A new topology for the set $\mathrm{C}^{\infty,G}(M,N)$ of G-equivariant smooth maps

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Suppose M and N are smooth manifolds, and let $\mathrm{C}^\infty(M,N)$ denote the set of smooth maps from M to N. The standard choices for a topology on $\mathrm{C}^\infty(M,N)$ are the weak topology, and the strong topology (also called the Whitney topology), see [Hi, Chapter 2] and [Ma]. We denote $\mathrm{C}^\infty(M,N)$ with the weak topology by $\mathrm{C}^\infty_{\mathrm{W}}(M,N)$, and with the strong topology by $\mathrm{C}^\infty_{\mathrm{S}}(M,N)$. When M is compact the weak and strong topologies are the same. If M is not compact the weak topology on $\mathrm{C}^\infty(M,N)$ does not control the behavior at infinity of the smooth maps from M to N, and in this case the strong topology is the right topology to use on $\mathrm{C}^\infty(M,N)$.

Now let G be a Lie group and let M and N be smooth G-manifolds. By $\mathrm{C}^{\infty,G}(M,N)$ we denote the set of G-equivariant smooth maps from M to N. Neither the weak topology nor the strong topology is in general a good topology for $\mathrm{C}^{\infty,G}(M,N)$. The weak topology has the limitations mentioned above, which in the equivariant case means that if the orbit space M/G is not compact the weak topology on $\mathrm{C}^{\infty,G}(M,N)$ does not control the behavior at infinity. The strong topology on $\mathrm{C}^{\infty,G}(M,N)$ is often completely inadequate. For example, if G is a non-compact Lie group and the action of G on M is proper, then the strong topology on $\mathrm{C}^{\infty,G}(M,N)$ is the discrete topology, see Proposition 4.7.

In this paper we define a new topology on $\mathrm{C}^{\infty,G}(M,N)$, *the strong-weak topology*, and we denote $\mathrm{C}^{\infty,G}(M,N)$ with this topology by $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$. When G is compact the strong-weak topology agrees with the strong topology, i.e.,

$$\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N) = \mathrm{C}^{\infty,G}_{\mathrm{S}}(M,N), \quad \text{if} \quad G \quad \text{is compact.}$$

On the other hand, if the orbit space M/G is compact, then the strong-weak topology agrees with the weak topology, i.e.,

$$\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N) = \mathrm{C}^{\infty,G}_{\mathrm{W}}(M,N), \quad \text{if} \quad M/G \quad \text{is compact,}$$

see Propositions 4.1 and 4.2. But in the general case, when G and M/G are non-compact, the strong-weak topology differs from both the weak topology and the strong topology, and in these cases the strong-weak topology is the right topology for $\mathrm{C}^{\infty,G}(M,N)$.

In order to illustrate the above claim we mention here that we prove in [I–Ka] that for a large class of Lie groups G (which includes all linear Lie groups and all connected Lie groups), every G-equivariant smooth map $f\colon M \to N$ between real analytic proper G-manifolds can be approximated arbitrarily well in the strong-weak topology by a G-equivariant real analytic map, that is, we prove that $\mathrm{C}^{\omega,G}_{\mathrm{SW}}(M,N)$ is dense in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$. For G a compact Lie group it was known before that $\mathrm{C}^{\omega,G}_{\mathrm{S}}(M,N)$ is dense in $\mathrm{C}^{\infty,G}_{\mathrm{S}}(M,N)$, see [M–S, Theorem 1.2] and [Ku].

We have here above given the discussion in the smooth case, i.e., in the C^∞ case, but it should be noted that we actually prove all our results in the C^r case, where $1 \le r \le \infty$ or $r = \omega$, and ω stands for real analytic.

The strong-weak topology may very well be of interest also in other cases than the case of G-equivariant maps between G-manifolds that we study in this paper. There are also some technical advantages for our work to have a strong-weak topology defined directly on the set $\mathrm{C}^r(M,N)$, $1 \le r \le \omega$. For these reasons we first define the strong-weak topology with respect to an arbitrary phase map $\rho\colon M \to \Omega$, and we denote $\mathrm{C}^r(M,N)$ with this topology by $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$, see Section 1. The strong-weak topology on the set $\mathrm{C}^{r,G}(M,N)$, $1 \le r \le \omega$, of G-equivariant C^r maps is then defined to be the topology one obtains by using the natural projection $p\colon M \to M/G$ as the phase map.

In Section 2 we prove basic properties of the strong-weak topology with respect to a phase map $\rho\colon M \to \Omega$, like continuity of induced maps and also a product theorem. Section 3 is concerned with proper G-spaces and fundamental sets, and here we also fix some notation that will be used later on in the paper.

In Section 4 we define the strong-weak topology on the set $\mathrm{C}^{r,G}(M,N)$ of G-equivariant C^r maps, $1 \le r \le \omega$. We show that the basic properties of the strong-weak topology established in Section 2 easily carry over to give the corresponding results for $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$, $1 \le r \le \omega$. We also prove in Proposition 4.7 the already mentioned fact that if G is a non-compact Lie group and M is a proper C^r G-manifold, then the strong topology on $\mathrm{C}^{r,G}(M,N)$ is the discrete topology.

The main results of the paper are proved in Sections 5, 6 and 7, and they are all established in the C^r case, where $1 \le r \le \omega$. For convenience of expression we restrict our discussion of these results here below to the smooth case, i.e., the C^∞ case.

Let us first consider the special case where a Lie group G acts on a smooth manifold M by a proper and free smooth action. Then the orbit space M/G is a smooth manifold, and there is a natural bijection between the set $\mathrm{C}^\infty(M/G,P)$ of smooth maps from M/G to a smooth manifold P and the set $\mathrm{C}^{\infty,G}(M,P)$ of G–invariant smooth maps from M to P. We prove the following.

Theorem A. *Let M be a smooth manifold on which a Lie group G acts by a proper and free smooth action, and let $p\colon M \to M/G$ be the projection. Suppose P is a smooth manifold, with no action of G. Then the natural bijection*

$$p^*\colon \mathrm{C}^\infty_{\mathrm{S}}(M/G,P) \to \mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,P), \quad f \mapsto f \circ p,$$

is a homeomorphism.

The result in Theorem A is established in Corollary 5.3. Proposition 5.4 gives a more general version of Corollary 5.3, and using this proposition and also other results proved in Sections 4 and 5, we are able to prove Theorem 5.5, which is one of the main results of this paper. The C^∞ case of Theorem 5.5 is the following.

Theorem B. *Let H be a closed subgroup of a Lie group G, and let N be a smooth H-manifold and P a smooth G-manifold. Then the canonical bijection*

$$\Theta\colon \mathrm{C}^{\infty,H}_{\mathrm{SW}}(N,P) \xrightarrow{\cong} \mathrm{C}^{\infty,G}_{\mathrm{SW}}(G \times_H N, P), \quad f \mapsto \mu(f),$$

is a homeomorphism.

Here $G \times_H N$ denotes the twisted product. Recall that if N is a smooth H-manifold, then $G \times_H N$ is a smooth G-manifold, and if $f\colon N \to P$ is an H-equivariant smooth map, where P is a smooth G-manifold, then $\mu(f)\colon G \times_H N \to P$, $[g,x] \mapsto gf(x)$, is a G-equivariant smooth map, see e.g. [I2, Section 4] for the details. The correspondence $f \mapsto \mu(f)$ gives a bijection between the sets $\mathrm{C}^{\infty,H}(N,P)$ and $\mathrm{C}^{\infty,G}(G \times_H N, P)$, and Theorem B says that if we give both sets the strong-weak topology this bijection is a homeomorphism.

Note the special case of Theorem B when H is a *compact* subgroup of G. In this case $\mathrm{C}^{\infty,H}_{\mathrm{SW}}(N,P) = \mathrm{C}^{\infty,H}_{\mathrm{S}}(N,P)$, see Proposition 4.1, and thus Theorem B gives us a homeomorphism

$$(*) \qquad \Theta\colon \mathrm{C}^{\infty,H}_{\mathrm{S}}(N,P) \xrightarrow{\cong} \mathrm{C}^{\infty,G}_{\mathrm{SW}}(G \times_H N, P), \quad f \mapsto \mu(f).$$

Let us here also point out the following. Suppose G is a non-compact Lie group and let H be a compact subgroup of G. Then $G \times_H N$ is a smooth proper G-manifold, see Lemma 3.10, and since G is non-compact it follows, see Proposition 4.7, that $\mathrm{C}^{\infty,G}(G \times_H N, P)$ with the strong topology, i.e., $\mathrm{C}^{\infty,G}_{\mathrm{S}}(G \times_H N, P)$, has the discrete topology. This shows that the bijection Θ in $(*)$ fails to be a homeomorphism if we use the strong topology on the set $\mathrm{C}^{\infty,G}(G \times_H N, P)$.

In Section 6 we prove that if M and N are smooth G-manifolds, and the action of G on M is proper, then the set of G-equivariant smooth immersions is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$, see Proposition 6.1. The analogous result also holds for G-equivariant submersions. Furthermore we prove in Proposition 6.4 that if M and N are smooth proper G-manifolds, then the set of G-equivariant smooth proper maps is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$.

In Section 7 we study G-equivariant C^r imbeddings. The main result here is Theorem 7.3, which in the C^∞ case is the following result.

Theorem C.1. *Let M and N be smooth proper G-manifolds, where G be a Lie group. Then the set of G-equivariant smooth imbeddings is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$.*

Combining Theorem C.1 with Proposition 6.4 we obtain.

Theorem C.2. *Let M and N be smooth proper G-manifolds, where G is a Lie group. Then the set of G-equivariant smooth closed imbeddings is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$.*

Finally as an easy consequence of Theorem C.2 we obtain the important result that the set of G-equivariant smooth diffeomorphisms is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$, see Theorem 7.5.

Theorem D. *Let G be a Lie group, and let M and N be smooth proper G-manifolds. Then the set of G-equivariant smooth diffeomorphisms is open in $\mathrm{C}^{\infty,G}_{\mathrm{SW}}(M,N)$.*

1. Definition of the strong-weak topology

Let U be an open subset of $\mathbb{R}^m$, and let $f\colon U \to \mathbb{R}^n$ be a C^r map, where $1 \le r < \infty$. If K is a compact subset of U we set

$$\|f\|_K^r = \max\Big\{ \Big|\frac{\partial^{|\alpha|} f_i(x)}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}\Big| \;\Big|\; x \in K, \quad 1 \le i \le n, \quad 0 \le |\alpha| \le r \Big\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is an m-tuple of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Now let M and N be C^r manifolds, where $1 \le r < \infty$. By $\mathrm{C}^r(M,N)$ we denote the set of all C^r maps from M to N. Suppose $f \in \mathrm{C}^r(M,N)$. Let (U,φ) be a chart in M, K a compact subset of U, (V,ψ) a chart in N such that $f(K) \subset V$, and $\varepsilon > 0$. Then we define

$$\begin{aligned}&\mathscr{N}^r(f; K, (U,\varphi), (V,\psi), \varepsilon)\\ &= \{h \in \mathrm{C}^r(M,N) \mid h(K) \subset V \;\text{ and }\; \|\psi \circ h \circ \varphi^{-1} - \psi \circ f \circ \varphi^{-1}\|_{\varphi(K)}^r < \varepsilon\}.\end{aligned}$$

We call such a set $\mathscr{N}^r(f; K, (U,\varphi), (V,\psi), \varepsilon)$ an *elementary neighborhood* of f.

We recall that the weak topology on $\mathrm{C}^r(M,N)$ has as a basis the family of all sets of the form

$$\mathscr{N}_{\mathrm{W}} = \bigcap_{i=1}^{s} \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i),$$

where s is finite. By $\mathrm{C}^r_{\mathrm{W}}(M,N)$ we denote $\mathrm{C}^r(M,N)$ with the weak topology. The strong topology (also called the Whitney topology) on $\mathrm{C}^r(M,N)$ has as a basis the family of all sets of the form

$$\mathscr{N}_{\mathrm{S}} = \bigcap_{i\in\Lambda} \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$$

where the family $\{K_i\}_{i\in\Lambda}$ is locally finite in M. We denote the set $\mathrm{C}^r(M,N)$ with the strong topology by $\mathrm{C}^r_{\mathrm{S}}(M,N)$. In the definition of the strong topology some authors require that the family $\{U_i\}_{i\in\Lambda}$ is locally finite in M, see e.g. [Hi, Section 2.1]. However this definition, and the one given above are equivalent as follows by the following lemma.

Lemma 1.1. *Let $\{A_i\}_{i\in\Lambda}$ be a locally finite family of subsets of a paracompact space M. Then each A_i has an open neighborhood U_i in M such that the family $\{U_i\}_{i\in\Lambda}$ is locally finite in X.*

Proof. See [Č, Theorem 30.C.10]. □

We shall now define the strong-weak topology, with respect to a phase map $\rho\colon M \to \Omega$.

Definition 1.2. A *phase map* for M is a continuous map $\rho\colon M \to \Omega$, where Ω is a topological space.

Definition 1.3. Let M and N be C^r manifolds, where $1 \le r < \infty$, and let $\rho\colon M \to \Omega$ be a phase map for M. Suppose $f \in \mathrm{C}^r(M,N)$. A *basic neighborhood with respect to $\rho\colon M \to \Omega$ of f* is a set of the form

(1) $$\hat{\mathscr{N}} = \bigcap_{i\in\Lambda} \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i),$$

where the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω.

Remark 1.4. Suppose $\mathscr{N} = \mathscr{N}^r(f; K, (U,\varphi), (V,\psi), \varepsilon)$ is an elementary neighborhood of $f \in \mathrm{C}^r(M,N)$, and let $f_0 \in \mathscr{N}$. Then there exists $\varepsilon_0 > 0$ such that for the elementary neighborhood $\mathscr{N}_0 = \mathscr{N}^r(f_0; K, (U,\varphi), (V,\psi), \varepsilon_0)$ of f_0 we have that $\mathscr{N}_0 \subset \mathscr{N}$.

Proof. We have that $d = \|\psi\circ f_0 \circ \varphi^{-1} - \psi \circ f \circ \varphi^{-1}\|^r_{\varphi(K)} < \varepsilon$. If we choose $\varepsilon_0 = \varepsilon - d$ the claim follows by the triangle inequality. $\square$

Remark 1.5. Let $\hat{\mathscr{N}}$ and $\hat{\mathscr{N}}'$ be basic neighborhoods with respect to $\rho: M \to \Omega$ of f and $f' \in \mathrm{C}^r(M,N)$, respectively, and let $f_0 \in \hat{\mathscr{N}} \cap \hat{\mathscr{N}}'$. Then there exists $\hat{\mathscr{N}_0}$, which is a basic neighborhood with respect to ρ of f_0, such that $\hat{\mathscr{N}_0} \subset \hat{\mathscr{N}} \cap \hat{\mathscr{N}}'$.

Proof. We have $\hat{\mathscr{N}} = \bigcap_{i\in\Lambda} \mathscr{N}_i$ and $\hat{\mathscr{N}}' = \bigcap_{j\in\Gamma} \mathscr{N}_j'$, where $\mathscr{N}_i = \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$, $i \in \Lambda$, and $\mathscr{N}_j' = \mathscr{N}^r(f'; K_j', (U_j', \varphi_j'), (V_j', \psi_j'), \varepsilon_j)$, $j \in \Gamma$, are elementary neighborhoods of f and f', respectively, and the families $\{\rho(K_i)\}_{i\in\Lambda}$ and $\{\rho(K_j')\}_{j\in\Gamma}$ are locally finite in Ω. By Remark 1.4 we find $\varepsilon_{i,0} > 0$ such that $\mathscr{N}_{i,0} = \mathscr{N}^r(f_0; K_i, (U_i,\varphi_i), (V,\psi_i),\ \varepsilon_{i,0}) \subset \mathscr{N}_i$, $i \in \Lambda$, and $\varepsilon_{j,0}' > 0$ such that and $\mathscr{N}_{j,0} = \mathscr{N}^r(f_0; K_j', (U_j', \varphi_j'), (V_j', \psi_j), \varepsilon_{j,0}') \subset \mathscr{N}_j'$, $j \in \Gamma$. Now the family $\{\rho(K_i), \rho(K_j')\}_{i\in\Lambda, j\in\Gamma}$ is locally finite in Ω, and hence

$$\hat{\mathscr{N}_0} = \bigcap_{i\in\Lambda} \mathscr{N}_{i,0} \cap \bigcap_{j\in\Gamma} \mathscr{N}_{j,0}$$

is a basic neighborhood of f_0, and $\hat{\mathscr{N}_0} \subset \hat{\mathscr{N}} \cap \hat{\mathscr{N}}'$. $\square$

It follows by Remark 1.5 that the family of all basic neighborhoods with respect to $\rho: M \to \Omega$ form a basis for a topology on $\mathrm{C}^r(M,N)$.

Definition 1.6. *The strong-weak topology on* $\mathrm{C}^r(M,N)$, $1 \le r < \infty$, *with respect to a phase map* $\rho: M \to \Omega$ is the topology which as a basis has the family of all basic neighborhoods with respect to $\rho: M \to \Omega$. We denote the set $\mathrm{C}^r(M,N)$ with this topology by $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$.

If M and N are smooth manifolds we define the strong-weak topology on $\mathrm{C}^\infty(M,N)$, with respect to a phase map $\rho: M \to \Omega$, to be the topology which is the union of all the relative topologies induced by the inclusions $\mathrm{C}^\infty(M,N) \to \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$, $1 \le r < \infty$. We denote $\mathrm{C}^\infty(M,N)$ with this topology by $\mathrm{C}^\infty_{\mathrm{SW}[\rho]}(M,N)$. If M and N are real analytic manifolds we define the strong-weak topology with respect to $\rho: M \to \Omega$ on $\mathrm{C}^\omega(M,N)$ to be the relative topology from $\mathrm{C}^\infty_{\mathrm{SW}[\rho]}(M,N)$.

Let M and N be C^r manifolds, where $1 \le r \le \omega$. Note that if $\rho_0: M \to \{x_0\}$ is the constant map onto one point, then

$$\mathrm{C}^r_{\mathrm{SW}[\rho_0]}(M,N) = \mathrm{C}^r_{\mathrm{W}}(M,N), \tag{2}$$

and if we as a phase map for M choose $\mathrm{id}: M \to M$, then

$$\mathrm{C}^r_{\mathrm{SW}[\mathrm{id}]}(M,N) = \mathrm{C}^r_{\mathrm{S}}(M,N). \tag{3}$$

The strong-weak topology, with respect to an arbitrary phase map $\rho\colon M \to \Omega$, is always at least as fine as the weak topology, and at most as fine as the strong topology. Thus the identity map on the set $\mathrm{C}^r(M,N)$ gives us continuous maps

$$\mathrm{C}^r_{\mathrm{S}}(M,N) \to \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N) \to \mathrm{C}^r_{\mathrm{W}}(M,N), \quad 1 \le r \le \omega.$$

As generalizations of the facts given in (2) and (3) we have Lemmas 1.7 and 1.9 below.

Lemma 1.7. *Suppose M and N are C^r manifolds, where $1 \le r \le \omega$. Let $\rho\colon M \to \Omega$ be a phase map for M, and assume that Ω is compact. Then*

$$\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N) = \mathrm{C}^r_{\mathrm{W}}(M,N).$$

Proof. It is enough to prove the claim in the case when $1 \le r < \infty$, and it follows by the above that we only need to show that every open subset of $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$ is open also in $\mathrm{C}^r_{\mathrm{W}}(M,N)$. Let $\hat{\mathscr{N}} = \bigcap_{i\in\Lambda} \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$ be a basic neighborhood of f in the strong-weak topology with respect to $\rho\colon M \to \Omega$. Then the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω, and since Ω is compact it follows that Λ is finite. Thus $\hat{\mathscr{N}}$ is an open neighborhood of f in the weak topology. □

We say that a map $f\colon X \to Y$ between topological spaces is *of finite type* if for each locally finite family $\{A_i\}_{i\in\Lambda}$ of subsets of X, the family $\{f(A_i)\}_{i\in\Lambda}$ is locally finite in Y.

Lemma 1.8. *Let $f\colon X \to Y$ be a proper map, where Y is locally compact. Then f is of finite type.*

Proof. Let $\{A_i\}_{i\in\Lambda}$ be a locally finite family of subsets of X. Suppose $y \in Y$ and let B be a compact neighborhood of y. Then $f^{-1}(B)$ is compact and hence $A_i \cap f^{-1}(B) \neq \emptyset$ for only finitely many $i \in \Lambda$. Thus also $f(A_i) \cap B \neq \emptyset$ for only finitely many $i \in \Lambda$. □

Lemma 1.9. *Let M and N be C^r manifolds, $1 \le r \le \omega$, and let $\rho\colon M \to \Omega$ be a proper map, where Ω is a locally compact space. Then*

$$\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N) = \mathrm{C}^r_{\mathrm{S}}(M,N).$$

Proof. We may assume that $1 \le r < \infty$, and we only need to show that each open subset of $\mathrm{C}^r_{\mathrm{S}}(M,N)$ is open in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$. Let $f \in \mathrm{C}^r(M,N)$, and let $\hat{\mathscr{N}}_{\mathrm{S}} = \bigcap_{i\in\Lambda} \mathscr{N}^r(f; K_i, (U_i,\varphi_i), (V_i,\psi_i), \varepsilon_i)$ be a basic neighborhood of f in the strong topology. and hence $\hat{\mathscr{N}}_{\mathrm{S}}$ is open in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$. □

2. Basic properties of the strong–weak topology

Lemma 2.1. *Let M, N and P be C^r manifolds, and let $h\colon N \to P$ be a C^r map, where $1 \le r < \infty$. Suppose $f \in \mathrm{C}^r(M,N)$, and let $\mathscr{N} = \mathscr{N}^r(h \circ f; K, (U,\varphi), (W,\omega), \varepsilon)$ be an elementary neighborhood of $h_*(f) = h \circ f \in \mathrm{C}^r(M,P)$. Then there exist elementary neighborhoods $\mathscr{M}_j = \mathscr{N}^r(f; K_j, (U,\varphi), (V_j,\psi_j), \varepsilon_j)$ of f, $1 \le j \le t$, such that $h_*(\bigcap_{j=1}^t \mathscr{M}_j) \subset \mathscr{N}$.*

Proof. Since K is compact, and $K \subset U$ and $(h \circ f)(K) \subset W$, we can find finitely many compact subsets K_j of K and charts (V'_j, ψ'_j) in N, $1 \le j \le t$, such that

(a) $K = \bigcup_{j=1}^t K_j$,

(b) $f(K_j) \subset V_j \subset \overline{V}_j \subset V'_j \subset h^{-1}(W)$, where V_j is open in N, and $\overline{V}_j$ is compact.

We denote $\psi_j = \psi'_j \mid V_j$. Then $\psi(V_j) \subset \psi'_j(\overline{V}_j)$, and $\|\omega \circ h \circ (\psi'_j)^{-1}\|^r_{\psi'_j(\overline{V}_j)} < \infty$. Hence there exists $\varepsilon_j > 0$ such that, if $\tilde{f}\colon M \to N$ is a C^r map with $\tilde{f}(K_j) \subset V_j$, then

$$\|\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1}\|^r_{\varphi(K_j)} < \varepsilon_j$$

implies that

$$\|(\omega \circ h \circ \psi_j^{-1})(\psi_j \circ \tilde{f} \circ \varphi^{-1} - \psi_j \circ f \circ \varphi^{-1})\|^r_{\varphi(K_j)} < \varepsilon, \quad 1 \le j \le t. \tag{1}$$

We define

$$\mathscr{M}_j = \mathscr{N}^r(f; K_j, (U,\varphi), (V_j,\psi_j), \varepsilon_j), \quad 1 \le j \le t.$$

Now suppose that $\tilde{f} \in \bigcap_{j=1}^t \mathscr{M}_j$. Then $(h \circ \tilde{f})(K) \subset W$, and by (1) it follows that

$$\|\omega \circ h \circ \tilde{f} \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}\|^r_{\varphi(K)} < \varepsilon.$$

Thus $h_*(\tilde{f}) = h \circ \tilde{f} \in \mathscr{N}^r(h \circ f; K, (U,\varphi), (W,\omega), \varepsilon) = \mathscr{N}$. ◻

If we take $N = P$ and $h = \mathrm{id}_N$ in Lemma 2.1, we see that the proof of Lemma 2.1 proves the following.

Remark 2.2. Let $f \in \mathrm{C}^r(M,N)$, where $1 \le r < \infty$ and let $\mathscr{N} = \mathscr{N}^r(f; K, (U,\varphi), (V,\psi,\varepsilon)$ be an elementary neighborhood of f. Suppose we are given charts (V_j,ψ_j) in N and compact subsets K_j of K, $1 \le j \le t$, such that

(a) $K = \bigcup_{j=1}^t K_j$,

(b) $f(K_j) \subset V_j \subset \overline{V}_j \subset V'_j \subset V$, where (V'_j,ψ'_j) are charts in N, and we have $\psi_j = \psi'_j | V'_j$ and $\overline{V}_j$ is compact, $1 \le j \le t$.

Then there exist $\varepsilon_j > 0$, $1 \le j \le t$, so that if $\mathscr{N}_j = \mathscr{N}^r(f; K_j, (U,\varphi), (V_j,\psi_j), \varepsilon_j)$, then $\bigcap_{j=1}^t \mathscr{N}_j \subset \mathscr{N}$. ◻

Lemma 2.3. *Let M, N and P be C^r manifolds, and let $f\colon M \to N$ be a C^r map, where $1 \le r < \infty$. Let $h \in \mathrm{C}^r(N,P)$, and let $\mathscr{N} = \mathscr{N}^r(h \circ f; K, (U,\varphi), (W,\omega), \varepsilon)$ be an elementary* *neighborhood of* $f^*(h) = h \circ f \in \mathrm{C}^r(M,P)$. *Then there exist elementary neighborhoods* $\mathscr{M}_j = \mathscr{N}^r(h; L_j, (V_j,\psi_j),(W,\omega),\varepsilon_j)$, $1 \le j \le t$, *of* h, *such that* $f^*(\bigcap_{j=1}^t \mathscr{M}_j) \subset \mathscr{N}$.

Proof. Since $f(K)$ is compact we can find finitely many compact subsets L_j of $f(K)$, and charts (V_j,ψ_j) in N, $1 \le j \le t$, such that

(a) $\bigcup_{j=1}^t L_j = f(K)$,
(b) $L_j \subset V_j$, $1 \le j \le t$.

Let us denote $K_j = K \cap f^{-1}(L_j)$. Then K_j is compact, $\bigcup_{j=1}^t K_j = K$ and $f(K_j) = L_j$. We have $\|\psi_j \circ f \circ \varphi^{-1}\|^r_{\varphi(K_j)} < \infty$, and hence there exists $\varepsilon_j > 0$ such that if $\tilde{h}: N \to P$ is a C^r map with $\tilde{h}(L_j) \subset W$, then

$$\|\omega \circ \tilde{h} \circ \psi_j^{-1} - \omega \circ h \circ \psi_j^{-1}\|^r_{\psi_j(L_j)} < \varepsilon_j$$

implies that

$$\|(\omega \circ \tilde{h} \circ \psi_j^{-1} - \omega \circ h \circ \psi_j^{-1})(\psi_j \circ f \circ \varphi^{-1})\|^r_{\varphi(K_j)} < \varepsilon, \quad 1 \le j \le t. \tag{2}$$

We define

$$\mathscr{M}_j = \mathscr{N}^r(h; L_j, (V_j,\psi_j),(W,\omega),\varepsilon_j), \quad 1 \le j \le t.$$

Now suppose that $\tilde{h} \in \bigcap_{j=1}^t \mathscr{M}_j$. Then we have that $(\tilde{h} \circ f)(K) \subset W$, and by (2) it follows that

$$\|\omega \circ \tilde{h} \circ f \circ \varphi^{-1} - \omega \circ h \circ f \circ \varphi^{-1}\|^r_{\varphi(K)} < \varepsilon, \quad 1 \le j \le t.$$

Thus $f^*(\tilde{h}) = \tilde{h} \circ f \in \mathscr{N}^r(h \circ f; K, (U,\varphi),(W,\omega),\varepsilon) = \mathscr{N}$. ◻

Proposition 2.4. *Let* M, N *and* P *be* C^r *manifolds and let* $h: N \to P$ *be a* C^r *map, where* $1 \le r \le \omega$. *Let* $\rho: M \to \Omega$ *be a phase map for* M. *Then the induced map*

$$h_*: \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N) \to \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,P), \quad f \mapsto h \circ f,$$

is continuous.

Proof. It is enough to prove the claim when r is finite, so we assume that $1 \le r < \infty$. Let $f \in \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$, and let $\hat{\mathscr{N}} = \bigcap_{i\in\Lambda} \mathscr{N}_i$ be any basic neighborhood of $h_*(f) = h \circ f$ in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,P)$. Here each $\mathscr{N}_i = \mathscr{N}^r(h \circ f; K_i, (U_i,\varphi_i),(W_i,\omega_i),\varepsilon_i)$, $i \in \Lambda$, is an elementary neighborhood of $h \circ f$, and the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω. By Lemma 2.1 there exist for each $\mathscr{N}_i$ finitely many elementary neighborhoods $\mathscr{M}_{i,j} = \mathscr{N}^r(f; K_{i,j}, (U_i,\varphi_i),(V_{i,j},\varphi_{i,j}),\varepsilon_{i,j})$, $1 \le j \le t(i)$, of f, such that $h_*(\bigcap_{j=1}^{t(i)} \mathscr{M}_{i,j}) \subset \mathscr{N}_i$. Here $K_i = \bigcup_{j=1}^{t(i)} K_{i,j}$. Then the family $\{\rho(K_{i,j}) \mid i \in \Lambda,\ 1 \le j \le t(i)\}$ is locally finite in Ω, and hence $\hat{\mathscr{M}} = \bigcap_{i\in\Lambda} \bigcap_{j=1}^{t(i)} \mathscr{M}_{i,j}$ is a basic neighborhood of f in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N)$. Furthermore we have that $h_*(\hat{\mathscr{M}}) \subset \bigcap_{i\in\Lambda} \mathscr{N}_i = \hat{\mathscr{N}}$. This completes the proof. ◻

Proposition 2.5. *Let* M, N *and* P *be* C^r *manifolds,* $1 \le r \le \omega$, *and let* $\rho: M \to \Omega$ *and* $\rho': N \to \Pi$ *be phase maps for* M *and* N. *Let* $f: M \to N$ *be a* C^r *map such that we have a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \rho\downarrow & & \downarrow\rho' \\ \Omega & \xrightarrow{\overline{f}} & \Pi \end{array}$$

where the map $\overline{f}\colon \Omega \to \Pi$ *is of finite type. Then*

$$f^*\colon \mathrm{C}^r_{\mathrm{SW}[\rho']}(N,P) \to \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,P), \quad h \mapsto h \circ f,$$

is continuous.

Proof. We may assume that $1 \le r < \infty$. Let $h \in \mathrm{C}^r_{\mathrm{SW}[\rho']}(N,P)$, and let $\hat{\mathscr{N}} = \bigcap_{i\in\Lambda}\mathscr{N}_i$ be any basic neighborhood of $f^*(h) = h \circ f$ in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,P)$. Here each $\mathscr{N}_i = \mathscr{N}^r(h \circ f; K_i, (U_i,\varphi_i), (W_i,\omega_i), \varepsilon_i)$, $i \in \Lambda$, is an elementary neighborhood of $f^*(h) = h\circ f$, and the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω. By Lemma 2.3 there exist for each $\mathscr{N}_i$ finitely many elementary neighborhoods $\mathscr{M}_{i,j} = \mathscr{N}^r(h; L_{i,j}, (V_{i,j},\psi_{i,j}), (W_i,\omega_i), \varepsilon_{i,j})$, $1 \le j \le t(i)$, of h such that $f^*(\bigcap_{j=1}^{t(i)} \mathscr{M}_{i,j}) \subset \mathscr{N}_i$. Here $L_i = \bigcup_{j=1}^{t(i)} L_{i,j} = f(K_i)$, for each $i \in \Lambda$, and $(V_{i,j},\psi_{i,j})$, $i \in \Lambda$, $1 \le j \le t(i)$ are charts in N.

Since $\overline{f}$ is of finite type the family $\{\overline{f}(\rho(K_i))\}_{i\in\Lambda} = \{\rho'(L_i)\}_{i\in\Lambda}$ is locally finite in Π. Hence also the family $\{\rho'(L_{i,j}) \mid i \in \Lambda,\ 1 \le j \le t(i)\}$ is locally finite in Π. Thus $\hat{\mathscr{M}} = \bigcap_{i\in\Lambda}\bigcap_{j=1}^{t(i)} \mathscr{M}_{i,j}$ is a basic neighborhood of h in $\mathrm{C}^r_{\mathrm{SW}[\rho']}(N,P)$. We now have that $f^*(\hat{\mathscr{M}}) \subset \bigcap_{i\in\Lambda}\mathscr{N}_i = \mathscr{N}$. This completes the proof. □

Proposition 2.6. *Let* M, N_1 *and* N_2 *be* C^r *manifolds,* $1 \le r \le \omega$, *and let* $\rho\colon M \to \Omega$ *be a phase map for* M. *Then the natural bijection*

$$\iota\colon \mathrm{C}^r_{\mathrm{SW}[\rho]}(M, N_1 \times N_2) \to \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N_1) \times \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N_2), \quad f \mapsto (q_1 \circ f, q_2 \circ f),$$

is a homeomorphism.

Proof. It follows by Proposition 2.4 that ι is continuous. The fact that ι^{-1} is continuous is seen as follows. It is enough to consider the case when r is finite. Let $f_j \in \mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N_j)$, $j = 1,2$, and set $f = \iota^{-1}(f_1,f_2) \in \mathrm{C}^r_{\mathrm{SW}[\rho]}(M, N_1 \times N_2)$. Suppose $\hat{\mathscr{N}} = \bigcap_{i\in\Lambda}\mathscr{N}_i$ is a basic neighborhood of f. We claim that there exist basic neighborhoods $\hat{\mathscr{N}_j}$ of f_j in $\mathrm{C}^r_{\mathrm{SW}[\rho]}(M,N_j)$, $j = 1,2$, such that $\iota^{-1}(\hat{\mathscr{N}_1} \times \hat{\mathscr{N}_2}) \subset \hat{\mathscr{N}}$. Using Remark 2.2 we see that we may assume that each of the elementary neighborhoods $\mathscr{N}_i$, $i \in \Lambda$, above is of the type where the chart in $N_1 \times N_2$ is a product chart. Furthermore it is readily seen that if $\mathscr{N} = \mathscr{N}^r(f; K, (U,\varphi), (V_1 \times V_2, \psi_1 \times \psi_2), \varepsilon)$, then $\iota^{-1}(\mathscr{N}^{(1)} \times \mathscr{N}^{(2)}) \subset \mathscr{N}$, where $\mathscr{N}^{(j)} = \mathscr{N}^r(f_j; K, (U,\varphi), (V_j,\psi_j), \varepsilon)$, $j = 1,2$. It now follows that our claim holds, and thus ι^{-1} is continuous. □

Lemma 2.7. *Let* M *and* N *be* C^r *manifolds, and* $f\colon M \to N$ *a* C^r *map,* $1 \le r \le \omega$, *and let* $\rho\colon M \to \Omega$ *be a phase map for* M. *Let* $\{B_i\}_{i\in\Lambda}$ *be a family of compact subsets of* M, *such that the family* $\{\rho(B_i)\}_{i\in\Lambda}$ *is locally finite in* Ω. *Suppose* V_i, $i \in \Lambda$, *are open subsets* *of N such that $f(B_i) \subset V_i$, for each $i \in \Lambda$. Then there exists an open neighborhood $\mathscr{N}_0$ of $f \in C^r_{\mathrm{SW}[\rho]}(M,N)$, such that if $h \in \mathscr{N}_0$ then $h(B_i) \subset V_i$, for each $i \in \Lambda$.*

Proof. Since B_i is compact we can find finitely many compact sets $K_{i,j}$, and charts $(U_{i,j}, \varphi_{i,j})$ in M and charts $(V_{i,j}, \psi_{i,j})$ in N, $1 \leq j \leq t(i)$, such that

(a) $K_{i,j} \subset U_{i,j}, \quad 1 \leq j \leq t(i)$,
(b) $f(U_{i,j}) \subset V_{i,j} \subset V_i, \quad 1 \leq j \leq s(i)$,
(c) $\bigcup_{j=1}^{t(i)} K_{i,j} = B_i$.

We set $\mathscr{N}_{i,j} = \mathscr{N}^r(f; K_{i,j}, (U_{i,j}, \varphi_{i,j}), (V_{i,j}, \psi_{i,j}), \infty)$, $1 \leq j \leq t(i)$. If $h \in \bigcap_{j=1}^{t(i)} \mathscr{N}_{i,j}$, then $h(B_i) \subset V_i$. Since the family $\{\rho(B_i)\}_{i \in \Lambda}$ is locally finite in Ω it follows that also the family $\{\rho(K_{i,j}) \mid i \in \Lambda,\ 1 \leq j \leq t(i)\}$ is locally finite in Ω. Therefore $\mathscr{N}_0 = \bigcap_{i \in \Lambda} \bigcap_{j=1}^{t(i)} \mathscr{N}_{i,j}$ is an open neighborhood of $f \in C^r_{\mathrm{SW}[\rho]}(M,N)$. We have that if $h \in \mathscr{N}_0$, then $h(B_i) \subset V_i$, for all $i \in \Lambda$. □

3. Proper G-spaces and fundamental sets

In this section we present some elementary facts about proper G-spaces and fundamental sets, and we also fix some notation that we will use in the following sections.

Each given topological space is in this section automatically assumed to be Hausdorff. Recall that a map $f: X \to Y$ is said to be *proper* if the inverse image of each compact set is compact. Clearly the composite of two proper maps is a proper map, and we also have the following converse.

Lemma 3.1. *Let $f: X \to Y$ and $h: Y \to Z$ be such that the composite $h \circ f: X \to Z$ is a proper map. Then:*

1) *$f: X \to Y$ is a proper map.*
2) *If $f: X \to Y$ is surjective, then the map $h: Y \to Z$ is proper.*

Proof. Direct verification. □

Let us here also record the following well-known fact.

Lemma 3.2. *If $f: X \to Y$ is a proper map and Y is locally compact, then f is a closed map.*

Proof. Each locally compact space is compactly generated, see e.g. [S, 2.2], and it is easy to see that Lemma 3.2 holds if Y is compactly generated, see e.g. [P2, Corollary]. □

Let G be a Lie group and let X be a Hausdorff space. Recall that an action $\Phi: G \times X \to X$, $(g,x) \mapsto gx$, of G on X is said to be *proper* if for every compact subset A of X we have that

$$G_{[A]} = \{g \in G \mid gA \cap A \neq \emptyset\}$$

is a compact subset of G.

Lemma 3.3. *The following are equivalent.*

1) *The action $\Phi: G \times X \to X$, $(g,x) \mapsto gx$, is a proper action.*

2) *The map* $\Phi^*: G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$, *is proper.*
3) *The restriction* $\Phi|: G \times A \to X$ *is a proper map for each compact subset* A *of* X.

Proof. See [I1, Proposition 1.4]. ▫

By a proper G-space M we will in the following always mean a *locally compact, second countable, Hausdorff space* M on which G acts by a proper action. Note that it follows by Lemmas 3.2 and 3.3, that each orbit Gx in a proper G-space M is closed, and moreover that GA is closed in M if A is a compact subset of M.

Lemma 3.4. *Let* M *be a proper* G*-space. Then there exist compact subsets* C_i, $i \geq 0$, *of* M/G *such that*

1) $C_0 \subset \mathring{C}_1 \subset \cdots \subset C_i \subset \mathring{C}_{i+1} \subset C_{i+1} \subset \cdots$,
2) $M/G = \bigcup_{i=0}^{\infty} C_i$.

Furthermore M/G *is paracompact and hence also normal.*

Proof. The well-known fact that M/G is Hausdorff, see [P1, Theorem 1.2.9], can be seen as follows. If Gx and Gy are two distinct orbits in M, we let A be a compact neighborhood of x such that $A \cap Gy = \emptyset$, and since GA is closed in M it follows that $p(\mathring{A})$ and $M - p(A)$ are disjoint open neighborhoods of $p(x)$ and $p(y)$.

Since the map $p: M \to M/G$ is open it also follows that M/G is both locally compact and second countable. As is well-known, and easy to see, this implies that M/G is the union of compact subsets C_i, $i \geq 0$, as above, and this in turn implies that M/G is paracompact, and moreover every paracompact space is normal. ▫

We shall now fix some notation that will be used throughout the rest of the paper. Suppose M is a proper G-space and let $p: M \to M/G$ be the projection onto the orbit space. We let C_i, $i \geq 0$, be compact subsets of M/G as in Lemma 3.4, and define

$$D_i = C_i - \mathring{C}_{i-1}, \quad i \geq 0,$$

$$\hat{V}_i = \mathring{C}_{i+1} - C_{i-2}, \quad i \geq 0,$$

$$\hat{D}_i = C_{i+1} - \mathring{C}_{i-2}, \quad i \geq 0,$$

where $\mathrm{C}_{-1} = \mathrm{C}_{-2} = \emptyset$. Then $\{D_i\}_{i \geq 0}$ is a locally finite family of compact subsets of M/G, and $\bigcup_{i=0}^{\infty} D_i = M/G$. We define

$$A_i = p^{-1}(C_i), \quad i \geq 0.$$

Then each A_i is a G-invariant closed subset of M, and we have that

(a) each $p(A_i) = A_i/G$ is compact, $i \geq 0$,
(b) $A_0 \subset \mathring{A}_1 \subset A_1 \subset \cdots \subset A_i \subset \mathring{A}_{i+1} \subset A_{i+1} \subset \cdots$,
(c) $M = \bigcup_{i=0}^{\infty} A_i$.

We denote

$$B_i = A_i - \mathring{A}_{i-1}, \quad i \geq 0$$

$$\hat{U}_i = \mathring{A}_{i+1} - A_{i-2}, \quad i \geq 0$$

$$\hat{B}_i = A_{i+1} - \mathring{A}_{i-2}, \quad i \geq 0,$$

where $A_{-1} = A_{-2} = \emptyset$. Then each B_i is a G-invariant closed subset of M and $p(B_i)$ is compact. We have that $M = \bigcup_{i=0}^{\infty} B_i$ and $B_i \cap B_j = \emptyset$ if $|i - j| \geq 2$.

Definition 3.5. Let M be a proper G-space, and let $p\colon M \to M/G$ be the projection onto the orbit space. We say that a closed subset E of M is a *closed fundamental set* in M if $GE = M$ and the restriction $p|\colon E \to M/G$ is a proper map. A closed fundamental set E in M is said to be *fat* if $G\mathring{E} = M$.

It is easy to see that if E is a subset of M such that $p|\colon E \to M/G$ is a proper map then E is closed in M.

Lemma 3.6. *Let M be a proper G-space. Then there exists a fat closed fundamental set E in M.*

Proof. Let the subsets D_i and $\hat{V}_i$, $i \geq 0$, of M/G be as above. Since D_i is compact we can find finitely many open subsets $U_{i,j}$, $1 \leq j \leq t(i)$, of M, where $\overline{U}_{i,j}$ is compact and $p(\overline{U}_{i,j}) \subset \hat{V}_i$, $1 \leq j \leq t(i)$, such that $D_i \subset \bigcup_{j=1}^{t(i)} p(U_{i,j})$. Then $M/G = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{t(i)} p(U_{i,j})$ and the family $\{p(\overline{U}_{i,j}) \mid i \geq 0,\ 1 \leq j \leq t(i)\}$ is locally finite in M/G. Setting $U = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{t(i)} U_{j,i}$ we have that $GU = M$. The family $\{\overline{U}_{i,j} \mid i \geq 0,\ 1 \leq j \leq t(i)\}$ is locally finite in M, and hence $\overline{U} = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{t(i)} \overline{U}_{i,j}$. It is now readily verified that $E = \overline{U}$ is a fat closed fundamental set in M. $\square$

If E is a closed fundamental set in M, we denote

$$E_i = E \cap A_i, \quad i \geq 0,$$

$$F_i = E \cap B_i, \quad i \geq 0.$$

Then E_i and F_i are compact subsets of E, $GE_i = A_i$ and $GF_i = B_i$, $i \geq 0$.

Lemma 3.7. *Let G and H be Lie groups, and let $\theta\colon G \to H$ be a homomorphism. Let $f\colon M \to N$ be a θ-equivariant proper map, where M is a proper G-space and N is a proper H-space. Then the induced map $\overline{f}\colon M/G \to N/H$ is proper.*

Proof. Let E' be a closed fundamental set in N, which exists by Lemma 3.6, and denote $E = f^{-1}(E')$. The diagram

$$\begin{tikzcd}
E \arrow[r, "f|"] \arrow[d, "p|"'] & E' \arrow[d, "q|"] \\
M/G \arrow[r, "\overline{f}"] & N/H
\end{tikzcd}$$

commutes, and both $f|$ and $q|$ are proper maps. Since $p|$ is surjective it now follows by Lemma 3.1 that $\overline{f}$ is a proper map. (Note that by Lemma 3.1 also $p|\colon E \to M/G$ is a proper map, and hence $E = f^{-1}(E')$ is a closed fundamental set in M.) $\square$

Lemma 3.8. *Let M and N be proper G-spaces, and let $f: M \to N$ be a G-equivariant map. Let A be a compact subset of M. Then GA is a closed subset of M and the map $f|: GA \to N$ is proper.*

Proof. The fact that GA is closed in M follows by Lemmas 3.2 and 3.3. Since the map $\mathrm{id} \times f: G \times A \to G \times f(A)$ is proper and $f(A)$ is compact, it follows by Lemma 3.3 that the map $G \times A \to N$, $(g, a) \mapsto gf(a)$, is proper. Since $gf(a) = f(ga)$ and the map $G \times A \to GA$, $(g, a) \mapsto ga$, is surjective it now follows by Lemma 3.1 that $f|: GA \to N$ is a proper map. □

Lemma 3.9. *Let M and N be proper G-spaces and suppose $f: M \to N$ is a G-equivariant map such that the induced map $\overline{f}: M/G \to N/G$ is proper. Then f is a proper map.*

Proof. Let E be a closed fundamental set in M, see Lemma 3.6, and let $p: M \to M/G$ and $q: N \to N/G$ be the projections. Suppose Y is a compact subset of N. Then $X^* = (\overline{f})^{-1}(q(Y))$ is a compact subset of M/G, and hence $X_0 = p^{-1}(X^*) \cap E$ is compact. By Lemma 3.8 the map $f|: GX_0 \to N$ is proper, and since $f^{-1}(Y) \subset GX_0$ it follows that $f^{-1}(Y)$ is compact. □

Now suppose that H is a closed subgroup of a Lie group G, and let N be an H-space. Recall that the twisted product $G \times_H N$ is the orbit space of the H-space $G \times N$, where H acts on $G \times N$ by $h(g, x) = (gh^{-1}, hx)$. Let $p: G \times N \to G \times_H N$ be the natural projection, and denote $p(g, x) = [g, x]$. There is a natural action of G on $G \times_H N$, given by $\overline{g}[g, x] = [\overline{g}g, x]$. The map

$$q: G \times_H N \to G/H, \quad [g, x] \mapsto gH,$$

is G-equivariant, and q is a locally trivial projection with fiber N. For if U is an open subset of G/H such that there is cross section $\sigma: U \to G$, then $\theta: U \times N \to q^{-1}(U)$, $(u, x) \mapsto [\sigma(u), x]$, is a homeomorphism. In particular we see that the twisted product $G \times_H N$ is Hausdorff, when N is Hausdorff. We also have the following.

Lemma 3.10. *Suppose H acts on N by a proper action. Then the action of G on $G \times_H N$ is proper.*

Proof. Let A be a compact subset of $G \times_H N$. We claim that $G_{[A]} = \{g \in G \mid gA \cap A \neq \emptyset\}$ is compact. Since $q(A)$ is compact there exists a compact subset L of G such that $\pi(L) = q(A)$, where $\pi: G \to G/H$ denotes the natural projection. Thus $A \subset LN$, and hence $A = LN \cap A \subset L(N \cap L^{-1}A)$. Since $B = N \cap L^{-1}A$ is a compact subset of N, we have that $H_{[B]} = \{h \in H \mid hB \cap B \neq \emptyset\}$ is compact. Now

$$G_{[A]} \subset G_{[LB]} = LG_{[B]}L^{-1} = LH_{[B]}L^{-1},$$

which shows that $G_{[A]}$ is compact. □

4. The strong-weak topology and G-equivariant maps

Let G be a Lie group and let M and N be C^r G-manifolds, where $1 \leq r \leq \omega$. By $\mathrm{C}^{r,G}(M,N)$ we denote the set of all G-equivariant C^r maps from M to N. Let $p\colon M \to M/G$ denote the projection onto the orbit space. We give the set $\mathrm{C}^r(M,N)$ the strong-weak topology with respect to $p\colon M \to M/G$, and obtain the space $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$. The strong-weak topology on $\mathrm{C}^{r,G}(M,N)$ is then defined to be the relative topology that $\mathrm{C}^{r,G}(M,N)$ obtains as a subset of $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$. We denote $\mathrm{C}^{r,G}(M,N)$ with this topology by $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.

Proposition 4.1. *Let G be a compact Lie group and let M and N be C^r G-manifolds, $1 \leq r \leq \omega$. Then*

$$\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N) = \mathrm{C}^{r,G}_{\mathrm{S}}(M,N).$$

Proof. Since G is compact the projection $p\colon M \to M/G$ is a proper map, and M/G is locally compact, see [B, Theorem I.3.1]. Hence the result follows by Lemma 1.9. □

Proposition 4.2. *Let G be a Lie group, and let M and N be C^r G-manifolds, $1 \leq r \leq \omega$, such that the orbit space M/G is compact. Then*

$$\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N) = \mathrm{C}^{r,G}_{\mathrm{W}}(M,N).$$

Proof. Follows by Lemma 1.7. □

Suppose now that M is a C^r G-manifold and that N is a C^r H-manifold, where G and H are Lie groups, and let $\theta\colon G \to H$ be a continuous homomorphism. We recall here that it follows that θ is real analytic, see e.g. [He, Theorem II.2.6]. We say that a map $f\colon M \to N$ is θ-equivariant if $f(gx) = \theta(g)f(x)$, for all $g \in G$ and $x \in X$. By $\mathrm{C}^{r,\theta}(M,N)$ we denote the set of all θ-equivariant C^r maps from M to N. We denote the set $\mathrm{C}^{r,\theta}(M,N)$ with the strong-weak topology with respect to the phase map $p\colon M \to M/G$ by $\mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,N)$.

Proposition 4.3. *Let M and N be C^r G–manifolds, and let P be a C^r H–manifold, where $1 \leq r \leq \omega$. Let $k\colon N \to P$ be a θ–equivariant C^r map. Then the induced map*

$$k_*\colon \mathrm{C}^{r,G}_{\mathrm{SW}}(M,N) \to \mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,P), \quad f \mapsto k \circ f,$$

is continuous.

Proof. This follows by Proposition 2.4, when we as a phase map for M choose the projection $p\colon M \to M/G$, and note that if $f\colon M \to N$ is a G-equivariant map, then $k_*(f) = k \circ f\colon M \to P$ is a θ-equivariant map. □

Proposition 4.4. *Let M be a C^r G–manifold, and let N and P be C^r H–manifolds, $1 \leq r \leq \omega$. Suppose $f\colon M \to N$ is a θ–equivariant C^r map such that the induced map $\overline{f}\colon M/G \to N/H$ is of finite type. Then the induced map*

$$f^*\colon \mathrm{C}^{r,H}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,P), \quad k \mapsto k \circ f, \tag{1}$$

is continuous.

Proof. Follows by Proposition 2.5, when we as phase maps for M and N choose the projections $p\colon M \to M/G$ and $q\colon N \to N/H$, and moreover note that if $k\colon N \to P$ is an H-equivariant map, then $f^*(k) = k \circ f\colon M \to P$ is a θ–equivariant map. $\square$

Corollary 4.5. *Let M and N be proper C^r G-manifolds, and let P be a C^r G-manifold, $1 \le r \le \omega$. Let $f\colon M \to N$ be a G-equivariant proper C^r map. Then the induced map*

$$f^*\colon \mathrm{C}^{r,G}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,G}_{\mathrm{SW}}(M,P), \quad k \mapsto k \circ f,$$

is continuous.

Proof. Follows by Lemmas 1.8, 3.7 and Proposition 4.4. $\square$

Proposition 4.6. *Let M, N_1 and N_2 be C^r G–manifolds, where $1 \le r \le \omega$, and let G act on the product $N_1 \times N_2$ by the diagonal action. Then the natural bijection*

$$\iota\colon \mathrm{C}^{r,G}_{\mathrm{SW}}(M, N_1 \times N_2) \to \mathrm{C}^{r,G}_{\mathrm{SW}}(M,N_1) \times \mathrm{C}^{r,G}_{\mathrm{SW}}(M,N_2), \quad f \mapsto (q_1 \circ f, q_2 \circ f),$$

is a homeomorphism.

Proof. This follows by Proposition 2.6, when we as a phase map for M use the projection $p\colon M \to M/G$, and also note that a map $f\colon M \to N_1 \times N_2$ is G-equivariant if and only if both components f_1 and f_2 are G-equivariant maps. $\square$

Proposition 4.7. *Let G be a non-compact Lie group, and let M and N be C^r G-manifolds, and assume that the action of G on M is proper. Then the topology on $\mathrm{C}^{r,G}_{\mathrm{S}}(M,N)$, $1 \le r \le \omega$, is the discrete topology.*

Proof. It is enough to prove the proposition in the case when r is finite, so we assume that $1 \le r < \infty$. Let $f\colon M \to N$ be a G-equivariant C^r map, and let

$$\mathscr{N} = \mathscr{N}^r(f; K, (U,\varphi), (V,\psi), \varepsilon) \tag{1}$$

be an elementary neighborhood of f. If $\mathscr{N}$ is as in (1) we set

$$\mathscr{N}(1/i) = \mathscr{N}^r(f; K, (U,\varphi), (V,\psi), \varepsilon/i), \quad i \ge 1. \tag{2}$$

If $g \in G$ and $\mathscr{N}$ is as in (1) we denote

$$\mathscr{N}_g = \mathscr{N}^r(f; gK, (gU, \varphi \circ g^{-1}), (gV, \psi \circ g^{-1}), \varepsilon). \tag{3}$$

We claim that

$$\mathscr{N} \cap \mathrm{C}^{r,G}(M,N) = \mathscr{N}_g \cap \mathrm{C}^{r,G}(M,N). \tag{4}$$

Suppose $h \in \mathrm{C}^{r,G}(M,N)$. Then $h(K) \subset V$ implies that $h(gK) \subset gV$, and conversely, and we also have that

$$\|(\psi\circ g^{-1})\circ h\circ(\varphi\circ g^{-1})^{-1} - (\psi\circ g^{-1})\circ f\circ(\varphi\circ g^{-1})^{-1}\|^r_{\varphi(K)} = \|\psi\circ h\circ\varphi^{-1} - \psi\circ f\circ\varphi^{-1}\|^r_{\varphi(K)},$$

and therefore (4) holds.

Next we observe that if $\mathscr{N}$ is as in (1) and

$$h \in \bigcap_{i=1}^{\infty} \mathscr{N}(1/i) \tag{5}$$

then $\|\psi \circ h \circ \varphi^{-1} - \psi \circ f \circ \varphi^{-1}\|^r_{\varphi(K)} = 0$, and hence in particular

$$h(x) = f(x), \quad \text{for each} \quad x \in K.$$

Since G is non-compact we can find elements $g_1, g_2, \dots$ of G, where $g_i \neq g_j$ if $i \neq j$, such that the set

$$L = \{g_i \mid i \geq 1\}$$

is a discrete closed subset of G. This is seen as follows. If the identity component G_0 of G is non-compact we have that $G_0 = \bigcup_{i=0}^{\infty} A_i$, where each A_i is compact and $A_i \subset \mathring{A}_{i+1},\ i \geq 0$. Then we can choose $g_i \in \mathring{A}_i - A_{i-1},\ i \geq 1$. If G_0 is compact then G/G_0 is infinite and we can pick each g_i from a different component of G.

Let K be a compact subset of M. We claim that the family $\{g_i K\}_{i \geq 1}$ is locally finite in M. Let $x \in M$ and let $\overline{V}$ be a compact neighborhood of x in M. Then

$$Q = \{g \in G \mid \overline{V} \cap gK \neq \emptyset\}$$

is a compact subset of G. Hence $Q \cap L$ is a compact discrete set, and thus a finite set. Therefore $\overline{V} \cap g_i K \neq \emptyset$ for only finitely many g_i.

Now let $\{K_j\}_{j \in \Lambda}$ be a family of compact subsets of M such that:

(a) $\bigcup_{j \in \Lambda} G K_j = M$.

(b) The family $\{G K_j\}_{j \in \Lambda}$ is locally finite in M.

(c) $K_j \subset U_j$, where (U_j, φ_j) is a chart in $M,\ j \in \Lambda$.

(d) $f(K_j) \subset V_j$, where (V_j, ψ_j) is a chart in $N,\ j \in \Lambda$.

Using the techniques of Section 3 it is easy to see that such a family $\{K_j\}_{j \in \Lambda}$ exists, compare with the proof of Lemma 7.2. Since the family $\{G K_j\}$ is locally finite in M, and for a fixed $j \in \Lambda$ the family $\{g_i K_j \mid i \geq 1\}$ is locally finite in M, it follows that the family $\{g_i K_j \mid i \geq 1,\ j \in \Lambda\}$ is locally finite in M.

We denote

$$\mathscr{N}_j = \mathscr{N}^r(f; K_j, (U_j, \varphi_j), (V_j, \psi_j), \varepsilon_j)$$

where for example $\varepsilon_j = 1$, for each $j \in \Lambda$. Using the notation given in (2) and (3) we set

$$\hat{\mathscr{N}_{\mathrm{S}}} = \bigcap_{j \in \Lambda} \bigcap_{i=1}^{\infty} \mathscr{N}_{j, g_i}(1/i).$$

Since the family $\{g_i K_j \mid i \geq 1,\ j \in \Lambda\}$ is locally finite in M we have that $\hat{\mathscr{N}_{\mathrm{S}}}$ is a basic neighborhood of f in the strong topology on $\mathrm{C}^r(M, N)$. It follows by (4) that

$$\hat{\mathscr{N}_{\mathrm{S}}} \cap \mathrm{C}^{r,G}(M, N) = \bigcap_{j \in \Lambda} \bigcap_{i=1}^{\infty} \mathscr{N}_j(1/i).$$

Now if $h \in \bigcap_{j\in\Lambda}\bigcap_{i=1}^{\infty} \mathscr{N}_j(1/i)$ then we have by (5) that

$$h(x) = f(x), \quad \text{for each} \quad x \in K_j, \quad \text{and all} \quad j \in \Lambda.$$

Since $M = \bigcup_{j\in\Lambda}^{\infty} GK_j$ it now follows that $h(x) = f(x)$, for every $x \in M$. Thus we have that

$$\hat{\mathscr{N}_{\mathrm{S}}} \cap \mathrm{C}^{r,G}(M,N) = \{f\},$$

and this shows that the strong topology on $\mathrm{C}^{r,G}(M,N)$ is the discrete topology. ◻

Let us here also record the following easy lemma.

Lemma 4.8. *Let M and N be C^r G-manifolds, $1 \le r \le \omega$, and assume that the action of G on M is proper. Let W be a G-invariant open subset of N. Then the set $\mathrm{C}^{r,G}(M,W)$ is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.*

Proof. Suppose $f \in \mathrm{C}^{r,G}(M,W)$. We let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a family of compact subsets of M such that:

(a) $\bigcup_{\alpha\in\Lambda} GK_\alpha = M$.
(b) The family $\{p(K_\alpha)\}_{\alpha\in\Lambda}$ is locally finite in M/G, where $p\colon M \to M/G$ is the projection.
(c) $K_\alpha \subset U_\alpha$, where $(U_\alpha, \varphi_\alpha)$ is a chart in $M, \alpha \in \Lambda$.
(d) $f(K_\alpha) \subset V_\alpha$, where (V_α, ψ_α) is a chart in N, $\alpha \in \Lambda$.

The existence of such a family $\{K_\alpha\}_{\alpha\in\Lambda}$ is easy to see using the techniques of Section 3, compare with the proof of Lemma 7.2. Since W is open in N and $f(K_\alpha) \subset W$, $\alpha \in \Lambda$, we may moreover assume that $V_\alpha \subset W$, for each $\alpha \in \Lambda$. Now the set

$$\hat{\mathscr{N}} = \bigcap_{\alpha\in\Lambda} \mathscr{N}^r(f; K_\alpha, (U_\alpha, \varphi_\alpha), (V_\alpha, \psi_\alpha), \varepsilon_\alpha),$$

where $\varepsilon_\alpha > 0$ are arbitrary (e.g. $\varepsilon_\alpha = \infty$), $\alpha \in \Lambda$, is a basic neighborhood of f with respect to $p\colon M \to M/G$. If

$$h \in \hat{\mathscr{N}}^G = \hat{\mathscr{N}} \cap \mathrm{C}^{r,G}(M,N),$$

then $h(M) = h(\bigcup_{\alpha\in\Lambda} GK_\alpha) = \bigcup_{\alpha\in\Lambda} Gh(K_\alpha) \subset \bigcup_{\alpha\in\Lambda} GV_\alpha \subset W$. Thus $\hat{\mathscr{N}}^G \subset \mathrm{C}^{r,G}(M,W)$, which proves our claim. ◻

5. Proofs of Theorems A and B

Let H be a closed subgroup of a Lie group G, and let N be a C^r H–manifold. We let $G \times H$ act on $G \times N$ by

$$(*) \qquad (G \times H) \times (G \times N) \to G \times N, \quad ((\overline{g}, h), (g, x)) \mapsto (\overline{g} g h^{-1}, hx).$$

Note that this is a diagonal action of the group $G \times H$ on the $G \times N$, where $G \times H$ acts on G by $(\overline{g}, h)g = \overline{g}gh^{-1}$, and on N by $(\overline{g}, h)x = hx$.

Lemma 5.1. *Let H be a closed subgroup of a Lie group G, and let N and P be C^r H–manifolds, $1 \le r \le \omega$. Then the map*

$$\chi: \mathrm{C}^{r,H}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, G \times P), \quad f \mapsto \mathrm{id} \times f,$$

is continuous. Here $G \times H$ acts on $G \times N$, and also on $G \times P$, as given in $(*)$.

Proof. It follows by Proposition 4.6 that it is enough to prove that the maps

$$\mathrm{C}^{r,H}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, G), \quad f \mapsto q_1 \circ (\mathrm{id} \times f), \tag{1}$$

and

$$\mathrm{C}^{r,H}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, P), \quad f \mapsto q_2 \circ (\mathrm{id} \times f), \tag{2}$$

are continuous. Here $q_1: G \times P \to G$ and $q_2: G \times P \to P$ are the projections. The group $G \times H$ acts on $G \times N$ and on G as above, and on P by $(\overline{g}, h)y = hy$. The map in (1) is the constant map onto the element $r_1 \in \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, G)$, where $r_1: G \times N \to G$ is the projection, and hence the map in (1) is continuous.

Observe that $q_2 \circ (\mathrm{id} \times f) = f \circ r_2$, where $r_2: G \times N \to N$ is the projection. Hence the map in (2) equals the map

$$r_2^*: \mathrm{C}^{r,H}_{\mathrm{SW}}(N,P) \to \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, P), \quad f \mapsto f \circ r_2. \tag{3}$$

Furthermore $\mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, P) = \mathrm{C}^{r,\rho}_{\mathrm{SW}}(G \times N, P)$, where $\rho: G \times H \to H$ denotes the projection. Note that $r_2: G \times N \to N$ is a ρ–equivariant map. Since the induced map $\overline{r}_2: (G \times N)/(G \times H) \to N/H$ is a homeomorphism, and thus in particular of finite type, it now follows by Proposition 4.4 that (3) is continuous. □

Lemma 5.2. *Let M be a proper free C^r H–manifold, $1 \le r \le \omega$, and let $p: M \to M/H$ be the projection. Suppose $\rho': M/H \to \Omega$ is a phase map for the manifold M/H, and give M the phase map $\rho = \rho' \circ p: M \to \Omega$. Let P be a C^r manifold. Then the map*

$$p^*: \mathrm{C}^{r}_{\mathrm{SW}[\rho']}(M/H, P) \to \mathrm{C}^{r,H}_{\mathrm{SW}[\rho]}(M, P), \quad k \mapsto k \circ p,$$

is a homeomorphism

Proof. It follows by Proposition 2.5 that p^* is continuous. Clearly p^* is a bijective map, and the inverse of p^* is the map

$$p_!: \mathrm{C}^{r,H}_{\mathrm{SW}[\rho]}(M, P) \to \mathrm{C}^{r}_{\mathrm{SW}[\rho']}(M/H, P), \quad f \mapsto \overline{f}.$$

We shall prove that $p_!$ is continuous. It is enough to consider the case $1 \le r < \infty$. Suppose $f \in \mathrm{C}^{r,H}_{\mathrm{SW}[\rho]}(M, P)$, and let $\hat{\mathscr{N}} = \bigcap_{i \in \Lambda} \mathscr{N}_i$ be a basic neighborhood of $p_!(f) =$

$\overline{f} \in \mathrm{C}^r_{\mathrm{SW}[\rho']}(M/H, P)$. Here $\mathscr{N}_i = \mathscr{N}^r(\overline{f}; L_i, (V_i, \psi_i), (W_i, \omega_i),\ \varepsilon_i)$, $i \in \Lambda$, are elementary neighborhoods of $\overline{f}$ and the family $\{\rho'(L_i)\}_{i\in\Lambda}$ is locally finite in Ω. We shall find an open neighborhood $\hat{\mathscr{M}}^H$ of f in $\mathrm{C}^{r,H}_{\mathrm{SW}[\rho]}(M, P)$, such that $p_!(\hat{\mathscr{M}}^H) \subset \hat{\mathscr{N}}$.

Since H acts properly and freely on M the projection $p\colon M \to M/H$ is a locally trivial C^r-smooth bundle with fiber H over M/H. It follows by Remark 2.2 that we may assume that each of the charts (V_i, ψ_i) in M/H is such that the bundle $p\colon M \to M/H$ is trivial over V_i. For each $i \in \Lambda$ we can then find product charts (U_i, φ_i) in M, so that the diagram

$$\begin{tikzcd} W_i \times \psi_i(V_i) \arrow[d, "pr_2"'] & U_i \arrow[l, "\varphi_i"', "\cong"] \arrow[d, "p|"] \\ \psi_i(V_i) & V_i \arrow[l, "\psi_i"', "\cong"] \end{tikzcd}$$

commutes, where φ_i is a C^r diffeomorphism onto $W_i \times \psi_i(V_i)$ and W_i is an open neighborhood of the origin in $\mathbb{R}^q$, $q = \dim H$. We set

$$K_i = \varphi_i^{-1}(\{0\} \times \psi_i(L_i)) \subset U_i.$$

Then $p(K_i) = L_i$ and $\varphi_i(K_i) = \{0\} \times \psi_i(L_i)$. If now $k\colon M \to P$ is an H-invariant C^r map, then $(k|U_i) \circ \varphi_i^{-1} = (\overline{k}|V_i) \circ \psi_i^{-1} \circ pr_2\colon W_i \times \varphi_i(U_i) \to P$, and hence

$$(4) \qquad \|\omega_i \circ k \circ \varphi_i^{-1} - \omega_i \circ f \circ \varphi_i^{-1}\|^r_{\varphi_i(K_i)} = \|\omega_i \circ \overline{k} \circ \psi_i^{-1} - \omega_i \circ \overline{f} \circ \psi_i^{-1}\|^r_{\psi_i(L_i)}.$$

We define

$$\mathscr{M}_i^H = \mathscr{N}^r(f; K_i, (U_i, \varphi_i), (W_i, \omega_i), \varepsilon_i) \cap \mathrm{C}^{r,H}(M, P).$$

Since (4) holds it follows that if $k \in \mathscr{M}_i^H$ then $\overline{k} \in \mathscr{N}_i$. Thus we have $p_!(\mathscr{M}_i^H) \subset \mathscr{N}_i$. Since $\rho(K_i) = (\rho' \circ p)(K_i) = \rho'(L_i)$, the family $\{\rho(K_i)\}_{i\in\Lambda}$ is locally finite in Ω, and hence $\hat{\mathscr{M}}^H = \bigcap_{i\in\Lambda} \mathscr{M}_i^H$ is an open neighborhood of f in $\mathrm{C}^{r,H}_{\mathrm{SW}[\rho]}(M, P)$. Now $p_!(\hat{\mathscr{M}}^H) \subset \bigcap_{i\in\Lambda} \mathscr{N}_i = \hat{\mathscr{N}}$, and hence $p_!$ is continuous. $\square$

If we in Lemma 5.2 choose the phase map ρ' to be $\mathrm{id}\colon M/H \to M/H$, we obtain Corollary 5.3 below. Note that Theorem A in the introduction is the C^∞-case of this corollary.

Corollary 5.3. *The map $p^*\colon \mathrm{C}^r_{\mathrm{S}}(M/H, P) \longrightarrow \mathrm{C}^{r,H}_{\mathrm{SW}}(M, P)$, $k \mapsto k \circ p$, is a homeomorphism.* $\square$

Proposition 5.4. *Let H be a closed normal subgroup of a Lie group G, and let $\pi\colon G \to G/H$ be the projection. Let M be a C^r G–manifold, $1 \le r \le \omega$, such that H acts properly and freely on M, and let $p\colon M \to M/H$ be the projection. Let P be a C^r G/H–manifold. Then*

$$p^*\colon \mathrm{C}^{r,G/H}_{\mathrm{SW}}(M/H, P) \to \mathrm{C}^{r,\pi}_{\mathrm{SW}}(M, P), \quad k \mapsto k \circ p,$$

is a homeomorphism.

Proof. We have that $(M/H)/(G/H) = M/G$, and if we let $q' : M/H \to (M/H)/(G/H)$ and $q : M \to M/G$ denote the projections onto the orbit spaces, then $q = q' \circ p$. Hence the result follows by Lemma 5.2, when we note that if $k : M/H \to P$ is a (G/H)-equivariant map then $p^*(k) = k \circ p : M \to P$ is a π-equivariant map, and conversely. $\square$

We are now ready to prove one of the main results of this paper, of which Theorem B in the introduction is the C^∞-case.

Theorem 5.5. *Let H be a closed subgroup of a Lie group G. Suppose N is a C^r H-manifold, and that P is a C^r G-manifold, $1 \le r \le \omega$. Then the canonical bijection*

$$\Theta : \mathrm{C}^{r,H}_{\mathrm{SW}}(N, P) \xrightarrow{\cong} \mathrm{C}^{r,G}_{\mathrm{SW}}(G \times_H N, P), \quad f \mapsto \mu(f), \tag{5}$$

is a homeomorphism.

Proof. Recall that if $f \in \mathrm{C}^{r,H}(N, P)$, then $\mu(f) : G \times_H N \to P$, $[g, x] \mapsto gf(x)$. We let $\Phi : G \times P \to P$, $(g, y) \mapsto gy$, denote the action of G on P. The group $G \times H$ acts on $G \times P$ by $(\overline{g}, h)(g, y) = (\overline{g}gh^{-1}, hy)$. Let $\pi : G \times H \to G$ be the projection. Then Φ is a π-equivariant map, since $\Phi(\overline{g}gh^{-1}, hy) = \overline{g}gy = \overline{g}\Phi(g, y)$. We let $p : G \times N \to G \times_H N$, $(g, x) \mapsto [g, x]$, be the natural projection. Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{C}^{r,H}_{\mathrm{SW}}(N, P) \arrow[r, "\chi"] \arrow[rrd, "\Theta"'] & \mathrm{C}^{r,G\times H}_{\mathrm{SW}}(G \times N, G \times P) \arrow[r, "\Phi_*"] & \mathrm{C}^{r,\pi}_{\mathrm{SW}}(G \times N, P) \\
& & \mathrm{C}^{r,G}_{\mathrm{SW}}(G \times_H N, P) \arrow[u, "p^*", "\cong"']
\end{tikzcd}$$

Here the map χ is continuous by Lemma 5.1, and Φ_* is continuous by Proposition 4.3. Since $H = \{e\} \times H$ is a closed normal subgroup of $G \times H$, and the action of H on $G \times N$, given by $(h, (g, x)) \mapsto (gh^{-1}, hx)$, is proper and free, it follows by Proposition 5.4 that p^* is a homeomorphism. Hence the map Θ in (5) is continuous. The inverse map Θ^{-1} equals

$$i^* : \mathrm{C}^{r,G}_{\mathrm{SW}}(G \times_H N, P) \to \mathrm{C}^{r,H}_{\mathrm{SW}}(N, P)$$

where $i : N \hookrightarrow G \times_H N$ denotes the natural inclusion. The induced map $\overline{i} : N/H \to (G \times_H N)/G$ is a homeomorphism, and thus is particular of finite type, and hence i^* is continuous by Proposition 4.4. This completes the proof. $\square$

6. The set of G–equivariant immersions (submersions) is open in the strong-weak topology

In this section G denotes an arbitrary Lie group.

Proposition 6.1. *Let M and N be C^r G-manifolds, $1 \le r \le \omega$, where the action of G on M is proper. Then the set of G-equivariant C^r immersions is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.*

Proof. Suppose $f: M \to N$ is a G-equivariant C^r immersion. By Lemma 3.6 we find a closed fundamental set E in M. We let the sets F_i, $i \ge 0$, and also other notation be as in Section 3. Since F_i is compact and $F_i \subset \hat{U}_i$, we can find finitely many charts $(U_{i,j}, \varphi_{i,j})$, $1 \le j \le t(i)$, in M, and open sets $W_{i,j}$, where $W_{i,j} \subset \overline{W}_{i,j} \subset U_{i,j} \subset \hat{U}_i$ and $\overline{W}_{i,j}$ is compact, such that the following hold:

(a) $f(U_{i,j}) \subset V_{i,j}$, where $(V_{i,j}, \psi_{i,j})$ is a chart in N, $1 \le j \le t(i)$.

(b) $F_i \subset \bigcup_{j=1}^{t(i)} W_{i,j}$.

Now $p(\overline{W_{i,j}}) \subset \hat{D}_i$, for $1 \le j \le t(i)$, and hence we have:

(c) The family $\{p(\overline{W_{i,j}}) \mid i \ge 0, 1 \le j \le t(i)\}$ is locally finite in M/G.

Each map $\psi_{i,j} \circ f \circ \varphi_{i,j}^{-1}: \varphi_{i,j}(U_{i,j}) \to \mathbb{R}^n$ is a C^r immersion. Therefore its derivative at each point of $\varphi_{i,j}(U_{i,j})$ is a linear injective map from $\mathbb{R}^m$ into $\mathbb{R}^n$. Since $\varphi_{i,j}(\overline{W}_{i,j})$ is compact, and the set of linear injective maps is open in the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, there exists $\varepsilon_{i,j} > 0$ such that if $h \in \mathscr{N}_{i,j} = \mathscr{N}^r(f; \overline{W}_{i,j}, (U_{i,j}, \varphi_{i,j}), (V_{i,j}, \psi_{i,j}), \varepsilon_{i,j})$, then $\psi_{i,j} \circ h \circ \varphi_{i,j}^{-1}: \varphi_{i,j}(W_{i,j}) \to \mathbb{R}^n$, and hence also $h|: W_{i,j} \to N$, is a C^r immersion.

Since (c) holds $\hat{\mathscr{N}} = \bigcap_{i=0}^{\infty} \bigcap_{j=1}^{t(i)} \mathscr{N}_{i,j}$ is a basic neighborhood of f in $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$, and if $h \in \hat{\mathscr{N}}$ then $h: W \to N$ is an immersion, where $W = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{t(i)} W_{i,j}$. We have $GW = M$. Now let $h \in \hat{\mathscr{N}} \cap \mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$. Then $h: M \to N$ is a G-equivariant C^r immersion. □

Proposition 6.2. *Let M and N be C^r G-manifolds, where $1 \le r \le \omega$, and the action of G on M is proper. Then the set of G-equivariant C^r submersions is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.*

Proof. The proof is completely analogous to that of Proposition 6.1. □

Let G and H be Lie groups and let $\theta: G \to H$ be a homomorphism. Suppose M is a proper C^r G-manifold and that N is a proper C^r H-manifold, where $1 \le r \le \omega$. Then we denote

$$\overline{\mathrm{P}}^{r,\theta}_{\mathrm{SW}}(M,N) = \{f \in \mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,N) \mid \overline{f}: M/G \to N/H \quad \text{is a proper map}\}.$$

Lemma 6.3. *The set $\overline{\mathrm{P}}^{r,\theta}_{\mathrm{SW}}(M,N)$ is open in $\mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,N)$, $1 \le r \le \omega$.*

Proof. Let $p: M \to M/G$ and $q: N \to N/H$ be the projections onto the orbit spaces. We adopt the notation introduced in Section 3 in such a way that we in the case of the smooth H-manifold N denote $D'_i \subset N/H$, and $B'_i \subset \hat{U}'_i \subset N$, etc.

Let $f \in \overline{\mathrm{P}}^{r,\theta}_{\mathrm{SW}}(M,N)$. Then $\overline{f}\colon M/G \to N/H$ is a proper map. We denote $D^{\#}_i = \overline{f}^{-1}(D'_i)$, $i \geq 0$. Then $\{D^{\#}_i\}_{i\geq 0}$ is a locally finite family of compact subsets of M/G. Let E be a closed fundamental set in M. We define

$$F^{\#}_i = p^{-1}(D^{\#}_i) \cap E.$$

Then $\{F^{\#}_i\}_{i\geq 0}$ is a locally finite family of compact subsets of M. Furthermore $p(F^{\#}_i) = D^{\#}_i$, and $f(F^{\#}_i) \subset q^{-1}(D'_i) = B'_i \subset \hat{U}'_i$, $i \geq 0$. By Lemma 2.7 we find an open neighborhood $\mathscr{N}$ of f in $\mathrm{C}^{r,\theta}_{\mathrm{SW}}(M,N)$ such that if $k \in \mathscr{N}$ then $k(F^{\#}_i) \subset \hat{U}'_i$, and hence $k(GF^{\#}_i) \subset H\hat{U}'_i = \hat{U}'_i$, $i \geq 0$. Thus $\overline{k}(D^{\#}_i) \subset D'_{i-1} \cup D'_i \cup D'_{i+1} = \hat{D}'_i$, $i \geq 0$.

Let X be a compact subset of N/H. Then $X \subset \bigcup_{i=0}^{m} D'_i$, for some $m \geq 0$. Hence $\overline{k}^{-1}(X) \subset \bigcup_{i=0}^{m+2} D^{\#}_i$, and thus $\overline{k}^{-1}(X)$ is compact. ◻

Proposition 6.4. *Let M and N be proper C^r G-manifolds. Then the set of proper G-equivariant C^r maps is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$, $1 \leq r \leq \omega$.*

Proof. Follows by Lemmas 3.7, 3.9 and 6.3. ◻

7. Proofs of Theorems C and D

Throughout this section G denotes an arbitrary Lie group. Let M be a proper G–manifold, and let $p\colon M \to M/G$ denote the projection onto the orbit space. We use the notation introduced in Section 3.

Lemma 7.1. *Suppose $f\colon M \to N$ is a G-equivariant topological imbedding, and let E be a closed fundamental set in M, and let the sets F_i, $i \geq 0$, be as in Section 3. Then there exist open subsets V_i, $i \geq 0$, of N such that*

1) $f(F_i) \subset V_i$, $i \geq 0$,
2) $\overline{V}_i$ *is compact*, $i \geq 0$,
3) $GV_i \cap GV_j = \emptyset$, *if* $|i-j| \geq 2$.

In order to give the proof of Lemma 7.1 we first establish Lemmas 7.1.a and 7.1.b.

Lemma 7.1.a. *Let $f\colon X \to Y$ be an imbedding, where X and Y are Hausdorff spaces and Y is normal. Let C_0 be a compact subset of X and U an open neighborhood of C_0 in X. Then there exist disjoint open neighborhoods W and W^* of $f(C_0)$ and $f(X-U)$ in Y.*

Proof. Since $f(C_0)$ is compact it is closed in Y. Since $f(U)$ is open in $f(X)$ there exists an open subset $\tilde{W}$ of Y, such that $f(U) = \tilde{W} \cap f(X)$. We now have that $f(C_0) \subset f(U) \subset \tilde{W}$ and $f(X-U) \subset Y - \tilde{W}$. Since Y is normal there exist disjoint open neighborhoods W and W^* of $f(C_0)$ and $Y - \tilde{W}$ in Y. Then W and W^* are as required. ◻

Lemma 7.1.b. *Let $f\colon X \to Y$ be an imbedding, where X and Y are Hausdorff spaces and Y is normal. Let C_i, $i \geq 0$, be an expanding sequence of compact subsets of X such that $C_i \subset \mathring{C}_{i+1}$, for all $i \geq 0$, and denote*

$$D_i = C_i - \mathring{C}_{i-1}, \quad i \geq 0.$$

Then there exist open subsets W_i of Y, such that

i) $f(D_i) \subset W_i, \quad i \geq 0,$

ii) $W_i \cap W_j = \emptyset$, *if* $|i-j| \geq 2$.

Proof. First we claim that there exist open subsets W_i' and W_i^*, $i \geq 0$, of Y, such that

(a) $f(C_i) \subset W_i'$ and $f(X - \mathring{C}_{i+1}) \subset W_i^*$, $i \geq 0$,

(b) $W_i' \cap W_i^* = \emptyset$, $i \geq 0$,

(c) $W_k^* \subset W_i^*$, for $0 \leq i \leq k$.

Let $i \geq 0$, and assume this holds for $j \leq i-1$. Then we construct W_i' and W_i^* as follows. By Lemma 7.1.a we find open subsets W_i' and $W_i'^*$ of Y such that $f(C_i) \subset W_i'$, $f(X - \mathring{C}_{i+1}) \subset W_i'^*$ and $W_i' \cap W_i'^* = \emptyset$. By the inductive assumption we have that $f(X - \mathring{C}_{i+1}) \subset f(X - \mathring{C}_i) \subset W_{i-1}^*$. We set

$$W_i^* = W_{i-1}^* \cap W_i'^*, \quad i \geq 0,$$

where $W_{-1}^* = Y$. Then $f(X - \mathring{C}_{i+1}) \subset W_i^*$, and hence W_i' and W_i^* are as required, and our claim follows.

We now define

$$W_i = W_{i-2}^* \cap W_i', \quad i \geq 0,$$

where $W_{-2}^* = W_{-1}^* = Y$. Since $D_i = C_i - \mathring{C}_{i-1}$, $i \geq 0$, it follows by (a) that

$$f(D_i) \subset W_i' \cap W_{i-2}^* = W_i, \quad i \geq 0.$$

Now suppose $|i-j| \geq 2$, say $j \geq i+2$. Then we obtain, using (b) and (c), that

$$W_i \cap W_j = W_{i-2}^* \cap W_i' \cap W_{j-2}^* \cap W_j' \subset W_i' \cap W_{j-2}^* \subset W_i' \cap W_i^* = \emptyset.$$

Thus the sets W_i, $i \geq 0$, are as required. □

Proof of Lemma 7.1. The orbit spaces M/G and N/G are Hausdorff and normal by Lemma 3.4, and the induced map $\overline{f}\colon M/G \to N/G$ is a clearly an imbedding. By Lemma 7.1.b there exist open subsets W_i, $i \geq 0$, of N/G such that

(a) $f(B_i) \subset q^{-1}(W_i)$, $i \geq 0$,

(b) $q^{-1}(W_i) \cap q^{-1}(W_j) = \emptyset, \quad |i-j| \geq 2.$

We have that $f(F_i) \subset f(B_i) \subset q^{-1}(W_i)$, and since $f(F_i)$ is compact there exists an open neighborhood V_i of $f(F_i)$ such that

$$f(F_i) \subset V_i \subset \overline{V}_i \subset q^{-1}(W_i),$$

and $\overline{V}_i$ is compact. Now $GV_i \subset q^{-1}(W_i)$, and we see that 1)–3) hold. □

Lemma 7.2. *Let M and N be proper C^r G-manifolds, $1 \leq r \leq \omega$, and let E be a fat closed fundamental set in M, which exists by Lemma 3.6. We use the natural projection*

$p\colon M \to M/G$ as a phase map for M, and consider the space $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$. Suppose $f\colon M \to N$ is a C^r imbedding. Then there exists an open neighborhood $\mathscr{M}$ of f in $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$, such that if $h \in \mathscr{M}$, then $h|\colon \mathring{E} \to N$ is a C^r imbedding and $h|\colon E \to N$ is a topological imbedding.

Proof. Let the sets $A_i, B_i, \hat{U}_i, \hat{B}_i$, and E_i, F_i, $i \geq 0$, be as in Section 3. Since F_i is compact and $F_i \subset \hat{U}_i$, we can cover each F_i with finitely many compact sets $K_{i,j}$, $1 \leq j \leq t(i)$, such that

(a) $K_{i,j} \subset W_{i,j} \subset \overline{W}_{i,j} \subset U_{i,j} \subset \hat{U}_i$,
(b) $f(U_{i,j}) \subset V_{i,j}$,

where $W_{i,j}$ is open in M, $\overline{W}_{i,j}$ is compact, $(U_{i,j}, \varphi_{i,j})$ is a chart in M, and $(V_{i,j}, \psi_{i,j})$ is a chart in N, $1 \leq j \leq t(i)$.

The map $\psi_{i,j} \circ f \circ \varphi_{i,j}^{-1}\colon \varphi_{i,j}(U_{i,j}) \to \mathbb{R}^n$ is a C^r imbedding and $\varphi_{i,j}(\overline{W}_{i,j})$ is compact. Hence there exist $\varepsilon_{i,j} > 0$, see [Hi, Lemma 2.1.3], such that if $h \in \mathrm{C}^r(M,N)$, and $h(\overline{W}_{i,j}) \subset V_{i,j}$, then the condition

$$\|\psi_{i,j} \circ h \circ \varphi_{i,j}^{-1} - \psi_{i,j} \circ f \circ \varphi_{i,j}^{-1}\|^1_{\varphi(\overline{W}_{i,j})} < \varepsilon_{i,j}$$

implies that $\psi_{i,j} \circ h \circ \varphi_{i,j}^{-1}\colon \varphi(W_{i,j}) \to \mathbb{R}^n$ is a C^r imbedding, and hence also that $h|\colon W_{i,j} \to N$ is a C^r imbedding. We define $\mathscr{N}_{i,j} = \mathscr{N}^r(f; \overline{W}_{i,j}, (U_{i,j}, \varphi_{i,j}), (V_{i,j}, \psi_{i,j}), \varepsilon_{i,j})$, for $i \geq 0$, $1 \leq j \leq t(i)$. The family $\{p(\overline{W}_{i,j}) \mid i \geq 0, 1 \leq j \leq t(i)\}$ is clearly locally finite in M/G, and hence $\hat{\mathscr{N}} = \bigcap_{i=0}^{\infty} \bigcap_{j=1}^{t(i)} \mathscr{N}_{i,j}$ is a basic neighborhood of f in $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$. We now have that if $h \in \hat{\mathscr{N}}$ then

$$h|\colon W_{i,j} \to N$$

is a C^r imbedding for all $i \geq 0$, $1 \leq j \leq t(i)$. In particular it follows that if $h \in \hat{\mathscr{N}}$ then

$$h|\colon W \to N$$

is a C^r immersion, where $W = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{t(i)} W_{i,j}$.

We have that $K_{i,j} \subset W_{i,j} \subset \hat{B}_i$, and we define

$$K^*_{i,j} = \hat{B}_i - W_{i,j}, \quad 1 \leq j \leq t(i).$$

Then $f(K_{i,j})$ and $f(K^*_{i,j})$ are compact and disjoint, and hence there exist disjoint open neighborhoods $Q_{i,j}$ and $Q^*_{i,j}$ of $f(K_{i,j})$ and $f(K^*_{i,j})$ in N. By Lemma 7.1 we find open subsets $V_i^{\#}$, $i \geq 0$, of N such that $f(F_i) \subset V_i^{\#}$, $i \geq 0$, and $V_i^{\#} \cap V_{i'}^{\#} = \emptyset$ if $|i - i'| \geq 2$. Note that the families $\{p(K_{i,j}) \mid i \geq 0, 1 \leq j \leq t(i)\}$, $\{p(K^*_{i,j}) \mid i \geq 0, 1 \leq j \leq t(i)\}$ and $\{p(F_i) \mid i \geq 0\}$ are locally finite in M/G. By applying Lemma 2.7 three times we find an open neighborhood $\mathscr{M}_0$ of f in $\mathrm{C}^r_{\mathrm{SW}[p]}(M,N)$ so that if $h \in \mathscr{M}_0$ then:

(i) $h(K_{i,j}) \subset Q_{i,j}$ and $h(K^*_{i,j}) \subset Q^*_{i,j}$, $\quad i \geq 0, \quad 1 \leq j \leq t(i)$,
(ii) $h(F_i) \subset V_i^{\#}$, $\quad i \geq 0$.

We shall now show that if $h \in \mathscr{M} = \hat{\mathscr{N}} \cap \mathscr{M}_0$ then

$$h|: E \to N \tag{1}$$

is injective. Let $x, y \in E$. Then $x \in K_{i,j}$, for some $i \geq 0$, and $1 \leq j \leq t(i)$. If $y \in W_{i,j}$, then $h(x) = h(y)$ implies that $x = y$, since $h|: W_{i,j} \to N$ is injective.

We now show that if $x \in K_{i,j}$ and $y \notin W_{i,j}$, then $h(x) \neq h(y)$. This is seen as follows. If $y \in \hat{B}_i$, then $y \in \hat{B}_i - W_{i,j} = K^*_{i,j}$. We now have that $h(x) \neq h(y)$, since $h(x) \in h(K_{i,j}) \subset Q_{i,j}$ and $h(y) \in h(K^*_{i,j}) \subset Q^*_{i,j}$, and $Q_{i,j} \cap Q^*_{i,j} = \emptyset$. If $y \in M - \hat{B}_i$, then $y \in (M - A_{i+1}) \cup \mathring{A}_{i-2}$, and thus $y \in F_{i'}$, where $|i' - i| \geq 2$. Hence $h(x) \neq h(y)$, since $h(x) \in h(F_i) \subset V_i^{\#}$, $h(y) \in h(F_{i'}) \subset V_{i'}^{\#}$, and $V_i^{\#} \cap V_{i'}^{\#} = \emptyset$, because $|i' - i| \geq 2$. Thus $h|E$ in (1) is injective.

Next we show that $h| : E \to N$ is a topological imbedding, i.e., that $h| : E \to h(E)$ is a homeomorphism, where $h(E)$ has the relative topology from N. We must show that $(h|)^{-1}: h(E) \to E$ is continuous. Let $h(x) \in h(E)$ and suppose that $h(x_n) \to h(x)$, where $x \in E$ and $x_n \in E$, $n \geq 1$. We claim that $x_n \to x$. Let V be an open neighborhood of $h(x)$ in $h(E)$ such that $V \cap V_i^{\#} \neq \emptyset$ for only finitely many i, and let m_0 be such that $V \cap V_i^{\#} = \emptyset$ for all $i \geq m_0 + 1$. If $h(x_n) \in V$, for $n \geq 1$, then $x_n \in F_0 \cup \cdots \cup F_{m_0} = E_{m_0}$, for $n \geq 1$, since $h(F_i) \subset V_i^{\#}$ and $V \cap V_i^{\#} = \emptyset$ if $i \geq m_0 + 1$. Since E_{m_0} is compact the map

$$h|: E_{m_0} \to h(E_{m_0})$$

is a homeomorphism, and hence $h(x_n) \to h(x)$ implies that $x_n \to x$, as we claimed. Thus $h|: E \to N$ is a topological imbedding.

Now $h|: \mathring{E} \to N$ is a topological imbedding and a C^r immersion, since $h|: W \to N$ is a C^r immersion. Thus $h|: \mathring{E} \to N$ is a C^r imbedding. □

We are now ready to prove the following theorem, which in particular gives us Theorem C.1 in the introduction.

Theorem 7.3. *Let M and N be proper C^r G-manifolds, $1 \leq r \leq \omega$, where G is a Lie group. Then the set of G-equivariant C^r imbeddings is open in $C^{r,G}_{\mathrm{SW}}(M, N)$.*

Proof. By Lemma 3.6 we find a fat closed fundamental set E of M. We let the sets F_i, $i \geq 0$, be defined as in Section 3. Suppose $f: M \to N$ is a G-equivariant C^r imbedding. By Lemma 7.1 there exist open subsets V_i, $i \geq 0$, of N such that each $\overline{V}_i$ is compact, and

(i) $f(F_i) \subset V_i$, $i \geq 0$,
(ii) $GV_i \cap GV_j = \emptyset$, if $|i - j| \geq 2$.

For each $i \geq 0$ we set

$$Z_i = \overline{V_{i-1} \cup V_i \cup V_{i+1}}.$$

Then Z_i is a compact subset of N, and since the action of G on N is proper we have that

$$Q_i = \{g \in G \mid gZ_i \cap Z_i \neq \emptyset\}$$

is a compact subset of G. We define

$$E^{\#} = \bigcup_{i=1}^{\infty} Q_i F_i.$$

Then $E^{\#}$ is a fat G-section subset of M, and $E \subset E^{\#}$.

It follows by Lemmas 2.7 and 7.2 that there exists an open neighborhood $\mathscr{M}$ of f in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$ such that if $h \in \mathscr{M}$, then the G-equivariant C^r map $h\colon M \to N$ has the following properties:

(a) $h(F_i) \subset V_i,\ i \geq 0$,
(b) $h|\colon \mathring{E}^{\#} \to N$ is a C^r imbedding,
(c) $h|\colon E^{\#} \to N$ is a topological imbedding.

We shall now show that if $h \in \mathscr{M}$, then

$$h\colon M \to N$$

is injective. Let $x, y \in M$ be such that $h(x) = h(y)$. Since $M = GE$ and $E = \bigcup_{i=0}^{\infty} F_i$ it follows that $x = gx_0$, and $y = g'y_0$, where $x_0 \in F_i,\ y_0 \in F_j$, some $i, j \geq 0$. Setting $g_0 = g^{-1}g' \in G$ we have that $h(x_0) = g_0 h(y_0) \in V_i \cap g_0 V_j$, hence $GV_i \cap GV_j \neq \emptyset$. Thus $|i-j| \leq 1$, and hence $h(x_0) \in Z_j \cap g_0 Z_j$, and therefore $g_0 \in Q_j$. Now $x_0 \in E^{\#}$ and $g_0 y_0 \in E^{\#}$, and since $h(x_0) = h(g_0 y_0)$ and $h|\colon E^{\#} \to N$ is injective, it follows that $x_0 = g_0 y_0$, and hence $x = y$.

Next we show that $h\colon M \to N$ is a topological imbedding, i.e., that $h\colon M \to h(M)$ is a homeomorphism, where $h(M)$ has the relative topology from N. We must prove that if $y \in h(M)$ and $y_n \to y$, where $y_n \in h(M)$, $n \geq 1$, then $h^{-1}(y_n) \to h^{-1}(y)$ in M. We denote $y = h(x)$ and $y_n = h(x_n)$, where $x \in M$ and $x_n \in M$, $n \geq 1$. Our claim is that $x_n \to x$.

Let V be an open neighborhood of y such that $V \cap GV_i \neq \emptyset$ for only finitely many i, and let m_0 be such that $V \cap GV_i = \emptyset$ for all $i \geq m_0 + 1$. Suppose $h(x_n) \in V$, for $n \geq 1$, then $x_n \in GF_0 \cup \cdots \cup GF_{m_0} = GE_{m_0}$, since $h(GF_i) \subset GV_i$ and $V \cap GV_i = \emptyset$ for all $i \geq m_0 + 1$. Since E_{m_0} is compact the map

$$h|\colon GE_{m_0} \to N$$

is a closed map, by Lemmas 3.2 and 3.8. Thus $h|\colon GE_{m_0} \to h(GE_{m_0})$ is a homeomorphism. Since $x_n \in GE_{m_0}$, $n \geq 1$, and $h(x_n) \to h(x)$, it now follows that $x_n \to x$. Thus $h\colon M \to N$ is a topological imbedding.

By (b) we have that $h|\colon \mathring{E}^{\#} \to N$ is a C^r imbedding, and hence in particular a C^r immersion. Since $G\mathring{E}^{\#} = M$, and since h is a G-equivariant map it now follows that $h\colon M \to N$ is a C^r immersion. Since we already showed that $h\colon M \to N$ is a topological imbedding, it now follows that $h\colon M \to N$ is a C^r imbedding. □

Theorem 7.4. *Let M and N be proper C^r G-manifolds, $1 \leq r \leq \omega$, where G is a Lie group. Then the set of G-equivariant C^r closed imbeddings is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.*

Proof. Follows by Theorem 7.3 and Proposition 6.4. □

Theorem 7.4 gives us Theorem C.2 in the Introduction. Theorem 7.5, which gives us Theorem D, follows from Theorem 7.4 in a standard way. For the sake of completeness we give the details below.

Theorem 7.5. *Let M and N be proper C^r G-manifolds, $1 \leq r \leq \omega$, where G is a Lie group. Then the set of G-equivariant C^r diffeomorphisms is open in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$.*

Proof. Let $f: M \to N$ be a G-equivariant C^r diffeomorphism. We claim that there exists an open neighborhood $\mathscr{M}$ of f in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$ such that if $h \in \mathscr{M}$, then $h: M \to N$ is a G-equivariant C^r diffeomorphism. Clearly it is enough to consider the case where $M = GM_0$ and $N = GN_0$, and M_0 and N_0 are connected components of M and N, and $f(M_0) = N_0$. By Theorem 7.4 we find an open neighborhood $\mathscr{M}$ of f in $\mathrm{C}^{r,G}_{\mathrm{SW}}(M,N)$ such that if $h \in \mathscr{M}$, then $h: M \to N$ is a G-equivariant closed C^r imbedding, and $h(M_0) \subset N_0$.

Since $\dim M = \dim N$ we have that $\dim M_0 = \dim N_0$, and thus $\dim h(M_0) = \dim N_0$. Hence it follows by invariance of domain that $h(M_0)$ is open in N_0. Now $h(M_0)$ is both open and closed in N_0 and hence $h(M_0) = N_0$, since N_0 is connected. Thus $h(M) = N$, and by the C^r inverse mapping theorem, see e.g. [N, Theorem 2.2.10], it follows that the map $h^{-1}: N \to M$ is a C^r map, and hence $h: M \to N$ is a G-equivariant C^r diffeomorphism. □

References

[B] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.

[Č] E. Čech, *Topological spaces*, John Wiley & Sons Ltd, 1966.

[He] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York and London, 1962.

[Hi] M. Hirsch, *Differential topology*, Springer - Verlag, New York – Heidelberg – Berlin, 1976.

[I1] S. Illman, *Subanalytic equivariant triangulations*, to appear.

[I2] S. Illman, *Every proper smooth action of a Lie group is equivalent to a real analytic action: a contribution to Hilbert's fifth problem*, Ann. of Math. Stud. **138** (1995), 189–220.

[I–Ka] S. Illman and M. Kankaanrinta, *Some basic results for real analytic proper G-manifolds*, Preprint, University of Helsinki, 1998.

[Ku] F. Kutzschebauch, *On the uniqueness of the analyticity of a G-action*, Manuscripta Math. **90** (1996), 17 - 22.

[Ma] J. Mather, *Stability of C^∞-mappings II, Infinitesimal stability implies stability*, Ann. of Math. (2) **89** (1969), 254 - 299.

[M–S] T. Matumoto and M. Shiota, *Unique triangulation of the orbit space of a differentiable transformation group and its applications*, Homotopy Theory and Related Topics, Adv Stud. Pure Math., vol **9**, Kinokuniya, Tokyo, 1987, 41 - 55.

[N] R. Narasimhan, *Analysis on real and complex manifolds*, North–Holland, Amsterdam, 1968.

[P1] R.S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295 - 323.

[P2] R.S. Palais, *When proper maps are closed*, Proc. Amer. Math. Soc., **24** (1970), 835 - 836.

[S] N. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133 - 152.

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