index $H > 1/2$. Lin defines the stochastic integral for integrand of the type $f_t = f(x_t)$, where $f \in C^1(\mathbb{R})$, as follows:*

$$\int_0^{x_T} f(u)du \doteq P - \lim_n (f^{(n)}, x),$$

where $(f^{(n)}, x) = \sum_{i \leq n} f(x_{t_{i-1}}) \Delta x_i$ (we refer to [6] *for more details). We also have that $f^{(n)} \to f$ in $L^2(P \otimes Leb([0,T])$ and hence*

$$\int_0^T f(x_s)dx_s = \bar{G} - \lim_n (f^{(n)}, x).$$

The random variable $\int_0^{x_T} f(u)du$ is thus a representative of the generalized stochastic integral $\int_0^T f(x_s)dx_s$. The same holds, when $f_t = f(x_t, t)$ and $f \in C^{1,2}(\mathbb{R} \times \mathbb{R})$.

Remark 3.2. *(* ***The approach of Föllmer****) Föllmer introduced the class of Dirichlet processes x as processes having zero conditional energy in* [4]*. He also showed that if $F \in C^2(\mathbb{R})$ there exists stochastic integral*

$$\int_0^T F_x(x_s)dx_s \doteq P - \lim_n (F_x^{(n)}(x), x).$$

Since fractional Brownian motion is a Dirichlet process, when $H > 1/2$, this approach to define integrals is the same as the approach of Lin.

Remark 3.3. *(* ***The approach of Dasgupta****) Dasgupta defined Wiener integrals for fractional Brownian motions and used these integrals for multiple Wiener integrals. Consider first the case $H > \frac{1}{2}$. Assume that $f^{(n)} \in L^s$ and $f^{(n)} \xrightarrow{L^2} f$. One can show that in this case the integral introduced by Dasgupta converges in probability to the limit integral $D - \int_0^T f_s dx_s$ and hence this integral is a representative of our integral.*

For $H < 1/2$ the integral of Dasgupta is defined for functions f satisfying $\int_\infty^\infty |\hat{f}(\lambda)|^2 \left(1+|\lambda|^2\right) d\lambda < 0$. This is not true for step functions and hence we cannot compare our integral with the integral introduced by Dasgupta.

Some other approaches to define stochastic integrals with respect to fractional Brownian motions have been studied for the case $H > 1/2$. Zähle defined stochastic integrals using fractional integrals and derivatives (see [11]) and Feyel and de La Pradelle ([3]) defined integrals for Hölder continuous f. If f is a simple process, then these integrals do not exist, and we cannot compare these definitions with ours. Finally, Decreusefond and Üstünel defined integrals using the kernel representation (3.1) ([2]).

5. (*Fractional Brownian motions and the property* **(AA)**) Again we have that the property **(AA)** holds for fractional Brownian motion for any $k > 1$.

Theorem 3.3. *The property* **(AA)** *holds for fractional Brownian motions x^H, $H \in (0,1)$.*

Proof We know from Theorem 3.1 the claim holds for $k = 1$. Assume now that $k > 1$ and let n_0, $w^z_{n_0}$, i_{n_0-1} and B be as in the proof of Theorem 3.1. Put $\alpha^c_i \doteq 1_B\alpha_i$. Note that $\alpha^c_i \in F_{t_{n_0-1}}$ for $i \leq n_0$. Denote by χ the random variable

$$\chi \doteq \sum_{i \leq n_0-1} \alpha^c_i (\Delta x^H_i)^k.$$

For the random variable χ we have that $\chi \in F_{n_0-1}$, and this fact is used below. Assume that $\sum_{i \leq n} \alpha_i \Delta(x^H_i)^k = 0$. With the above notation we have from this assumption that also

$$\chi + \alpha^c_{n_0} \sum_{r=0}^{k} \binom{k}{r} (i_{n_0-1})^{k-r} (w^z_{n_0})^r = 0. \tag{3.14}$$

Write the expression in (3.14) as

$$\left(\chi + \alpha^c_{n_0} \sum_{r=0, r \text{ even}}^{k} \binom{k}{r} (i_{n_0-1})^{k-r} (w^z_{n_0})^r\right) + \left(\alpha^c_{n_0} \sum_{r=0, r \text{ odd}}^{k} \binom{k}{r} (i_{n_0-1})^{k-r} (w^z_{n_0})^r\right) \doteq \chi_1 + \chi_2. \tag{3.15}$$

The random variable $w^z_{n_0}$ is a Gaussian random variable with zero expectation and hence for odd r $E(w^z_{n_0})^r = 0$ and by conditioning on F_{n_0-1} in (3.15) it is easy to see that $E(\chi_1\chi_2) = 0$. So from this we can conclude that $E\chi_2^2 = 0$, using also (3.14) and (3.15). But

$$\chi_2^2 = \alpha^2_{n_0} (\gamma_1 + \gamma_2)$$

with

$$\gamma_1 \doteq \sum_{r=0, r \text{ odd}}^{k} \left(\binom{k}{r} (i_{n_0-1})^{k-r} \left(w^z_{n_0}\right)^r\right)^2 \tag{3.16}$$

and

$$\gamma_2 \doteq \sum_{r \neq q, r, q \text{ odd}} \left(\binom{k}{r}\binom{k}{q} (i_{n_0-1})^{2k-r-q} \left(w^z_{n_0}\right)^{r+q}\right). \tag{3.17}$$

All the terms in (3.17) all non-negative and since $r + q$ is even, the same holds for the expression (3.17), too. Note also that if $r = 1$, then

$$k^2 \left(i_{n_0-1}\right)^{2k-2} \left(w^z_{n_0}\right)^2 > 0$$

almost surely. But at the same time $E\left(\alpha_{n_0}^2\left(\gamma_1+\gamma_2\right)\right)=0$. Hence $\alpha_{n_0}=0$ almost surely. From this follows that $\alpha_i=0$ almost surely for all $i\leq n$. We have shown that fractional Brownian motion x^H satisfies the property **(AA)** for all $k\geq 1$. $\square$

Example 3.5. ***(Fractional Brownian motions)*** *Assume that x is a fractional Brownian motion with index $H\in(0,1)$. If $H>1/2$, then the above constructed $x_T^{n,N}\stackrel{P}{\to}x_T$ and zero is a representative of the sum $\sum_{l=2}^N\int_0^T dx_s^{(l)}$.*

If $H<1/2$, then zero is the representative of the sum

$$\sum_{l=\lfloor\frac{1}{H}\rfloor+1}^{N}\int_0^T dx_s^{(l)}.$$

Note that here the sequence of random variables $x_T^{n,N}$ does not converge.

In the next section we show that more can be said, when we consider integrals of the type $\sum_{l=1}^N\int_0^T f_l(s)dx_s^{(l)}$, x is a fractional Brownian motion with Hurst index H.

4. Ito formula for x^H, $H<1/2$, and applications

1. (*Definitions*) If $f\in L^2(P\otimes Leb)$, f is predictable, π is a partition, then f_π is the step function $f_\pi(t)=\sum_i f(t_{i-1})1_{(t_{i-1},t_i]}(t)$.

Define the the class of functions Φ as follows: $\vec{f}\in\Phi$ if the following conditions are satisfied:

- **i:** $\vec{f}\doteq(f^i:i\geq 1)$, where $f^i\in L^2(P\otimes Leb)$, f^i is predictable and $\sum_i||f^i||_{L^2(P\otimes Leb)}<\infty$.
- **ii:** $\vec{f}$ is uniformly tight: $P\{\sup_{t\leq T}\sup_i|f^i(t)|>C\}\to 0$ as $C\to\infty$.
- **iii:** The random variable u defined by $u\doteq\sum_i(f_\pi^i,x^{(i)})$ does not depend on the partition π and the series converges absolutely with probability one, when $\vec{f}\in\Phi$.

Write $(\vec{f},\vec{x})$ for the sum $\sum_i(f_\pi^i,x^{(i)})$, and put $\mathcal{U}\doteq\{u:u=(\vec{f},\vec{x}),f\in\Phi\}$ and Φ_p is the projection of Φ to the first p coordinates.

The following example shows that $\mathcal{U}$ is not empty.

Example 4.1. *Assume that $f\in C_b^\infty(\mathrm{I\!R})$: then*

$$f(x_T)-f(0)=\sum_{k=1}^{(n)}\Delta f(x_{t_i})$$

and if $f^k\doteq\frac{1}{k!}f^{(k)}$, $k\geq 1$, then

$$f(x_T)-f(0)=(\vec{f},\vec{x}),$$

$f(x_T)-f(0)\in\mathcal{U}$ and , $\vec{f}\in\Phi$, $(f^1,\dots,f_p)\in\Phi_p$ for any $p\geq 1$.

Lemma 4.1. *If $u\in\mathcal{U}$, $u=(\vec{f},\vec{x})$ with $u=0$, then $f^i=0$, $i\geq 1$.*

Proof Since u does not depend from the partition, take first the partition $\{0, T\}$. The random variable u has a representation

$$u = \sum_i f_0^i (x_T)^i, \tag{4.1}$$

where f_0^i are real numbers, since F_0 is the trivial σ-algebra, But since $u = 0$ from (4.1) it follows that for almost all $y \in \mathbb{R}$ we have that $\sum_i \alpha_i y^i = 0$ and hence $\alpha_i = 0$ for all $i \geq 1$.

Next, consider the partition $\{0, t, T\}$. We have that

$$u = \sum_i f_0^i x_t^i + \sum_i f_t^i (x_T - x_t)^i = 0.$$

>From the above we get that $f_0^i = 0$ for all $i \geq 1$ and hence also $f_t^i = 0$ for all $i \geq 1$. □

2. (*Ito formula*) The following is an analogue of Ito formula in this context.

Theorem 4.1. *Assume that the Hurst index H satisfies $H < 1/2$. There exists one-to-one correspondence between $\mathcal{U}$ and the set*

$$\mathcal{V} \doteq \{v : v \doteq \sum_{i=1}^{\lfloor \frac{1}{H} \rfloor} (f^i, x^{(i)})\}.$$

Proof We must show that there exists one-to-one correspondence between $\mathcal{U}$ and $\Phi_{\lfloor \frac{1}{H} \rfloor}$. Assume that $f \in \Phi_{\lfloor \frac{1}{H} \rfloor}$. Then there exists a vector $\vec{g} \in \Phi$ such that $f^i = g^i$ for $i \leq \lfloor \frac{1}{H} \rfloor$. Assume that $\vec{h}$ is another element from Φ such that $f^i = h^i$ for $i \leq \lfloor \frac{1}{H} \rfloor$. Put $u \doteq (\vec{g}, \vec{x})$ and $v \doteq (\vec{h}, \vec{x})$. Then

$$u - v = \sum_{i = \lfloor \frac{1}{H} \rfloor + 1}^{\infty} (g^i - h^i, x^{(i)}).$$

Since u and v are independent of the partition π take a partition π such that $\mathrm{mesh}(\pi) < 1$. Then for any $\epsilon > 0$ we have that

$$P\{|u - v| > \epsilon\} \leq P(D) + P\{|u - v| > \epsilon, \Omega \setminus D\} \tag{4.2}$$

and D is the set $D \doteq \{\sup_{t \leq T} \sup_i |f_t^i - g_t^i| \geq C\}$. But

$$P\{|u - v| > \epsilon, \Omega \setminus D\} \leq \frac{C}{\epsilon} \sum_{i > \frac{1}{H}} E \sum_k (\Delta x_k)^i$$

and since

$$E \sum_k (\Delta x_k)^i \leq T(\mathrm{mesh}(\pi))^{i-1}$$

we have that

$$P\{|u - v| > \epsilon, \Omega \setminus D\} \to 0$$

as $\mathrm{mesh}(\pi) \to 0$. By property iii of Φ we can choose C such that $P(D) < \delta$ for any $\delta > 0$. Use these estimates in (4.2) to conclude that $u = v$.

On the other hand, if $u = (\vec{f}, \vec{x}) = (\vec{h}, \vec{x})$ we have from Lemma 4.1 that $\vec{f} = \vec{h}$.

To finish, note that from Example 4.1 it follows that the random variable $f(x_T) - f(0)$is a representative of $\sum_{i=1}^{\lfloor \frac{1}{H} \rfloor} \frac{1}{i!} \int_0^T f^{(i)}(x_s) dx_s^{(i)}$. □

Example 4.2. ***(Fractional Doleans exponent)*** *Assume that* $\lfloor \frac{1}{H} \rfloor = 2p$*, where* $p \in \mathbb{N}$*. Then the random variable* y_t*,* $y_t = \exp(x_t^H - \frac{1}{(2p)!}t) - 1$ *is a representative of*

$$\sum_{i=1}^{2p-1} \frac{1}{i!} \int_0^t y_s dx_s^{(i)}.$$

We say that y *is the Doleans exponent of* x^H*.*

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References

[1] Dasgupta, A., (1998) Fractional Brownian motion: its properties and applications to stochastic integration, Ph. Thesis, University of North Carolina, 97 pages.
[2] Decreusefond, L. and Üstünel, A. S. (1998), Stochastic analysis of the fractional Brownian motion, *Potential Analysis*, to appear.
[3] Feyel, D. and de La Pradelle, A. (1996) Fractional Integrals and Brownian Processes, Preprint, Universite d'Evry, 15 pages.
[4] Föllmer, H. (1981), Dirichlet processes in Stochastic Integrals, *Lecture Notes in Mathematics*, **851**, 476-478.
[5] Huang, S. and Cambanis, S. (1978), Stochastic and multiple Wiener integrals for Gaussian processes. *Annals of Probability*, **6**, 585-614.
[6] Lin, S.J. (1996), Stochastic analysis of fractional Brownian motion. *Stochastics and Stochastics Reports*, **55**, 121-140.
[7] Memim, J. and Valkeila, E. (1998), A note on Wiener integrals for fractional Brownian motions. *Manuscript.*
[8] Norros, I., Valkeila, E. and Virtamo, J. (1998), An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, *Bernoulli*, to appear.
[9] Novikov, A. and Valkeila, E. (1998), On some maximal inequalities for fractional Brownian motions, *Statistics & Probability Letters*, to appear.
[10] Protter, P. (1990), *Stochastic Integration and Differential Equations*, Springer, Berlin Heidelberg New York.
[11] Zähle, M. (1998), Integration with respect to fractal functions and stochastic calculus. I. Probability Theory and Related Fields, **111**, 333-374.

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