Some basic results for real analytic proper G-manifolds

Sören Illman and Marja Kankaanrinta

By a real analytic proper G-manifold M we mean a real analytic manifold M on which a Lie group G acts by a real analytic and proper action. In this paper we address three basic questions concerning real analytic proper G-manifolds. They are:

(i) Given a real analytic proper G-manifold M, does there exist a G-invariant real analytic Riemannian metric on M?

(ii) Let $f: M \to N$ be a G-equivariant C^r smooth map, $1 \leq r \leq \infty$, between two real analytic proper G-manifolds. Can one then always approximate f by a G-equivariant real analytic map $h: M \to N$?

(iii) Suppose G is a linear Lie group and let M be a real analytic proper G-manifold with only finitely many isotropy types. Does there then exist a G-equivariant real analytic imbedding of M into some linear representation space for G?

We say that a Lie group G is *good* if it can be imbedded as a closed subgroup in a Lie group with only finitely many connected components. Note in particular that every linear Lie group, that is a group which is isomorphic to a closed subgroup of some $\mathrm{GL}(n)$, is a good Lie group. The class of good Lie groups is however strictly larger than the class of all linear Lie groups. It is well-known that there exist for example connected Lie groups that are not linear, the first example of such a Lie group is given in [Bi].

It is known that every smooth proper G-manifold, where G is an arbitrary Lie group, can be given a G-invariant smooth Riemannian metric, see [P, Theorem 4.3.1]. In the smooth case one can use partition of unity techniques, and it is precisely the fact that one in the real analytic case cannot have any partition of unity that makes question (i), and also questions (ii) and (iii), much more difficult to solve than the corresponding questions in the smooth case.

The answer to question (i) is known to be affirmative if G is compact, see [Ka1, Theorem 1.17], and in the case when G is a Lie group with only finitely many connected components, see [Ku1, Satz 2.5.6], and also [Ka3, Theorem 5,1]. We prove in this paper that if G is a good Lie group, then every real analytic proper G-manifold M has a G-invariant real analytic Riemannian metric, see Theorem I. This settles question (i) for a very large class of Lie groups, including all linear Lie groups. The proof of this result is in fact obtained by a simple trick, using the twisted product construction, see Proposition 1.2.

Theorem I. *Let M be a real analytic proper G-manifold, where G is a good Lie group. Then M has a G-invariant real analytic Riemannian metric.*

If N is a G-invariant real analytic closed submanifold of M, then there exist a G-invariant open neighborhood W of N in M and a G–equivariant real analytic retraction $q: W \to N$.

The second part of Theorem I is used in the proof of Theorem II.

In Theorem II we prove that if G is a good Lie group, then every G-equivariant C^r smooth map $f\colon M \to N$, $1 \le r \le \infty$, where M and N are real analytic proper G-manifolds, can be approximated arbitrarily well in the *strong-weak topology* by a G-equivariant real analytic map $h\colon M \to N$. For the definition and properties of the strong-weak topology we refer to the preceding paper [I–Ka].

Theorem II. *Let M and N be real analytic proper G-manifolds, where G is a good Lie group. Then $\mathrm{C}^{\omega,G}_{\mathrm{SW}}(M,N)$ is dense in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$, $1 \le r \le \infty$.*

In the case when G is compact the strong-weak topology agrees with the strong topology (i.e., the Whitney topology), see [I–Ka, Proposition 4.1], and for G a compact Lie group it was known before that $\mathrm{C}^{\omega,G}_{\mathrm{S}}(M,N)$ is dense in $\mathrm{C}^{r,G}_{\mathrm{S}}(M,N)$, $1 \le r \le \infty$, see [M–S, Theorem 1.2] and [Ku2].

Theorem II is proved in Section 2. At the heart of the proof lies the same trick as the one in the proof of Theorem I, where we use the twisted product construction to enlarge the transformation group from a good Lie group G to a Lie group J with only finitely many connected components. But the proof of Theorem II requires much more, it uses many basic results from the preceding paper [I–Ka], and in particular it uses Theorem B, or more precisely Theorem 5.5, of [I–Ka], which in turn is one of the main results of the paper [I–Ka].

Recall that it is proved in [I, Theorem A] that each smooth proper G-manifold, where G is an arbitrary Lie group, is equivalent to a real analytic proper G-manifold. The question of the uniqueness, of the obtained real analytic proper G-manifold, is then a natural and important question. It is proved in [Ku2] that uniqueness holds if G is a Lie group with only finitely many connected components. As a corollary of Theorem II we are now able to prove that uniqueness holds if G is a good Lie group.

Corollary II.a. *Let G be a good Lie group, and let M and N be real analytic proper G-manifolds. Suppose that M and N are G-equivariantly C^r diffeomorphic, where $1 \le r \le \infty$. Then M and N are G-equivariantly real analytically isomorphic.*

Proof. Suppose $f\colon M \to N$ is a G-equivariant C^r-diffeomorphism, where $1 \le r \le \infty$. By [I–Ka, Theorem 7.5] we know that there exists an open neighborhood $\mathscr{U}$ of f in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$ such that if $h \in \mathscr{U}$, then $h\colon M \to N$ is a G-equivariant C^r diffeomorphism. By Theorem II there exists an $h \in \mathscr{U}$, which is real analytic. It now follows by the real analytic inverse function theorem, see e.g. [N, Theorem 2.2.10], that $h^{-1}\colon N \to M$ is real analytic, and thus $h\colon M \to N$ is a G-equivariant real analytic isomorphism. ◻

We also obtain the following result as a corollary of Theorem II.

Corollary II.b. *Let G be a good Lie group, and let M and N be real analytic proper G-manifolds. Suppose $f\colon M \to N$ is a G-equivariant [closed] C^r imbedding, $1 \le r \le \infty$, and let $\mathscr{U}$ be an open neighborhood of f in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$. Then there exists a G-equivariant real analytic [closed] imbedding $h\colon M \to N$, such that $h \in \mathscr{U}$.*

Proof. In the case of imbeddings this follows by Theorem II and [I–Ka, Theorem 7.3], and in the case of closed imbeddings by Theorem II and [I–Ka, Theorem 7.4]. ◻

Now suppose G is a linear Lie group, and let M be a real analytic proper G-manifold, with only finitely many isotropy types. It has been an open question for some time if there then always exists a G-equivariant real analytic imbedding of M into some linear representation space $\mathbb{R}^n(\rho)$ for G. This was known before to hold in the case when G is compact, see [M–S, Theorem 1.1], and in the case when G is a linear Lie group with only finitely many connected components, see [Ku1, Satz 2.5.7], and also [Ka3, Corollary 4.2]. In the general case, when G is a linear Lie group, it was known before that one can always find a G-equivariant subanalytic imbedding, see [Ka1, Theorem 6.7], and it was also known that there exists a G-equivariant smooth imbedding of M into some linear representation space, see [Ka2].

In this paper we prove the following general G-equivariant real analytic imbedding theorem, which in particular shows that the answer to question (iii) is affirmative.

Theorem III. *Let G be a linear Lie group and let M be a real analytic proper G-manifold, with only finitely many G-isotropy types. Then there exists a G-equivariant real analytic closed imbedding $h: M \to \mathbb{R}^n(\rho)$, where $\mathbb{R}^n(\rho)$ is a linear representation space for G. Moreover this can be constructed so that there is a G-invariant open neighborhood U of $h(M)$ in $\mathbb{R}^n(\rho)$, and G acts properly on U.*

Theorem III is proved in Section 3. The key new ingredient in the proof is the same trick as the one that we use in the proofs of Theorems I and II. The idea of the trick is to enlarge the transformation group from a given linear Lie group G to, in this case, a special linear group $\mathrm{SL}(m)$, by using the twisted product construction.

In Section 0 we give some preliminaries, and we also present a new proof of the real analytic version of Abels' theorem. Suppose J is a Lie group with only finitely many connected components, and let K be a maximal compact subgroup of J, see e.g. [Ho, Theorem XV.3.1]. Abels' theorem [A] says that if M is a smooth proper J-manifold then there always exists a global smooth K-slice S in M. Expressed in other words this means that M is J-equivariantly diffeomorphic to a twisted product $J \times_K S$, where S is a K-invariant smooth submanifold of M. There is a real analytic version of Abels' theorem, and this result was first proved in [Ku1, Satz 2.5.3], see also [HHK, Section VI.2]. Under the assumption that J is a linear Lie group with only finitely many connected components a proof is also given in [Ka3, Theorem 4.1]. Since the real analytic version of Abels' theorem is very important for the present paper, it is used in all three Sections 1, 2 and 3, we present an alternative proof of it, see Theorem 0.6.

0. Preliminaries and the real analytic version of Abels' theorem

Let G be a Lie group and let X be a Hausdorff space. We recall that an action $\Phi: G \times X \to X$, $(g, x) \mapsto gx$, of G on X is said to be proper if for every compact subset A of X,

$$G_{[A]} = \{g \in G \mid gA \cap A \neq \emptyset\}$$

is a compact subset of G. In [P] a slightly different definition of a proper action is used, but the two definitions agree if X is locally compact, see [P, Theorem 1.2.9].

Let H be a closed subgroup of the Lie group G, and let Y be an H-space. Recall that the twisted product $G \times_H Y$ is defined to be the orbit space of the H-space $G \times Y$, where H acts on $G \times Y$ by $h(g,x) = (gh^{-1}, hx)$. We let $p: G \times Y \to G \times_H Y$ be the natural projection and denote $p(g,x) = [g,x]$. There is a natural action of G on $G \times_H Y$, given by $\overline{g}[g,x] = [\overline{g}g, x]$.

Lemma 0.1. *Let G be a Lie group, and let H be a closed subgroup of G. If N is a real analytic proper H-manifold, then the twisted product $G \times_H N$ is a real analytic proper G-manifold.*

Proof. For the fact that $G \times_H N$ is a real analytic G-manifold, see for example [I, Section 4], and since the action of H on N is proper we have by [I–Ka, Lemma 3.10] that the action of G on $G \times_H N$ is proper. $\square$

We now recall the notion of a slice. A subset S of a G-space X is said to be an H-slice in X if:

(a) S is an H-invariant subset of X.

(b) GS is open in X.

(c) The map $\mu: G \times_H S \to GS$, $[g,x] \mapsto gx$, is a G-equivariant homeomorphism.

If X is a C^r G-manifold, $1 \le r \le \omega$, and S is a C^r submanifold of X, we say that S is a C^r H-slice in X if (a) and (b) hold, and the map μ in (c) is a G-equivariant C^r isomorphism. We say that an H-slice S is a slice at $x \in X$, if $x \in S$ and $G_x = H$. A slice S is called a global slice in X if $GS = X$.

The definition of a slice given above is the same as for example in [Br], [Ko] and [Kaw], (in [Br] and [Kaw] the transformation group G is assumed to be a compact Lie group). In [P] and [A–S] the notion of a slice is defined by another definition, and Lemma 0.2 below shows that the two definitions are equivalent.

Lemma 0.2. *Let S be a subset [a C^r submanifold] of a G-space [a C^r G-manifold, $1 \le r \le \omega$] X, and let H be a closed subgroup of G. Then the following conditions are equivalent:*

1. *S is an H-slice [a C^r H-slice] in X.*
2. *GS is an open subset of X and there exists a G-equivariant map [C^r map] $\rho: GS \to G/H$ such that $\rho^{-1}(eH) = S$.*

Proof. See [I, Lemma 5.2], where the proof is given in the C^∞ case, but the same proof applies in the C^r case, $1 \le r \le \omega$, as well as in the purely topological case. $\square$

The following lemma is Lemma 4.2.3 in [A–S]. We use Lemma 0.3 in Section 1.

Lemma 0.3. *Let $f: X \to Y$ be a G-equivariant map, where X and Y are locally compact proper G-spaces. Suppose $x \in X$ is such that $G_{f(x)} = G_x$ and f is a local homeomorphism at x. Then there exists a G-invariant open neighborhood V of x such that $f|: V \to h(V)$ is a homeomorphism, and $h(V)$ is open in Y.*

Proof. We know by [P, Proposition 2.3.1] that there exists a slice S' at $f(x)$ in Y. Let us denote $G_x = G_{f(x)} = H$. Then S' is a H-slice, and using Lemma 0.2 we see that $S = f^{-1}(S')$ is an H-slice in X. Since $G_x = H$ we have that S is a slice at x.

Since H is compact and f is a local homeomorphism there exists an H-invariant open neighborhood U of x such that $f|: U \to f(U)$ is a homeomorphism and $f(U)$ is open in Y. Then $S_1 = U \cap S$ is a slice at x, and $S_1' = f(U) \cap S'$ is a slice at $f(x)$, and $f_1 = f|: S_1 \to S_1'$ is an H-equivariant homeomorphism. Thus $\hat{f}_1: G \times_H S_1 \to G \times_H S_1'$ is a G-equivariant homeomorphism, which means that $f|: GS_1 \to GS_1'$ is a G-equivariant homeomorphism. Moreover GS_1 and GS_1' are open in X and Y, respectively. □

Let us here recall the following result.

Theorem 0.4. *Let K be a compact Lie group, and let M and N be real analytic K-manifolds. Then $\mathrm{C}^{\omega,K}_{\mathrm{S}}(M,N)$ is dense in $\mathrm{C}^{\infty,K}_{\mathrm{S}}(M,N)$.*

Proof. This is proved in [M–S, Theorem 1.2] in the case when the number of K-isotropy types in N is finite, and in the general case in [Ku2, Corollary and Remark]. □

We shall now consider Abels' theorem and its real analytic version. Theorem 0.5 below was proved in [A]. We recall that if J is a Lie group with only finitely many connected components, then there exists a maximal compact subgroup K of J, and any two maximal compact subgroups K and K' of J are conjugate in J, see e.g. [Ho, Theorem XV.3.1].

Theorem 0.5. (H. Abels, 1974). *Let J be a Lie group with only finitely many connected components, and let K be a maximal compact subgroup of J. Suppose P is a C^∞ proper J-manifold. Then there exists a global C^∞ K-slice in P, i.e., there exists a K-invariant C^∞ submanifold S of P such that the map*

$$\mu: J \times_K S \to P, \quad [h,x] \mapsto hx,$$

is a J-equivariant C^∞ diffeomorphism.

The real analytic version of Abels' theorem is proved in [Ku1, Satz 2.5.3], see also [HHK, Section VI.2], and under the assumption that J is a linear Lie group with only finitely many connected components a proof is also given in [Ka3, Theorem 4.1]. We shall here give an alternative easy proof of the real analytic version of Abels' theorem.

Theorem 0.6. *Let J be a Lie group with only finitely many connected components, and let K be a maximal compact subgroup of J. Suppose P is a real analytic proper J-manifold. Then there exists a K-invariant real analytic submanifold $\tilde{S}$ of P such that the map*

$$\mu: J \times_K \tilde{S} \to P, \quad [h,x] \mapsto hx,$$

is a J-equivariant real analytic isomorphism.

Proof. By Theorem 0.5 there exists a global C^∞ K-slice S in P. By [I, Lemma 6.1] there exists a K-invariant open neighborhood W of S in P and an open neighborhood $\mathscr{W}$ of the inclusion map $i: S \to W$ in $\mathrm{C}^{\infty,K}_{\mathrm{S}}(S,W)$, such that if $\tilde{i} \in \mathscr{W}$ then $\tilde{i}: S \to W$ is a K-equivariant C^∞-imbedding and:
(a) $\tilde{S} = \tilde{i}(S)$ is a C^∞ K-slice in P,
(b) $J\tilde{S} = JS = P$.

By [I, Theorem A] there exists a real analytic structure α on S, compatible with the smooth structure on S, such that the action of K on S_α is real analytic. (Since K is compact this follows in fact already by [M–S, Theorem 1.3].) Now W is a real analytic K-manifold, and by Theorem 0.4 there exists an $\tilde{i} \in \mathscr{W} \cap \mathrm{C}^{\omega,K}_{\mathrm{S}}(S_\alpha, W)$. The map

$$\mu(\tilde{i})\colon J \times_K S_\alpha \to P, \quad [h, x] \mapsto h\tilde{i}(x),$$

is a J-equivariant C^∞-diffeomorphism by (a) and (b), and since $\tilde{i}$ is real analytic it follows that $\mu(\tilde{i})$ is real analytic, see e.g. [I, Lemma 4.1]. It now follows by the real analytic inverse function theorem, see e.g. [N, Theorem 2.2.10], that the map $\mu(\tilde{i})^{-1}$ is real analytic, and hence $\mu(\tilde{i})$ is a J-equivariant real analytic isomorphism. Therefore $\tilde{i}(S_\alpha) = \tilde{S}$ is a K-invariant real analytic submanifold of P, and the map

$$\tilde{\mu}\colon J \times_K \tilde{S} \to P, \quad [h, y] \mapsto hy,$$

is a J-equivariant real analytic isomorphism. □

1. G-invariant real analytic Riemannian metrics

Let P be a real analytic proper J-manifold, where J is a Lie group with only finitely many connected components. Using the real analytic version of Abels' theorem, i.e., Theorem 0.6, it is easy to show that P has a J-invariant real analytic Riemannian metric, and this is done in [Ku1, Satz 2.5.6]. For the sake of completeness we give this result here below as Lemma 1.1, and the proof we give is essentially the same as in [Ku1]. For a different proof of the result in Lemma 1.1 see [Ka3, Theorem 5.1].

Lemma 1.1. *Let J be a Lie group with only finitely many connected components, and let P be a real analytic proper J-manifold. Then P has a J-invariant real analytic Riemannian metric.*

Proof. By Theorem 0.6 we may assume that P is of the form $P = J \times_K S$, where K is a maximal compact subgroup of J and S is a real analytic K-manifold. Since K is compact there exists a K-invariant real analytic Riemannian metric $\alpha\colon \mathrm{T}S \oplus \mathrm{T}S \to \mathbb{R}$ on S, see [Ka1, Theorem 1.17]. Here $\mathrm{T}S$ is the tangent bundle of S, and $\mathrm{T}S \oplus \mathrm{T}S$ denotes Whitney sum. Now $\mathrm{T}P = J \times_K \mathrm{T}S$ and $\mathrm{T}P \oplus \mathrm{T}P = J \times_K (\mathrm{T}S \oplus \mathrm{T}S)$. Hence $\mu(\alpha)\colon J \times_K (\mathrm{T}S \oplus \mathrm{T}S) \to \mathbb{R}$, $[h, (u, v)] \mapsto \alpha(u, v)$, is a J-invariant Riemannian metric on P, and the map $\mu(\alpha)$ is real analytic since α is real analytic, see e.g. [I, Lemma 4.1]. □

Next we show that by a simple trick, using the twisted product construction, we can extend Lemma 1.1 to hold for all real analytic G-manifolds, where G is a good Lie group.

Proposition 1.2. *Let G be a good Lie group, and let M be a real analytic proper G-manifold. Then M has a G-invariant real analytic Riemannian metric.*

Proof. Since G is a good Lie group we may consider G as a closed subgroup of a Lie group J with only finitely many connected componets. We form the twisted product

$$P = J \times_G M.$$

Then P is a real analytic proper J-manifold, by Lemma 0.1. Hence there exists by Lemma 1.1 a J-invariant real analytic Riemannian metric on P. Since M is a G-invariant real analytic submanifold of P we obtain an induced G-invariant real analytic Riemannian metric on M. $\square$

Proposition 1.2 proves the first part of Theorem I. We shall now prove the second part of Theorem I, more precisely we prove the result in Proposition 1.4 below. The proof we give is completely analogous to the one given in [Mi, Theorem 3.6], in the non-equivariant case. (The result in this reference is actually for the C^∞-case, but this difference plays no essential role here, since we are assuming the existence of a G-invariant real analytic Riemannian metric.) We also use the following equivariant version of the topological lemma that Milnor uses in his proof, see [Mi, p. 58–59].

Lemma 1.3. *Let X and Y be locally compact, second countable and Hausdorff proper G-spaces, where G is a Lie group, and let A be a G-invariant closed subset of X. Let $f: X \to Y$ be a G-equivariant map, which is a local homeomorphism, and assume that $f|: A \to f(A)$ is a homeomorphism, and that $G_x = G_{f(x)}$, for all $x \in A$. Then there exists a G-invariant open neighborhood W of A in X such that $f|: W \to f(W)$ is a G-equivariant homeomorphism, and $f(W)$ is open in Y.*

Proof. It follows by [P, Theorem 1.2.9] that the orbit spaces X/G and Y/G are Hausdorff, and furthermore X/G and Y/G are locally compact and second countable, since the projection onto the orbit space is an open map. By Lemma 0.3 each $x \in A$ has a G-invariant open neighborhood U_x such that $f|: U_x \to f(U_x)$ is a G-equivariant homeomorphism, and $f(U_x)$ is open in Y. We set $U = \bigcup_{x \in A} U_x$, and $V = f(U)$. The induced map $\overline{f}|: U/G \to V/G$ is a local homeomorphism and $\overline{f}|: A/G \to f(A)/G$ is a homeomorphism. It now follows by the lemma in [Mi, p. 58–59] (see also e.g. [Kaw, Lemma 4.9]), that there exists an open neighborhood U^* of A/G in U/G such that $\overline{f}|: U^* \to f(U^*)$ is a homeomorphism, and $f(U^*)$ is open in V/G. Let $W = p^{-1}(U^*)$, where $p: X \to X/G$ is the natural projection. Then W and $f|: W \to f(W)$ are as required. $\square$

Proposition 1.4. *Let M be a real analytic proper G-manifold, where G is a Lie group, and let N be a G-invariant real analytic closed submanifold of M. Assume that M has a G-invariant real analytic Riemannian metric. Then there exist a G-invariant open neighborhood W of N in M and a G-equivariant real analytic retraction $q: W \to N$.*

Proof. The tangent bundle $\mathrm{T}M$ of M is a real analytic G-manifold and, by identifying M with the zero-section of $\mathrm{T}M$, we consider M as a G-invariant real analytic closed submanifold of $\mathrm{T}M$. Then the natural projection $\pi: \mathrm{T}M \to M$ is a retraction onto M, and π is both G-equivariant and real analytic. Since the action of G on M is proper it follows that the action of G on $\mathrm{T}M$ is proper.

Since M has a G-invariant real analytic Riemannian metric the exponential map is defined on a G-invariant open neighborhood U of the zero-section M in $\mathrm{T}M$, and $\exp: U \to M$ is a G-equivariant real analytic map, and $\exp|M = \mathrm{id}_M$.

Let $(\mathrm{T}N)^\perp$ be the orthogonal complement of $\mathrm{T}N$ in $(\mathrm{T}M)|_N$, where $(\mathrm{T}M)|_N$ denotes the restriction of the tangent bundle $\mathrm{T}M$ to N. The bundle $\pi|: (\mathrm{T}N)^\perp \to N$ is canonically isomorphic to the normal bundle of N in M. Now N, which we have identified with the zero-section of $(\mathrm{T}N)^\perp$, is a G-invariant real analytic closed submanifold of the real analytic proper G-manifold $(\mathrm{T}N)^\perp$, and we have that $\dim(\mathrm{T}N)^\perp = \dim M$. Now $V = U \cap (\mathrm{T}N)^\perp$ is a G-invariant open neighborhood of N in $(\mathrm{T}N)^\perp$, and

$$\exp| \colon V \to M$$

is a G-equivariant real analytic map, and $\exp|N = \mathrm{id}_N$. It is not difficult to verify that at each point $x \in N$ the differential of $\exp|V$ is an isomorphism, see e.g. [Br, p. 306]. Hence there exists an open neighborhood V_1 of N in V, such that the differential of $\exp|V$ is an isomorphism at every $y \in V_1$. We may assume that V_1 is G-invariant, since if the differential is an isomorphism at a point y, then it is also an isomorphism at the point gy, for all $g \in G$. The real analytic inverse function theorem, see e.g. [N, Theorem 2.2.10], now tells us that $\exp| \colon V_1 \to M$ is a real analytic local isomorphism. By Lemma 1.3 we find a G-invariant open neighborhood V_0 of N in $(\mathrm{T}N)^\perp$, where $V_0 \subset V_1$, such that

$$\exp| \colon V_0 \to \exp(V_0) = W$$

is a G-equivariant real analytic isomorphism, and W is a G-invariant open neighborhood of N in M. Now $q = (\pi|) \circ (\exp|)^{-1} \colon W \to N$ is a G-equivariant real analytic retraction. $\square$

Note that Propositions 1.2 and 1.4 together prove Theorem I.

2. Approximation of G-equivariant smooth maps by real analytic ones

In this section we prove Theorem II. First we establish the following result.

Proposition 2.1. *Suppose J is a Lie group with only finitely many connected components. Let P and Q be real analytic J-manifolds, and assume that the action of J on P is proper. Then $\mathrm{C}^{\omega,J}_{\mathrm{SW}}(P,Q)$ is dense in $\mathrm{C}^{r,J}_{\mathrm{SW}}(P,Q)$, $1 \le r \le \infty$.*

Proof. Let K be a maximal compact subgroup of J. By Theorem 0.6 we may assume that P is of the form $P = J \times_K S$, where S is a K-invariant real analytic submanifold of P. If $f \colon S \to Q$ is a K-equivariant C^r map, $1 \le r \le \omega$, then

$$\mu(f) \colon J \times_K S \to Q, \quad [j,x] \mapsto jf(x),$$

is a J-equivariant C^r map, and note in particular the case $r = \omega$, i.e., if f is real analytic then also $\mu(f)$ is real analytic, see e.g. [I, Lemma 4.1]. By [I–Ka, Theorem 5.5] the map

$$\Theta \colon \mathrm{C}^{r,K}_{\mathrm{S}}(S,Q) \to \mathrm{C}^{r,J}_{\mathrm{SW}}(J \times_K S, Q), \quad f \mapsto \mu(f),$$

is a homeomorphism, and moreover Θ maps $\mathrm{C}^{\omega,K}_{\mathrm{S}}(S,Q)$ onto $\mathrm{C}^{\omega,J}_{\mathrm{SW}}(J \times_K S, Q)$. Since $\mathrm{C}^{\omega,K}_{\mathrm{S}}(S,Q)$ is dense in $\mathrm{C}^{r,K}_{\mathrm{S}}(S,Q)$, $1 \le r \le \infty$, by Theorem 0.4, it now follows that $\mathrm{C}^{\omega,J}_{\mathrm{SW}}(J \times_K S, Q)$ is dense in $\mathrm{C}^{r,J}_{\mathrm{SW}}(J \times_K S, W)$, which proves our claim. $\square$

We remark here that for the above proof it was enough to know that Θ is bijective and continuous, and that $\Theta(\mathrm{C}_{\mathrm{S}}^{\omega,K}(S,Q)) \subset \mathrm{C}_{\mathrm{SW}}^{\omega,J}(J \times_K S, Q)$. However, the proof of Theorem II *does use* the full conclusion of [I–Ka, Theorem 5.5] that Θ is a homeomorphism, for the proof below uses the continuity of the inverse map Θ^{-1}, given by [I–Ka, Proposition 4.4].

Proof of Theorem II. Let M and N be real analytic proper G-manifolds, where G is a good Lie group. Suppose $f\colon M \to N$ is a G-equivariant C^r map, $1 \le r \le \infty$, and let $\mathscr{U}$ be an open neighborhood of f in $\mathrm{C}_{\mathrm{SW}}^{r,G}(M,N)$. We shall show that there exists a G-equivariant real analytic map $h\colon M \to N$, such that $h \in \mathscr{U}$.

Since G is a good Lie group we may consider G as a closed subgroup of a Lie group J with only finitely many connected components. We can then form the twisted products $J \times_G M$ and $J \times_G N$, and we define

$$\hat{f}\colon J \times_G M \to J \times_G N, \quad [j,x] \mapsto [j, f(x)].$$

Then $J \times_G M$ and $J \times_G N$ are real analytic proper J-manifolds, by Lemma 0.1, and $\hat{f}$ is a J-equivariant C^r map.

We now consider the space $J \times_G N$ with the induced action of G. Then $J \times_G N$ is a real analytic proper G-manifold, and we regard N as a G-invariant real analytic closed submanifold of $J \times_G N$ through the canonical imbedding

$$i_2\colon N \to J \times_G N, \quad y \mapsto [e,y].$$

By Theorem I there exist a G-invariant open neighborhood W of N in $J \times_G N$, and a G-equivariant real analytic retraction $q\colon W \to N$. By [I–Ka, Proposition 4.3] the induced map

$$q_*\colon \mathrm{C}_{\mathrm{SW}}^{r,G}(M,W) \to \mathrm{C}_{\mathrm{SW}}^{r,G}(M,N), \quad \alpha \mapsto q \circ \alpha, \tag{1}$$

is continuous. Since $\mathrm{C}_{\mathrm{SW}}^{r,G}(M,W)$ is open in $\mathrm{C}_{\mathrm{SW}}^{r,G}(M, J \times_G N)$, see [I–Ka, Lemma 4.8], it now follows that the set

$$\mathscr{V} = q_*^{-1}(\mathscr{U})$$

is open in $\mathrm{C}_{\mathrm{SW}}^{r,G}(M, J \times_G N)$. We denote

$$i_1\colon M \to J \times_G M, \quad x \mapsto [e,x].$$

The induced map $\bar{i}_1\colon M/G \to (J \times_G M)/J$ is a homeomorphism, and hence in particular of finite type, and thus we have by [I–Ka, Proposition 4.4] that

$$i_1^*\colon \mathrm{C}_{\mathrm{SW}}^{r,J}(J \times_G M, J \times_G N) \to \mathrm{C}_{\mathrm{SW}}^{r,G}(M, J \times_G N), \quad \beta \mapsto \beta \circ i_1,$$

is continuous. Hence

$$\mathscr{W} = (i_1^*)^{-1}(\mathscr{V})$$

is an open subset of $\mathrm{C}^{r,J}_{\mathrm{SW}}(J \times_G M, J \times_G N)$. Moreover $\hat{f} \in \mathscr{W}$, since $q \circ \hat{f} \circ i_1 = f \in \mathscr{U}$, and thus $\mathscr{W}$ is in particular non-empty. By Proposition 2.1 we know that there exsits an $\tilde{h} \in \mathscr{W}$, which is real analytic. We define

$$h = q_*(i_1^*(\tilde{h})).$$

Then $h \in \mathscr{U}$, and $h = q \circ \tilde{h} \circ i_1 \colon M \to N$ is a G–equivariant real analytic map, since the maps $i_1, \tilde{h}$ and q are G-equivariant and real-analytic. □

3. Existence of G-equivariant real analytic imbeddings into representation spaces

In this section we prove Theorem III. We shall first recall some standard facts about the space $\mathrm{S}(n)$ of all symmetric $n \times n$-matrices over $\mathbb{R}$. The space $\mathrm{S}(n)$ is an $\frac{1}{2}n(n+1)$-dimensional vector space over $\mathbb{R}$, and there is a linear action of $\mathrm{GL}(n)$ on $\mathrm{S}(n)$ given by

$$\Phi \colon \mathrm{GL}(n) \times \mathrm{S}(n) \to \mathrm{S}(n), \quad (a, b) \mapsto aba^t, \tag{1}$$

where a^t denotes the transpose of the matrix $a \in \mathrm{GL}(n)$. Note that the isotropy subgroup at the identity matrix $e \in \mathrm{S}(n)$ is $\mathrm{GL}(n)_e = \mathrm{O}(n)$. It is well-known, and easy to see, that the $\mathrm{GL}(n)$-orbit $\mathrm{GL}(n)e = \{aa^t \mid a \in \mathrm{GL}(n)\}$ equals the set $\mathrm{S}^+(n)$ of all positive definite symmetric $n \times n$-matrices over $\mathbb{R}$, and this set is open in $\mathrm{S}(n)$. Furthermore

$$\alpha \colon \mathrm{GL}(n)/\mathrm{O}(n) \to \mathrm{S}^+(n), \quad a\mathrm{O}(n) \mapsto aa^t,$$

is a $\mathrm{GL}(n)$-equivariant real analytic isomorphism. Since $\mathrm{O}(n)$ is compact the action of $\mathrm{GL}(n)$ on $\mathrm{GL}(n)/\mathrm{O}(n)$ is proper, and hence the action of $\mathrm{Gl}(n)$ on $\mathrm{S}^+(n)$ is proper.

We now restrict the action of $\mathrm{GL}(n)$ in (1) to the subgroup $\mathrm{SL}(n)$, that is we consider the action

$$\Phi| \colon \mathrm{SL}(n) \times \mathrm{S}(n) \to \mathrm{S}(n), \quad (a, b) \mapsto aba^t. \tag{2}$$

Then $\mathrm{SL}(n)_e = \mathrm{SO}(n)$ and the $\mathrm{SL}(n)$-orbit $\mathrm{SL}(n)e = \{aa^t \mid a \in \mathrm{SL}(n)\}$ equals the set of all positive definite symmetric $(n \times n)$-matrices over $\mathbb{R}$ with determinant equal to 1, and this set is closed in $\mathrm{S}(n)$. Since the action of $\mathrm{GL}(n)$ on $\mathrm{S}^+(n)$ is proper, it follows that the action of $\mathrm{SL}(n)$ on $\mathrm{S}^+(n)$ is proper. Therefore

$$\beta \colon \mathrm{SL}(n)/\mathrm{SO}(n) \to \mathrm{S}(n), \quad a\mathrm{SO}(n) \mapsto aa^t,$$

is an $\mathrm{SL}(n)$-equivariant real analytic closed imbedding, and $\mathrm{S}^+(n)$ is an $\mathrm{SL}(n)$-invariant open neighborhood of $\mathrm{Im}\,\beta$ in $\mathrm{S}(n)$, such that the action of $\mathrm{SL}(n)$ on $\mathrm{S}^+(n)$ is proper.

We now choose a linear isomorphism $\gamma \colon \mathrm{S}(n) \to \mathbb{R}^s$, where $s = \frac{1}{2}n(n+1)$, and identify $\mathrm{S}(n)$ with $\mathbb{R}^s$ through γ. Then (2) gives a linear action of $\mathrm{SL}(n)$ on $\mathbb{R}^s$, and hence a linear representation $\sigma\colon \mathrm{SL}(n) \to \mathrm{GL}(s)$, and we denote the corresponding linear representation space for $\mathrm{SL}(n)$ by $\mathbb{R}^s(\sigma)$. We set

$$\gamma(e) = v \quad \text{and} \quad \gamma(\mathrm{S}^+(n)) = W. \tag{3}$$

Then

$$\beta\colon \mathrm{SL}(n)/\mathrm{SO}(n) \to \mathbb{R}^s(\sigma), \quad a\mathrm{SO}(n) \mapsto av, \tag{4}$$

where $av = \sigma(a) \cdot v$, is an $\mathrm{SL}(n)$-equivariant real analytic closed imbedding, and W is an $\mathrm{SL}(n)$-invariant open neighborhood of $\operatorname{Im} \beta$ in $\mathbb{R}^s(\sigma)$ and the action of $\mathrm{SL}(n)$ on W is proper.

We will also use the imbedding

$$\iota\colon \mathrm{GL}(n) \to \mathrm{SL}(2n) \tag{5}$$

of $\mathrm{GL}(n)$ as a closed subgroup of $\mathrm{SL}(2n)$ given by

$$\iota(a) = \begin{bmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{bmatrix}.$$

We use the following lemma in the proof of Proposition 3.2.

Lemma 3.1. *Let H be a closed subgroup of a Lie group G, and let $f\colon N \to M$ be an H-equivariant real analytic closed imbedding, where N is a real analytic H-manifold and M is a real analytic G-manifold. Then the map*

$$f^*\colon G \times_H N \to (G/H) \times M, \quad [g, x] \mapsto (gH, gf(x)),$$

is a G-equivariant real analytic closed imbedding.

Proof. The twisted products $G \times_H N$ and $G \times_H M$ are real analytic G-manifolds, see e.g. [I, Section 4] for details, and $\hat{f}\colon G \times_H N \to G \times_H M$, $[g, x] \mapsto [g, f(x)]$, is a G-equivariant real analytic closed imbedding. Hence we only need to show that the map

$$\psi\colon G \times_H M \to (G/H) \times M, \quad [g, y] \mapsto (gH, gy),$$

is a G-equivariant real analytic isomorphism. The map ψ is G-equivariant and real analytic, and also a bijection. The inverse map is given by

$$\psi^{-1}\colon (G/H) \times M \to G \times_H M, \quad (gH, y) \mapsto [g, g^{-1}y].$$

Each point in G/H has an open neighborhood U such that there exists a real analytic cross section $\sigma\colon U \to G$, see e.g. [He, Lemma II.4.1]. Hence $\psi^{-1}|(U \times M)$ is real analytic, since $\psi^{-1}(u, y) = [\sigma(u), \sigma(u)^{-1}y]$, for every $(u, y) \in U \times M$. This completes the proof. □

Proposition 3.2. *Let P be a real analytic proper* $\mathrm{SL}(n)$*-manifold, $n \geq 2$, with a finite number of isotropy types. Then there exists an* $\mathrm{SL}(n)$*-equivariant real analytic closed imbedding $h: P \to \mathbb{R}^t(\theta)$, where $\mathbb{R}^t(\theta)$ is a linear representation space for* $\mathrm{SL}(n)$.

Moreover, this can be constructed so that there is an $\mathrm{SL}(n)$*- invariant open neighborhood U of $h(P)$ in $\mathbb{R}^t(\theta)$, and the action of* $\mathrm{SL}(n)$ *on U is proper.*

Proof. By Theorem 0.6 we may suppose that P is of the form

$$P = \mathrm{SL}(n) \times_{\mathrm{SO}(n)} N,$$

where N is an $\mathrm{SO}(n)$-invariant real analytic submanifold of P. By [P, Proposition 4.4.2] the number of $\mathrm{SO}(n)$-isotropy types in P, when regarded as an $\mathrm{SO}(n)$-space, is finite and hence in particular the number of $\mathrm{SO}(n)$-isotropy types in N is finite. Since $\mathrm{SO}(n)$ is compact there exists by [M–S, Theorem 1.1] an $\mathrm{SO}(n)$-equivariant real analytic closed imbedding

$$f: N \to \mathbb{R}^k(\rho),$$

where $\mathbb{R}^k(\rho)$ is a linear representation space for $\mathrm{SO}(n)$. By [P, Theorem 3.1] we find a linear representation space $\mathbb{R}^r(\tau)$ for $\mathrm{SL}(n)$ which contains $\mathbb{R}^k(\rho)$ as an $\mathrm{SO}(n)$-invariant linear subspace. Now $f: N \to \mathbb{R}^r(\tau)$, and by Lemma 3.1 the map

$$f^*: \mathrm{SL}(n) \times_{\mathrm{SO}(n)} N \to (\mathrm{SL}(n)/\mathrm{SO}(n)) \times \mathbb{R}^r(\tau), \quad [g, x] \mapsto (g\mathrm{SO}(n), gf(x)),$$

is an $\mathrm{SL}(n)$-equivariant real analytic closed imbedding. We set

$$\beta \times \mathrm{id}: (\mathrm{SL}(n)/\mathrm{SO}(n)) \times \mathbb{R}^r(\tau) \to \mathbb{R}^s(\sigma) \times \mathbb{R}^r(\tau) = \mathbb{R}^t(\theta),$$

where β is as in (4), and $t = s + r$ and $\theta = \sigma \oplus \tau$. Then

$$h = (\beta \times \mathrm{id}) \circ f^*: P \to \mathbb{R}^t(\theta)$$

is an $\mathrm{SL}(n)$-equivariant real analytic closed imbedding. If W is as in (3), then $U = W \times \mathbb{R}^r(\tau)$ is an $\mathrm{SL}(n)$-invariant open neighborhood of $\mathrm{Im}(h)$, and the action of $\mathrm{SL}(n)$ on U is proper. $\square$

Proof of Theorem III. Let G be a linear Lie group, and let M be a real analytic proper G-manifold with only finitely many isotropy types. Using (5) above we may consider G as a closed subgroup of some $\mathrm{SL}(m)$, $m \geq 2$. We now form the twisted product

$$P = \mathrm{SL}(m) \times_G M.$$

Then P is a real analytic proper $\mathrm{SL}(m)$-manifold, by Lemma 0.1. Since the number of G-isotropy types in M is finite, it follows directly that the number of $\mathrm{SL}(m)$-isotropy types in P is finite. Hence there exist by Proposition 3.2 an $\mathrm{SL}(m)$-equivariant real analytic closed imbedding

$$h: P \to \mathbb{R}^t(\theta),$$

where $\mathbb{R}^t(\theta)$ is a linear representation space for $\mathrm{SL}(m)$, and an $\mathrm{SL}(m)$-invariant open neighborhood U of $h(P)$ in $\mathbb{R}^t(\theta)$, such that the action of $\mathrm{SL}(m)$ on U is proper. Then

$$h|: M \to \mathbb{R}^t(\theta|G)$$

is a G-equivariant real analytic closed imbedding of M into the linear representation space $\mathbb{R}^t(\theta|G)$ for G, and U is a G-invariant open neighborhood of $h(M)$ in $\mathbb{R}^t(\theta|G)$, and the action of G on U is proper. $\square$

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Department of Mathematics
P.O. Box 4
FIN–00014 University of Helsinki
Finland