ON BELINSKII CONFORMALITY IN A PRESCRIBED SET OF POINTS

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Abstract

The local behavior of plane quasiconformal mappings is investigated. In particular, generalizing the well-known Reich–Walczak problem, we study the possibility for a quasiconformal mapping to be conformal in the sense of Belinskii at a prescribed point or in a prescribed set of points when the modulus of the complex dilatation is a fixed measurable function. The case of arbitrary countable sets is considered.

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1 Introduction

By one of the analytic definitions of quasiconformality, cf., e.g., [LV], p. 176, [A], p. 24, 33, a sense-preserving homeomorphism of the complex plane $f:\mathbb{C}\to\mathbb{C}$, of the class $W^1_{1,loc}$, is called Q–quasiconformal, $Q\in[1,\infty)$, if it satisfies the Beltrami equation

$$f_{\overline{z}}=\mu(z)f_z \qquad \text{a.e.} \tag{1.1}$$

where, as usual, $f_{\overline{z}}=(f_x+if_y)/2$, $f_z=(f_x-if_y)/2$, $z=x+iy$, and $\mu:\mathbb{C}\to\mathbb{C}$ is a measurable function such that

$$p(z)=\frac{1+|\mu(z)|}{1-|\mu(z)|}\le Q \qquad \text{a.e.} \tag{1.2}$$

The quantities $\mu(z)$ and $p(z)$ are called the complex dilatation and the dilatation of the mapping f at the point z, respectively. Conversely, given $Q\in[1,\infty)$ and a measurable function μ of a subdomain D of the plane satisfying the inequality (1.2), there exists a Q-quasiconformal homeomorphism of D whose complex dilatation is equal to μ a.e. [LV, p. 204].

In the direction of investigations of differential and closely connected geometric properties of quasiconformal mappings, a series of fundamental results has been obtained. O. Teichmüller, H. Wittich, P.P. Belinskii and O. Lehto have established that for a quasiconformal mapping f, the integral condition

$$\int\limits_{|z-z_0|<\delta} \frac{|\mu(z)|}{|z-z_0|^2} dxdy < \infty \ , \tag{1.3}$$

for some $\delta > 0$, implies the conformality of f at the point z_0, see, e.g., [T], [W], [B$_1$], [B$_2$], [L] and [LV].

In [RW], it was conjectured that, for whatever the fixed dilatation $p(z)$, it is possible to choose the argument $\arg \mu(z)$ of the complex dilatation in such a way that the corresponding quasiconformal mappings be conformal at an arbitrary prescribed point $z_0 \in \mathbb{C}$. In other words, if $K(z) : \mathbb{C} \to [1, Q]$ is an arbitrary measurable function, then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the local dilatation

$$p(z) = K(z) \qquad \text{a.e. in } \ \mathbb{C} \tag{1.4}$$

such that f is conformal at the point z_0.

In [RW], a particular solution of the problem has been given, too. More precisely, E. Reich and H. Walczak have shown that, for every measurable function $\varphi(r) : (0,1) \to [1, Q]$, there is a quasiconformal mapping f of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself that has the dilatation

$$p(z) = \varphi(|z|) \qquad \text{a.e. in } \ \mathbb{D} \tag{1.5}$$

and that is conformal at the origin, see also [LV], p. 248–249.

We consider similar problems with respect to the so–called conformality in the sense of Belinskii at a single point as well as in a prescribed set of points. A related notion of the so–called asymptotic rotation is studied by J.A. Jenkins and M. Brakalova [BJ$_1$], [BJ$_2$]. Both notions inherit the main geometrical properties of the usual conformality.

2 On conformality in the sense of Belinskii

A mapping f is called conformal at a point z_0, if there is a nonzero derivative of f in the sense of the complex analysis, $f'(z_0) \neq 0$. In other words, f has a total differential at z_0,

$$\Delta f = f'(z_0)\Delta z + o(|\Delta z|) \tag{2.1}$$

with a nonsingular linear part. The main geometric properties of a mapping f which is conformal at a point are the preservation of the angles, infinitesimal disks and the moduli of infinitesimal rings centered at the point.

As the following example of Shabat [Sh]

$$w = z(1 - \log|z|), \quad |z| < \delta < 1 \tag{2.2}$$

shows, the mapping $w = f(z), f(0) = 0$, may be nondifferentiable in the usual sense if the complex dilatation $\mu(z)$ is continuous with

$$\mu(z) \to 0 \quad \text{as } z \to 0 \ . \tag{2.3}$$

However, if $\mu(z)$ is continuous at a point z_0, then, as was first observed by Belinskii, see [B$_1$], p. 41, the mapping $w = f(z)$ is differentiable in the sense that

$$\Delta w = A(\rho)(\Delta z + \mu_0 \Delta \overline{z} + o(\rho)), \tag{2.4}$$

where $\mu_0 = \mu(z_0)$ and $A(\rho)$ depends on $\rho = |\Delta z + \mu_0 \Delta \overline{z}|$, while $o(\rho)/\rho \to 0$ as $\rho \to 0$. Here we must stress that in this case, as opposed to the usual differentiability, $A(\rho)$ may fail to have a limit value as $\rho \to 0$ but

$$\lim_{\rho \to 0} \frac{A(t\rho)}{A(\rho)} = 1 \tag{2.5}$$

for each fixed $t > 0$, see [R$_1$], p. 290.

The differentiability in the sense (2.4) together with the additional condition (2.5) is called *differentiability in the sense of Belinskii.* Also, μ_0 in the relation (2.4) will not necessarily be equal to $\mu(z_0)$ if $\mu(z)$ is not continuous at the point z_0. If $\mu_0 = 0$, the mapping f is said to be *conformal in the sense of Belinskii* at z_0. Thus, the conformality in the sense of Belinskii means that at the point z_0

$$\Delta w = A(\rho)(\Delta z + o(\rho)), \tag{2.6}$$

where $\rho = |\Delta z|$ and (2.5) holds.

In [R$_1$], see also [GR], a few criteria of conformality in the sense of Belinskii were found. One of these criteria, a necessary and sufficient condition, is *(complex) asymptotic homogeneity* of a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, $f(0) = 0$, at the origin, i.e.

$$\lim_{z \to 0} \frac{f(z\zeta)}{f(z)} = \zeta \tag{2.7}$$

for each fixed $\zeta \in \mathbb{C}$. Moreover, the pointwise limit relation (2.7) is equivalent to one which is uniform in each compact subset of $\mathbb{C}$ with respect to the parameter ζ.

Simultaneously, conformality in the sense of Belinskii is equivalent to *(real) asymptotic homogeneity* of a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, $f(0) = 0$, at the origin, i.e.

$$\lim_{z \to 0} \frac{f(zt)}{f(z)} = t \tag{2.8}$$

for each fixed $t > 0$. Sometimes, we write the relations (2.7) and (2.8) in the form

$$(2.9)\quad f(z\zeta) \sim \zeta f(z), \qquad f(zt) \sim t f(z) \qquad \text{as} \qquad z \to 0\ .$$

Another equivalent condition is the limit relation

$$(2.10)\qquad \lim_{z',z\to 0} \left\{ \frac{f(z')}{f(z)} - \frac{z'}{z} \right\} = 0$$

under z', $z \in \mathbb{C}$, $|z'| < \delta |z|$, for each fixed $\delta > 0$.

In particular, under $|z'| = |z|$ we obtain from (2.10) that

$$(2.11)\qquad \lim_{r\to 0} \frac{\max\limits_{|z|=r} |f(z)|}{\min\limits_{|z|=r} |f(z)|} = 1\,.$$

Geometrically, the condition (2.11) means that infinitesimal disks at 0 are transformed by f into infinitesimal disks.

However, the condition (2.7) of asymptotic homogeneity is much stronger than the condition (2.11). From (2.7) we obtain also *asymptotic preservation of angles* between rays coming from the origin in the direction of the corresponding points,

$$(2.12)\qquad \lim_{z\to 0} \{\arg f(z\zeta) - \arg f(z)\} = \arg \zeta\ ,$$

and *preservation of the moduli of infinitesimal rings*,

$$(2.13)\qquad \lim_{z\to 0} \frac{|f(z\zeta)|}{|f(z)|} = |\zeta|\ ,$$

for each fixed $\zeta \in \mathbb{C} \setminus \{0\}$.

Note also that the pointwise limit relations (2.12) and (2.13) are equivalent to the uniform limit relations (2.12) and (2.13) with respect to the parameter $\zeta \in \mathbb{C}, c^{-1} \le |\zeta| \le c$, and, in particular, in the unit circle $|\zeta| = 1$.

2.14. Remark. These two properties (2.12) and (2.13) are characteristic for conformality in the sense of Belinskii. Thus, the main geometric properties of usual conformality are inherited by the Belinskii conformality. Moreover, the initial definition (2.6) may be considered as a generalized differentiability.

The following simple sufficient condition for differentiability in the sense of Belinskii may be formulated in terms of the complex dilatation, see [R_1]. If $\mu(z)$ is approximately continuous at a point z_0, then f is differentiable in the sense of Belinskii at the point. In particular, if $\mu(z_0) = 0$, then f is conformal in the sense of Belinskii at the point.

Recall that $\mu(z)$ is *approximately continuous at a point* z_0, if there is a measurable set E such that $\mu(z) \to \mu(z_0)$ as $z \to z_0$ along the set E and z_0 is *a density point of the set* E, i.e.

$$\lim_{\varepsilon \to 0} \frac{\operatorname{meas} E \cap D(z_0, \varepsilon)}{\operatorname{meas} D(z_0, \varepsilon)} = 1 \ , \tag{2.15}$$

where $D(z_0, \varepsilon)$ is the open disk with the center z_0 and the radius ε. Because $|\mu(z)| \leq 1$, i.e. $\mu \in L^\infty$, it is clear that the points of approximate continuity coincide with *Lebesgue points* z_0 of the function μ :

$$\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int\limits_{|z - z_0| < \varepsilon} |\mu(z) - \mu(z_0)| dxdy = 0 \ . \tag{2.16}$$

The condition of approximate continuity is advantageously distinguished from the condition (1.3) by the circumstance that, for an arbitrary measurable function, almost all points are its points of approximate continuity, see, e.g., [S].

2.17. Remark. Another more restrictive notion that is close to conformality in the sense of Belinskii was defined by M. Brakalova and J.A. Jenkins [BJ_1], [BJ_2]. They say that $f(z)$ is *asymptotically a rotation on circles as* $z \to 0$, if, firstly,

$$|f(z)| \sim A|z| \quad \text{as} \ \ z \to 0, \ \ A > 0 \tag{2.18}$$

and, secondly, for an appropriate choice of the arguments,

$$\arg f(re^{i\vartheta_2}) - \arg f(re^{i\vartheta_1}) - (\vartheta_2 - \vartheta_1) \tag{2.19}$$

tends to zero uniformly in ϑ_1 and ϑ_2 as r tends to zero. Thus, the main difference of conformality in the sense of Belinskii from the asymptotic rotations is in more general relation (2.5) instead (2.18) with fixed $A > 0$.

The positive solution to the Reich–Walczak problem in the case of a single point with respect to conformality in the sense of Belinskii has been given in [R_2].

2.20. Theorem. *Let $K(z) : \mathbb{C} \to [1, Q]$ be an arbitrary measurable function. Then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the local dilatation $p(z) = K(z)$ a.e. such that f is conformal in the sense of Belinskii at a prescribed point $z_0 \in \mathbb{C}$.*

In this connection, the following natural generalization of the problem has appeared.

Problem. Describe all measurable sets $E \subset \mathbb{C}$ which has the BRW property: for every measurable function $K(z) : \mathbb{C} \to [1, Q]$ there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the dilatation $p(z) = K(z)$ a.e. such that f is conformal in the sense of Belinskii at all points $z \in E$.

2.21. Remark. A necessary condition for the BRW property is the restriction that $\operatorname{meas} E = 0$. This follows because by (2.11) $K(z) = 1$ a.e. in E that is impossible simultaneously for all measurable functions $K(z) : \mathbb{C} \to [1, Q]$, if $\operatorname{meas} E > 0$.

The authors had the following three hypotheses with respect to the problem.

Conjecture A. An arbitrary countable set $E \subset \mathbb{C}$ has the BRW property.

Conjecture B. An arbitrary compact set $E \subset \mathbb{C}$ of zero measure has the (uniform) BRW property.

Conjecture C. An arbitrary set $E \subset \mathbb{C}$ of zero measure has the BRW property.

We prove the first hypothesis in the following section.

3 The case of countable sets

In this section we show that the generalized Reich–Walczak problem with respect to the Belinskii conformality has always a positive solution for all countable sets. More precisely, we prove the following statement.

3.1. Theorem. *Let $K(z) : \mathbb{C} \to [1, Q]$ be a measurable function and let $S = \{z_n\}_{n=1}^{\infty}$ be a countable subset of $\mathbb{C}$. Then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the local dilatation $p(z) = K(z)$ a.e. such that f is conformal in the sense of Belinskii at all points $z_n, n = 1, 2, \cdots$.*

In particular, we have the following interesting consequences.

3.2. Corollary. *Let $K(z) : \mathbb{C} \to [1, Q]$ be an arbitrary measurable function. Then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the local dilatation $p(z) = K(z)$ a.e. such that f is conformal in the sense of Belinskii at all points $z \in \mathbb{C}$ with rational coordinates.*

3.3. Corollary. *Let $Q > 1$ be an arbitrary number. Then there is a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the local dilatation $p(z) = Q =$ constant a.e. such that f is conformal in the sense of Belinskii in an arbitrary prescribed countable subset of $\mathbb{C}$.*

The proof of Theorem 3.1 is based on Theorem 2.20 and two lemmas. The first of them is the following *covering lemma* and the second one, the so–called *comparison lemma*, will be formulated below.

3.4. Lemma. *Let $S = \{z_n\}_{n=1}^{\infty}$ be an arbitrary countable subset of $\mathbb{C}$. Then there is a disjoint collection of closed sets $E_n \subset \mathbb{C}$ such that $z_n \in E_n$ is a density point of E_n for each $n = 1, 2, \cdots$.*

Note that the set S may be everywhere dense in $\mathbb{C}$, for example, as above the set of all points of $\mathbb{C}$ with rational coordinates. Hence Lemma 3.4 is nontrivial. Furthermore, this lemma is precise as the following remark shows.

3.5. Remark. Let $S \subset \mathbb{C}$ be an arbitrary *uncountable* set. Then there is no disjoint collection of measurable sets $E(z)$, $z \in S$, such that $z \in E(z)$ are density points for the corresponding $E(z)$.

Indeed, if the set S is uncountable, then by the well–known Lindelöf–Young theorem there is at least one condensation point $z_0 \in S$, see, e.g., [K], p. 251. Let D_0 be the open unit disk with the center z_0. Then the set $S_0 = S \cap D_0$ is uncountable by the definition of a condensation point.

Assume now that $E(z)$ is a disjoint cover of S such that every $z \in S$ is a density point of $E(z)$. Then we have that $\operatorname{meas} E_0(z) > 0$ for all $z \in S_0$ where $E_0(z) = E(z) \cap D(z_0)$. Hence for all $n = 1, 2, \cdots$, the number K_n of all points in the set

$$S_n = \left\{ z \in S_0 : \frac{\pi}{n+1} < \operatorname{meas} E_0(z) \le \frac{\pi}{n} \right\} \tag{3.6}$$

is finite and, moreover,

$$S_0 = \bigcup_{n=1}^{\infty} S_n \ . \tag{3.7}$$

Consequently, S_0 must be countable. Thus, the above assumption leads to a contradiction.

Proof of Lemma 3.4. First of all, we prove that there is a collection of closed sets L_n, generally speaking, with a nonempty intersection such that, for each $n = 1, 2, \cdots$, the point $z_n \in L_n$ is a density point for L_n and $L_n \cap S \setminus \{z_n\} = \emptyset$.

Indeed, denote by $D(z_n, r_m)$ the open disk with the center z_n and the radius $r_m = 1/m$, $m = 0, 1, 2, \cdots$. As well–known, every countable subset of $\mathbb{C}$ has zero Lebesgue measure and hence by the regularity of the measure, see, e.g., [S], there exist open sets $\Omega_n^{(m)}, m = 0, 1, 2, \cdots$, such that

$$S \cap D(z_n, r_m) \setminus \{z_n\} \subset \Omega_n^{(m)} \subset D(z_n, r_m) \setminus \{z_n\} \tag{3.8}$$

where

$$\operatorname{meas} \Omega_n^{(m)} \leq 2^{-m} \tag{3.9}$$

and we may assume that, for each fixed $n = 1, 2, \cdots$, the sequence of the sets $\Omega_n^{(m)}, m = 0, 1, 2, \cdots$ is decreasing.

Set $L_n = \mathbb{C} \setminus \Omega_n$ where

$$\Omega_n = \bigcup_{m=0}^{\infty} \Omega_n^{(m)} \setminus \overline{D(z_n, r_{m+1})} \ . \tag{3.10}$$

We see directly by the construction that $z_n \in L_n,\ L_n \cap S \setminus \{z_n\} = \emptyset$ and L_n is closed for each $n = 1, 2, \cdots$. It remains to show that z_n is a density point for L_n. To verify this we write

$$\alpha_{n,\rho} = \frac{\operatorname{meas} D(z_n, \rho) \setminus L_n}{\operatorname{meas} D(z_n, \rho)} = \frac{\operatorname{meas} D(z_n, \rho) \cap \Omega_n}{\operatorname{meas} D(z_n, \rho)} \leq$$

$$\leq \frac{\sum\limits_{m=k}^{\infty} \operatorname{meas} \Omega_n^{(m)}}{\operatorname{meas} D(z_n, \rho)} \leq \frac{k^2}{\pi} \sum_{m=k-3}^{\infty} 2^{-m} \leq \frac{k^2}{2^{k-3}} \to 0 \qquad \text{as} \quad k \to \infty$$

where $k = 1 + [\rho^{-1}]$ and $[\rho^{-1}]$ is the integral part of the number $1/\rho$. Thus, for each fixed $n = 1, 2, \cdots$, we have the limit relation $\alpha_{n,\rho} \to 0$ as $\rho \to 0$. It just means that the point z_n is a density point of the set L_n.

Now we construct by induction mutually disjoint closed sets E_n making use of L_n. More precisely, we let $E_1 = L_1 \cap C_1$ where C_1 is a closed disk with the center z_1. If $E_1, \cdots, E_n$ have already been given we set

$$E_{n+1} = L_{n+1} \cap C_{n+1} \tag{3.11}$$

where C_{n+1} is the closed disk with the center z_{n+1} and the radius

$$\rho_{n+1} = \frac{1}{2} \min_{z \in \bigcup\limits_{k=1}^{n} E_k} |z_{n+1} - z| \ . \tag{3.12}$$

Immediately by the construction we see that E_n are closed, mutually disjoint and $z_n \in E_n$ are density points of the corresponding E_n for each fixed $n = 1, 2, \cdots$.

3.13. Remark. Note that if two quasiconformal mappings f and g have a common complex dilatation $\mu(z)$ in a neighborhood of a point z_0 then in the neighborhood $f = A \circ g$ where A is a conformal mapping. Thus, all differential properties of a quasiconformal mapping at a point are defined only by the complex dilatation of the mapping in a neighborhood of the point and do not depend on normalizations.

To recognize the Belinskii conformality we need a more precise *comparison lemma.*

3.14. Lemma. *Let f and $g : \mathbb{C} \to \mathbb{C}, f(0) = g(0) = 0$, be Q–quasiconformal mappings with the complex dilatations μ and ν such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int\limits_{|z|<\varepsilon} |\mu(z) - \nu(z)| dxdy = 0 \ . \tag{3.15}$$

Then

$$g(z) = \varphi(f(z)) \ , \quad z \in \mathbb{C} \tag{3.16}$$

where the quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ is conformal in the sense of Belinskii at the origin.

3.17. Corollary. *Under the conditions of Lemma 3.14, the mapping f is conformal in the sense of Belinskii at the origin if and only if g is.*

3.18. Corollary. *Let f and $g : \mathbb{C} \to \mathbb{C}, f(0) = g(0) = 0$, be Q–quasiconformal mappings with the complex dilatations μ and ν such that*

$$\mu(z) = \nu(z), \ \ z \in E \subset \mathbb{C}, \tag{3.19}$$

where E is an arbitrary measurable set for which 0 is a density point. Then the mapping f is conformal in the sense of Belinskii at the origin if and only if g is.

3.20. Remark. Thus, the property of a quasiconformal mapping to be conformal in the sense of Belinskii at a point $z_0 \in \mathbb{C}$ is entirely determined by the values of its complex dilatation on an arbitrary set $E \subset \mathbb{C}$ for which z_0 is a density point.

In what follows, $\mathfrak{F}_{\mathfrak{Q}}$ denotes the class of all the Q-quasiconformal self-mappings of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ normalized in the following way

$$f(0)=0, \quad f(1)=1, \quad f(\infty)=\infty \quad . \tag{3.21}$$

Proof of Lemma 3.14. First of all, the complex dilatation of the mapping $\varphi = g\circ f^{-1}$ is

$$\kappa(z) = \left\{ \frac{\mu-\nu}{1-\mu\overline{\nu}} \cdot \frac{f_w}{\overline{f_w}} \right\} \circ f^{-1}(z), \quad z\in\mathbb{C}.$$

Thus, the family of the mappings

$$\varphi(z;t) = \varphi(tz)/\varphi(t) \in \mathfrak{F}_{\mathfrak{Q}}, \ \ \mathfrak{t} > \mathfrak{o},$$

corresponds to the family of their complex dilatations

$$\kappa(z;t) = \kappa(tz) = \alpha(h(z;t);t), \ \ t>0.$$

Here

$$h(z;t) = f^{-1}(tz)/f^{-1}(t) \in \mathfrak{F}_{\mathfrak{Q}}, \ \ \mathfrak{t} > \mathfrak{o},$$

and

$$\alpha(w;t) = \left\{ \frac{\mu-\nu}{1-\mu\overline{\nu}} \cdot \frac{f_w}{\overline{f_w}} \right\} \circ (w\zeta(t)), \quad w\in\mathbb{C}$$

where $\zeta(t) = f^{-1}(t)\to 0$ as $t\to 0$.

However, making changes of variables $\varepsilon = R|\zeta(t)|$ and $z = w\zeta(t)$ in (3.15), we obtain that

$$\lim_{t\to 0} \frac{1}{\pi R^2} \int\limits_{|w|<R} |\mu(w\zeta(t)) - \nu(w\zeta(t))| du dv = 0$$

for each fixed $R>0$. Here $w = u+iv$. It follows that

$$\mu(w\zeta(t)) - \nu(w\zeta(t)) \to 0 \qquad \text{as } \ t\to 0$$

in measure in the disk $D(0,R)$, for each $R>0$, i.e. in $\mathbb{C}$. Hence we conclude that

$$\alpha(w;t)\to 0 \qquad \text{as } \ t\to 0$$

in measure in $\mathbb{C}$ because

$$\left| \left\{ \frac{1}{1-\mu\overline{\nu}} \cdot \frac{f_w}{\overline{f_w}} \right\} \circ (w\zeta(t)) \right| \le \frac{1}{1-q^2}$$

where $q = (Q-1)/(Q+1)$.

Further, since the homeomorphisms $h(z;t)\in\mathfrak{F}_{\mathfrak{Q}}, \mathfrak{t}>\mathfrak{o}$, are locally absolutely equicontinuous, see [G], also

$$\kappa(z;t)\to 0 \qquad \text{as } \ t\to 0$$

in measure in $\mathbb{C}$.

Finally, in view of the sequential compactness of the class $\mathfrak{F}_{\mathfrak{Q}}$, see, e.g., [LV], p. 176, and the well–known Bers–Bojarskii convergence theorem, see, e.g., [Be], [Bo] and [LV],

$$\varphi(z;t) = \frac{\varphi(tz)}{\varphi(t)} \to z, \qquad \text{as } t \to 0,$$

that by (2.8) is equivalent to conformality in the sense of Belinskii of the mapping φ at the origin.

We complete the section by a proof of the main result.

Proof of Theorem 3.1. First, by Theorem 2.20, for each $n = 1, 2, \cdots$ there is a quasiconformal mapping $f_n : \mathbb{C} \to \mathbb{C}$ with the dilatation $p_n(z) = K(z)$ a.e. in $\mathbb{C}$ such that f_n is conformal in the sense of Belinskii at the point z_n. Let μ_n be the complex dilatation of $f_n, n = 1, 2, \cdots$.

Then by the covering lemma there is a disjoint collection of measurable sets $E_n \subset \mathbb{C}$ such that $z_n \in E_n$ is a density point of the corresponding E_n for each $n = 1, 2, \cdots$.

Finally, by the existence theorem, see, e.g., [A], [Bo] and [LV], there is a Q–quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ with the complex dilatation

$$\mu(z) = \mu_n(z), \;\; z \in E_n \;\; ,$$

and, say,

$$\mu(z) = k(z), \;\; z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} E_n,$$

where $k(z) = (K(z)-1)/(K(z)+1)$. By the comparison lemma f is conformal in the sense of Belinskii at each point $z_n, n = 1, 2, \cdots$.

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