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THE DENSITY TOPOLOGY IS MAXIMALLY RESOLVABLE

Abstract

We prove that the real line is the union of $2^{\aleph_0}$ disjoint sets each meeting each nonempty set open in the density topology in $2^{\aleph_0}$ points.

A topological space containing two disjoint dense sets is said to be resolvable [4]. For example, $\mathbb{R}$, the real line with the usual topology, is resolvable as $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense. Recently Dontchev, Ganster, and Rose [2] proved that $\mathbb{R}_{\mathrm{d}}$, the real line equipped with the (Lebesgue) density topology, is resolvable. In fact, for a set $D \subset \mathbb{R}$ the sets D and $\mathbb{R} \setminus D$ are dense in $\mathbb{R}_{\mathrm{d}}$ if and only if $m(A) = 0$ for each measurable set A contained either in D or in $\mathbb{R} \setminus D$; such a set D exists and is nonmeasurable [3, 16.E].

Let X be a nonempty topological space with no isolated points, and let $\Delta(X) = \min\{ \operatorname{card} U \mid U \text{ is a nonempty open set in } X \}$ be the dispersion character of X [4] (then $\Delta(X) \geq 2$). Ceder [1] calls X *maximally resolvable* if X is the union of $\Delta(X)$ disjoint dense sets each meeting each nonempty open set in at least $\Delta(X)$ points. Let $w(X) = \min\{ \operatorname{card} \mathcal{B} \mid \mathcal{B} \text{ is a base for } X \}$ be the weight of X. In [1, Theorem 3] Ceder proved that if $\aleph_0 \leq w(X) \leq \Delta(X)$, then X is maximally resolvable. Thus, $\mathbb{R}$ is maximally resolvable as $w(\mathbb{R}) = \aleph_0$ and $\Delta(\mathbb{R}) = 2^{\aleph_0}$. In [2] it is asked whether $\mathbb{R}_{\mathrm{d}}$ is maximally resolvable. We apply Ceder's theorem to prove that this is really the case.

We first recall the definition of the density topology on the set $\mathbb{R}$; see [6], [7]. For a measurable set $A \subset \mathbb{R}$ let $\varphi(A)$ denote the set of all density points of A in $\mathbb{R}$, that is, of all points $x \in \mathbb{R}$ for which $\lim_{h \to 0+}(m(A \cap [x-h, x+h])/2h) = 1$. Then a set $A \subset \mathbb{R}$ is open in $\mathbb{R}_{\mathrm{d}}$ if and only if A is measurable and $A \subset \varphi(A)$. The density topology is finer than the usual topology. By [7, 2.7], for a set A in $\mathbb{R}_{\mathrm{d}}$ the following conditions are equivalent: (1) A is of measure zero; (2) A is nowhere dense; and (3) A is closed and discrete. It follows that $\mathbb{R}_{\mathrm{d}}$ is neither

Key Words: density topology, maximally resolvable
Mathematical Reviews subject classification: Primary 54A25; Secondary 28A05

first countable [7, 2.11] nor locally compact [7, 3.6(1)], and thus the maximal resolvability of $\mathbb{R}_{\mathrm{d}}$ is not covered by [1, Theorems 7 and 8].

The next lemma is known; see [7, 4.10] for $w(\mathbb{R}_{\mathrm{d}})$ and compare [5, footnote on p. 135] for $\Delta(\mathbb{R}_{\mathrm{d}})$. However, we present a proof for the benefit of the reader.

Lemma 1. *We have* $w(\mathbb{R}_{\mathrm{d}}) = \Delta(\mathbb{R}_{\mathrm{d}}) = 2^{\aleph_0}$.

PROOF. By the proof of [6, 22.9] the family $\mathcal{B}_0 = \{\,\varphi(F) \mid F \text{ is closed in } \mathbb{R}\,\}$ is a base for $\mathbb{R}_{\mathrm{d}}$. As here $\mathbb{R} \setminus F$ is the union of countably many open intervals, we conclude that $\operatorname{card} \mathcal{B}_0 \le 2^{\aleph_0}$. On the other hand, the standard Cantor set C has measure zero and is thus discrete in $\mathbb{R}_{\mathrm{d}}$. Hence, if $\mathcal{B}$ is a base for $\mathbb{R}_{\mathrm{d}}$, then for each $x \in C$ there is $U_x \in \mathcal{B}$ with $U_x \cap C = \{x\}$, which implies that $\operatorname{card} \mathcal{B} \ge \operatorname{card} C = 2^{\aleph_0}$. Consequently, $w(\mathbb{R}_{\mathrm{d}}) = 2^{\aleph_0}$.

We have, of course, $\Delta(\mathbb{R}_{\mathrm{d}}) \le \operatorname{card} \mathbb{R} = 2^{\aleph_0}$. To show that $\Delta(\mathbb{R}_{\mathrm{d}}) \ge 2^{\aleph_0}$, let $A \subset \mathbb{R}_{\mathrm{d}}$ be open and nonempty. Then $m(A) > 0$. Hence, by [5, 10.30] there is a compact set $F \subset \mathbb{R}$ with $F \subset A$ and $m(F) > 0$. By [5, 6.66] there are a perfect set $P \subset \mathbb{R}$ and a countable set N with $F = P \cup N$; then $m(P) = m(F) > 0$ and thus $P \neq \emptyset$. Now $\operatorname{card} P = 2^{\aleph_0}$ by [5, 6.65]. Hence, $\operatorname{card} A = 2^{\aleph_0}$. It follows that $\Delta(\mathbb{R}_{\mathrm{d}}) = 2^{\aleph_0}$. □

Our result follows immediately from Ceder's theorem and Lemma 1.

Theorem 2. *The real line equipped with the density topology is the union of* $2^{\aleph_0}$ *disjoint dense sets each meeting each nonempty open set in* $2^{\aleph_0}$ *points.*

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