On some maximal inequalities for fractional Brownian motions

Alexander Novikov*
The University of Newcastle
Department of Statistics

Esko Valkeila†
University of Helsinki
Department of Mathematics

Abstract

We prove some maximal inequalities for fractional Brownian motions. These extend the Burkholder-Davis-Gundy inequalities for fractional Brownian motions. The methods are based on the integral representations of fractional Brownian motions with respect to a certain Gaussian martingale in terms of beta kernels.

keywords: fractional Brownian motions, maximal inequalities

AMS-classification: 60G15, 60G40, 60H99

*Supported by Suomalainen Tiedeakatemia, the Research Grants Committee of the University of Helsinki and RMS New Staff Grant

†Corresponding author: Department of Mathematics, P.O. Box 4, FIN-00014 University of Helsinki, FINLAND

1 Introduction

1. (*Fractional Brownian motions*) We work in probability space (Ω, F, P). Fractional Brownian motion $Z = Z^H$ with Hurst index H is a continuous Gaussian process with stationary increments and with the following properties:

i $Z_0 = 0$.

ii $EZ_t = 0$ for all $t \geq 0$.

iii $EZ_tZ_s = \frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right)$ for all $s, t \geq 0$.

The standard Brownian motion B is a fractional Brownian motion with Hurst index $H = 1/2$.

2. (*Some maximal inequalities*) Denote by $\mathbf{F} = (F_t)_{t\geq 0}$ the filtration generated by the fractional motion Z: $F_t = \sigma(Z_s^H : s \leq t)$. For any process X denote by X^* the supremum process: $X_t^* = \sup_{s\leq t} |X_s|$.

Fractional Brownian motion is a self-similar process: $Z_{at} \stackrel{d}{=} a^H Z_t$, where $a > 0$ is a constant and the notation $X \stackrel{d}{=} Y$ means that the random variable X has the same distribution as the random variable Y. Even more is true: from the self-similarity it follows for the supremum process Z^* that $Z_{at}^* \stackrel{d}{=} a^H Z_t^*$. For any $p > 0$ we have then the following result using self-similarity:

Theorem 1.1 *Let $T > 0$ be a constant and Z a fractional Brownian motion with Hurst index H. Then*

$$E\left(Z_T^*\right)^p = K(p,H)T^{pH}, \tag{1}$$

where $K(p,H) = E\left(Z_1^*\right)^p$.

The value of the constant $K(p,H)$ is not known to us. The following generalizes the above for fractional Brownian motions indexed with stopping times. Note that in the results below we do not have any upper bound for the case $H < 1/2$, but we conjecture that we have a similar result for the upper bound as for the other cases, and the proof is similar to the proof of the lower bound for the case $H > 1/2$.

Theorem 1.2 *Let τ be a stopping time with respect to the filtration $\mathbf{F}$. Then for any $p > 0$ and $H \in (1/2, 1)$ we have that*

$$c(p,H)E\left(\tau^{pH}\right) \leq E\left((Z_\tau^*)^p\right) \leq C(p,H)E\left(\tau^{pH}\right), \tag{2}$$

and for any $p > 0$ and $H \in (0, 1/2)$ we have that

$$c(p,H)E\left(\tau^{pH}\right) \leq E\left((Z_\tau^*)^p\right), \tag{3}$$

where the constants $c(p,H), C(p,H) > 0$ depend only from parameters p, H.

3. (*Discussion and comments*) Recall the classical Burkholder-Davis-Gundy (B-D-G) inequalities for the standard Brownian motion: for any stopping time τ with respect to the filtration generated by the Brownian motion B and $p > 0$ we have:

$$c(p)E\tau^{\frac{p}{2}} \leq E\left((B_\tau^*)^p\right) \leq C(p)E\tau^{\frac{p}{2}}. \tag{4}$$

The inequalities in (2) generalize the B-D-G inequalities to fractional Brownian motions. B-D-G have a long history and we cite only some works in this area. Maybe the first general results were due to Novikov ($p > 1/2$) and Burkholder (see Burkholder (1973) and Novikov (1971)).

It is well known that B-D-G inequalities generalize to arbitrary martingales M: one replaces τ in (4) by the stopped angle bracket process $[M, M]_\tau$. For a discussion and more references see Dellacherie and Meyer (1982), Chapter VII, or Liptser and Shiryaev (1986), Chapter I.

With an arbitrary martingale M there are also predictable moment bounds: Novikov has given upper bounds for purely discontinuous martingales in Novikov (1975) with $p > 1$, Dzhaparidze and Valkeila have given predictable upper and lower bounds for $p > 2$ for locally square integrable martingale Dzhaparidze and Valkeila (1990) and Lebedev has given predictable upper bounds for an arbitrary vector valued locally square integrable martingale M for $p > 1$ Lebedev (1996).

It is worthwhile to note that the process Z is not a semimartingale, except when it is the standard Brownian motion. For this see Liptser and Shiryaev (1986), p. 232, for the case $H > 1/2$.

2 Auxiliary results

1. (*Integral representations*) We give some recently obtained integral representation between a fractional Brownian motion Z and the so-called fundamental martingale M. The martingale M might also be called Molchans martingale (see Molchan and Golosov (1969)).

The integrals below can be defined by integration by parts, where the singularities of the kernels do not cause problems, due to the Hölder continuity of the fractional Brownian motion Z (see Norros et al. (1997), Lemma 2.1).

Put $w(t,s) \doteq \frac{c}{C}s^{-\alpha}(t-s)^{-\alpha}$ for $s \in (0,t)$ and $w(t,s) = 0$ for $s > t$, where $\alpha \doteq H - \frac{1}{2}$,

$$C \doteq \sqrt{\frac{H}{\left(H-\frac{1}{2}\right)B(H-\frac{1}{2}, 2-2H)}}$$

and

$$c \doteq \frac{1}{B(H+\frac{1}{2}, \frac{3}{2}-H)},$$

where the beta coefficient $B(\mu, \nu)$ for $\mu, \nu > 0$ is defined by

$$B(\mu,\nu) \doteq \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

Proposition 2.1 *For $H \in (0,1)$ define M by*

$$M_t = \int_0^t w(t,s)dZ_s. \tag{5}$$

Then M is a Gaussian martingale, $\mathbf{F}^M = \mathbf{F}$ with variance $\langle M \rangle_t = \frac{C^2}{4H^2(2-2H)} t^{2-2H}$.

For the proof we refer to Norros et al. (1997), Proposition 2.1.

For later use denote by c_2 the constant

$$c_2 \doteq \sqrt{\frac{C^2}{4H^2(2-2H)}}.$$

Put $Y_t \doteq \int_0^t s^{\frac{1}{2}-H} dZ_s$. Then $Z_s = \int_0^t s^{H-\frac{1}{2}} dY_s$. Moreover, by definition

$$M_t = \frac{c}{C} \int_0^t (t-s)^{\frac{1}{2}-H} dY_s,$$

and by Norros et al. (1997), Theorem 3.2,

$$Y_t = 2H \int_0^t (t-s)^{H-\frac{1}{2}} dM_s. \tag{6}$$

2. *Martingale inequalities* Let N be a continuous martingale. We have the following from the B-D-G inequalities:

Proposition 2.2 *For any $p > 0$ and stopping time τ there exists constants $c_p, C_p > 0$ such that*

$$c_p E \langle N, N \rangle_\tau^{p/2} \le E\left((N_\tau^*)^p\right) \le C_p E \langle N, N \rangle_\tau^{p/2}. \tag{7}$$

Corollary 2.1 *For any $p > 0$ and stopping time τ there exists constants $c_p, C_p > 0$ such that*

$$c_p c_2^p E \tau^{p(1-H)} \le E((M_\tau^*)^p) \le C_p c_2^p E \tau^{p(1-H)}. \tag{8}$$

Proof The angle bracket of the martingale M is $\langle M \rangle_t = c_2^2 t^{2-2H}$. Apply now (7) with $\langle M \rangle$ and stopping time τ to obtain (8) .

3 Proofs, $H \ge 1/2$

1. (*An upper bound for $H \ge 1/2$*) We have that $Z_t = \int_0^t s^{H-\frac{1}{2}} dY_s$. Use integration by parts to get the upper estimate for Z^*:

$$Z_t^* \le 2t^\alpha Y_t^*.$$

For the process Y use the representation (6) to get the estimate

$$Y_t^* \leq 4Ht^\alpha M_t^*. \tag{9}$$

>From these two upper bounds we derive the following upper bound

$$Z_t \leq 8HM_t^* t^{2\alpha}. \tag{10}$$

Note that (10) is valid for any stopping time τ:

$$Z_\tau^* \leq 8HM_\tau^* \tau^{2\alpha}.$$

Hence for any $p > 0$ we have that

$$E\left(Z_\tau^*\right)^p \leq (8H)^p\, E\left(\tau^{2\alpha p}\left(M_\tau^*\right)^p\right). \tag{11}$$

With Hölder's inequality, (11) with $q = \frac{H}{2\alpha} = \frac{H}{2H-1} > 1$, and $r = \frac{H}{1-H}$ we get

$$E\left(\tau^{2\alpha p}\left(M_\tau^*\right)^p\right) \leq \left(E\tau^{2\alpha q p}\right)^{\frac{1}{q}}\left(E\left(M_\tau^*\right)^{pr}\right)^{\frac{1}{r}}. \tag{12}$$

Apply now (8) to obtain

$$E\left(M_\tau^*\right)^{pr} \leq c_2^p C_p E\tau^{\frac{(1-2\alpha)pr}{2}} = c_2^p C_p E\tau^{pH}. \tag{13}$$

To finish, note that the right hand side inequality in (2) for $H > 1/2$ follows from (12), and (13) with constant $C \doteq C(p,H) = (8H)^p\, c_2^p C_p$ and C_p is the constant in the Burkholder-Davis-Gundy inequality.

2. (*A lower bound for* $H > 1/2$) Pick the constants $a, b \in (0,1)$ with $a < b$. Then, using Proposition 2.1 we obtain that

$$M_t = \int_0^{at} w(t,s)dZ_s + \int_{at}^{bt} w(t,s)dZ_s + \int_{bt}^{t} w(t,s)dZ_s. \tag{14}$$

Integrating by parts one can estimate the middle term in (14):

$$\begin{aligned} \left|\int_{at}^{bt} w(t,s)dZ_s\right| &= \frac{c}{C}\Big| Z_{bt}t^{-2\alpha}b^{-\alpha}(1-b)^{-\alpha} - Z_{at}t^{-2\alpha}a^{-\alpha}(1-a)^{-\alpha} + \\ &\qquad \int_{at}^{bt} Z_s d\left((t-s)^{-\alpha}s^{-\alpha}\right)\Big| \\ &\leq C(a,b,H)t^{-2\alpha}Z_t^*, \end{aligned}$$

where t is fixed and

$$C(a,b,H) \doteq 2\frac{c}{C}\left(b^{-\alpha}\left(1-b\right)^{-\alpha} + a^{-\alpha}\left(1-a\right)^{-\alpha}\right).$$

Put $\xi_t \doteq t^{2\alpha}|M_t|$ and from the above we can conclude that

$$\xi_t \leq t^{2\alpha}|M_t(a) + M_t(1-b)| + C(a,b,H)Z_t^*, \tag{15}$$

where $M_t(a) \doteq \int_0^{at} w(t,s)dZ_s$ and $M_t(1-b) \doteq \int_{bt}^t w(t,s)dZ_s$.

Lemma 3.1 *With the above notation and for any $p > 0$ and stopping time τ*

$$E\left(\xi_\tau^*\right)^p \geq z_p E\tau^{pH}, \tag{16}$$

Proof First we prove (16) for the case $p = 2$. Integration by parts gives

$$\xi_t^2 = \int_0^t s^{4\alpha} dM_s^2 + \int_0^t M_s^2 ds^{4\alpha}$$

i.e.

$$\xi_t^2 = \int_0^t \left(c_2^2 \left(1 - 2\alpha\right) s^{2\alpha} + 4\alpha s^{4\alpha - 1} M_s^2\right) ds + 2\int_0^t s^{4\alpha} M_s dM_s. \tag{17}$$

Hence for bounded stopping times τ

$$E\left(\xi_\tau^*\right)^2 \geq c_2^2 \frac{1 - 2\alpha}{1 + 2\alpha} E\tau^{2H}. \tag{18}$$

For general τ apply first (18) to the stopping time $\tau \wedge n$. Use now Fatou's lemma to obtain (16) for arbitrary τ with $p = 2$.

Consider now the case $p < 2$. Note first that the inequality (16) with $p = 2$ means that the predictable process $\left(\xi^*\right)^2$ dominates the process I^{2H}, where $I_t = t$. Using the domination results due to Lenglart we obtain for $p < 1$

$$E\left(\xi_\tau^*\right)^p \geq z_p E\tau^{pH}$$

with

$$z_p \doteq \frac{\left(c_2^2 \frac{1-2\alpha}{1+2\alpha}\right)^p (4 - p)}{2 - p} \tag{19}$$

(see Dellacherie and Meyer (1982), VI.113, for the Lenglart results).

To finish the proof of the lemma consider the case $p > 2$. We modify the approach developed in Novikov (1971). Take $k > 0, \delta > 0$ and define the process η by

$$\eta_t \doteq \delta + kt^{2H} + \xi_t^2.$$

>From (17) we obtain that

$$\langle \eta \rangle_t = 4\frac{c^4}{C^4} \int_0^t M_s^2 s^{6\alpha} ds;$$

use this, (17) and the Ito formula to have

$$\begin{aligned} \eta_t^{\frac{p}{2}} &= \delta^{\frac{p}{2}} + \int_0^t \{\frac{p}{2} \eta_s^{\frac{p}{2}-1} [\left(c_2^2 \frac{2 - 2H}{2H} + k2H\right) s^{2H-1} + 4\alpha s^{4H-3} M_s^2] + \\ &+ \frac{p(p-2)}{2} \eta_s^{\frac{p}{2}-2} s^{6\alpha} M_s^2\} ds + N_t, \end{aligned} \tag{20}$$

where N is a continuous martingale, $N_0 = 0$. >From (20) we obtain for any bounded stopping time τ

$$\begin{aligned} E\left(\eta_\tau^{\frac{p}{2}}\right) &\geq \delta^{\frac{p}{2}} + E\int_0^\tau \frac{p}{2}\eta_s^{\frac{p}{2}-1}(c_2^2\frac{2-2H}{2H} + 2kH)s^{2H-1}ds \\ &\geq \delta^{\frac{p}{2}} + E\int_0^\tau \frac{p}{2}\left(ks^{2H}\right)^{\frac{p}{2}-1}(1+2kH)s^{2H-1}ds \\ &= \delta^{\frac{p}{2}} + \frac{k^{\frac{p}{2}-1}\left(c_2^2\frac{2-2H}{2H} + 2kH\right)}{H}E\tau^{pH}. \end{aligned} \tag{21}$$

Use again Fatou's lemma to conclude that (21) holds for arbitrary stopping time τ and for $\delta = 0$. So we have

$$E\left(\xi_\tau^2 + k\tau^{2H}\right)^{\frac{p}{2}} \geq \frac{k^{\frac{p}{2}-1}\left(c_2^2\frac{2-2H}{2H} + 2kH\right)}{H}E\tau^{pH}. \tag{22}$$

Use the inequality

$$\left(\xi_\tau^2 + k\tau^{2H}\right)^{\frac{p}{2}} \leq 2^{\frac{p}{2}-1}\left(\xi_\tau^p + k^{\frac{p}{2}}\tau^{pH}\right)$$

in (22) to obtain

$$E\xi_\tau^p \geq z_p(k)E\tau^{pH}$$

with

$$z_p(k) \doteq k^{\frac{p}{2}}\left(\frac{2^{-\frac{p}{2}+1}\left(\frac{c_2^2(2-2H)}{k2H} + 2H\right)}{H} - 1\right).$$

We finish the proof of Lemma 3.1 by noting that $z_p(k) > 0$ for small enough k.

To continue with the proof of the lower bound note that using (15) and (16) we have the following inequality

$$C_p z_p E\tau^{pH} \leq \left(C\left(a,b,H\right)\right)^p E\left(Z_\tau^*\right)^p + E\left(\tau^{2\alpha}|M_\tau(a) + M_\tau(1-b)|\right)^p. \tag{23}$$

Next we show that any stopping time τ

$$|M_\tau(a)| \leq 8H\left(\frac{a}{1-a}\right)^{H-1/2} M_\tau^* \tag{24}$$

Note first that integration by parts gives

$$M_t(a) = \left(t\left(1-a\right)\right)^{1/2-H} Y_{at} - \int_0^{at} Y_s d\left(t-s\right)^{1/2-H}.$$

Hence

$$|M_t(a)| \leq 2\left(t\left(1-a\right)\right)^{1/2-H} Y_{at}^*;$$

the estimate (9) now gives the inequality $|M_t(a)| \leq 8H\left(\frac{a}{1-a}\right)^{H-1/2} M_{at}^*$ and since this holds for all t, we have shown that (24) is true.

Next we prove that

$$|M_\tau(1-b)| \le 8H\left(\frac{1-b}{b}\right)^{H-1/2} M_\tau^*. \tag{25}$$

Note first that with fixed t the process $\tilde{Z}_s \doteq Z_t - Z_{t-s}$ is a fractional Brownian motion with Hurst index H, $s \in (0,t)$. Then, introducing the substitution $u = t - s$ we have that

$$M_t(1-b) = \int_{tb}^{t}(t-s)^{1/2-H}s^{1/2-H}dZ_s = \int_0^{(1-b)} u^{1/2-H}(t-u)^{1/2-H}d\tilde{Z}_u.$$

Hence, similarly to above we obtain

$$|M_t(1-b)| \le 8H\left(\frac{1-b}{b}\right)^{H-1/2}\tilde{M}_t^*,$$

where $\tilde{M}$ is obtained from $\tilde{Z}$ using (5). It remains to note that for each t $\tilde{M}_t^* = M_t^*$ and we have (25).

Similarly to (11) and (12) we get, using also (24) that for any stopping time τ

$$E\left(\tau^{2H-1}M_\tau(a)\right)^p \le \left(\frac{a}{1-a}\right)^{p(H-1/2)}(8H)^pC(H,p)E\tau^{pH} \tag{26}$$

and with (25)

$$E\left(\tau^{2H-1}M_\tau(1-b)\right)^p \le \left(\frac{1-b}{b}\right)^{p(H-1/2)}(8H)^pC(H,p)E\tau^{pH}. \tag{27}$$

Use (26) and (27) in (23) to obtain

$$E\left(Z_\tau^*\right)^p \ge \frac{[c_pz_p - 2^{\frac{p}{2}-1}\left(\left(\frac{a}{1-a}\right)^{p(H-1/2)} + \left(\frac{1-b}{b}\right)^{p(H-1/2)}\right)]}{\left(C\left(a,b,H\right)\right)^p}E\tau^{pH}. \tag{28}$$

It is clear that by taking small enough a and by taking b close to one the coefficient on the right hand side of (28) is strictly positive.

4 Proofs, $H < 1/2$

1. (*A lower bound for* $H < \frac{1}{2}$) We use the representation (5)

$$M_t = \frac{c}{C}\int_0^t s^{-\alpha}\left(t-s\right)^{-\alpha}dZ_s.$$

Integration by parts gives the estimate

$$M_t^* \le \frac{c}{C}B(1/2-H, 3/2-H)Z_t^*2t^{1-2H}. \tag{29}$$

Hence for any stopping time τ and any $p>0$ we have from (29)

$$(M_\tau^*)^{\frac{pH}{1-H}} \le \left(\frac{c}{C}2B(1/2-H,3/2-H)\right)^{\frac{pH}{1-H}} (Z_\tau^*)^{\frac{pH}{1-H}}\tau^{\frac{(1-2H)pH}{1-H}}. \tag{30}$$

By Proposition 2.2 we have

$$E\left(M_\tau^*\right)^{\frac{pH}{1-H}} \ge c_{p,H}E\tau^{pH}. \tag{31}$$

To finish, apply Hölder's inequality with $q=\frac{1-H}{H}$ and $r=\frac{1-H}{1-2H}$ to the right hand side of (30) to get

$$E\left(Z_\tau^*\right)^p \ge c(p,H)E\tau^{pH}$$

with

$$c(p,H)=c_{p,H}^{\frac{1-H}{H}}2^{-p}\left(B\left(1/2-H,3/2-H\right)\right)^{-p}\frac{C^p}{c^p}.$$

Acknowledgement

The authors are greateful to an anonumous referee for pointing out an error in the constants of Lemma 3.1.

References

Burkholder, D. L. (1973), Distribution function inequalities for martingales, *Annals of Probability*, **1**,19-42.

Decreusefond, L. and Üstünel, A. S. (1997), Stochastic analysis of the fractional Brownian motion, *Potential Analysis*, to appear.

Dellacherie, C. and Meyer, P.-A. (1982), *Probabilities and Potential B*, (North-Holland, Amsterdam).

Dzhaparidze, K. and Valkeila, E. (1990), On the Hellinger Type Distances for Filtered Experiments, *Probability Theory and Related Fields* **85**, 105-117.

Lebedev, V.A. (1996), *Martingales, convergence of probability measures and stochastic equations*, in Russian, (MAI, Moscow).

Liptser, R.S. and Shiryaev, A.N. (1986) *Theory of martingales*, in Russian, (Nauka, Moscow).

Molchan, G. and Golosov, J. (1969), Gaussian stationary processes with asymptotic power spectrum. *Soviet Math. Dokl.* **10**, 134-137.

Norros, I., Valkeila, E. and Virtamo, J. (1997), An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, *Bernoulli*, to appear.

Novikov, A.A. (1971), On moment inequalities for stochastic integrals, *Theory of Probability and its Applications* **16**, 538-541.

Novikov, A.A. (1975), On discontinuous martingales, *Theory of Probability and its Applications* **20**, 11-26.