More on weak covering properties of weak topologies

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ABSTRACT. In the first part of the paper, we study covering properties of weak topologies of Banach spaces, especially of weak or point–wise topologies of $C(K)$–spaces (here K is a compact space), and we answer questions posed by A.V. Arkhangel'skii and R.W. Hansell as well as a question from our previous paper. Our main results are the following:

– If a Banach space admits a WLUR norm, then it is hereditarily σ–relatively metacompact in its weak topology.

– The space $C_p(\beta\omega)$ is not σ–relatively metacompact.

– The Banach space ℓ^∞ is not σ–relatively metacompact in its weak topology.

– ($\diamondsuit$) There exists a non–separable scattered compact space K such that the point–wise cellularity of $C_p(K)$ is ω.

In the second part of the paper, we introduce the class of *supmetrizable* spaces containing all Banach spaces with the weak topology as well as all $C(K)$–spaces with the pointwise topology. We establish covering properties of subspaces of supmetrizable spaces and we prove the cardinal equality $e(X) = \ell(X)$ for every such subspace. We show that all supmetrizable spaces are monolithic and angelic.

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Introduction. In this paper, we pursue further the study, which we began in [DoJuPe], of weak covering properties of Banach spaces equipped with weak topology and function spaces equipped with pointwise topology. We show that if the norm of a Banach space B is "weakly locally uniformly convex" (WLUR), then the unit sphere of B is semi–metrizable in the relative weak topology; from this we derive the result that the weak topology of a WLUR–normable Banach space is hereditarily σ–relatively metacompact. In [DoJuPe], we showed that it is consistent with ZFC that the space $C_p(\beta\omega)$ is not σ–relatively metacompact. In this paper we show that this can be strengthened to an absolute result: the spaces $C_p(\beta\omega)$ and ℓ^∞_w fail to be σ–relatively metacompact in ZFC. This result answers a question of R.W. Hansell.

In the second section of the paper, we consider "strongly ccc" spaces, i.e., spaces in which every point–finite open family is countable. We answer (consistently) some questions of A.V. Arhangel'skij [Ar_4] with an example, under the set–theoretic axiom $\diamondsuit$, of a non–metrizable scattered compact space K with $C_p(K)$ strongly ccc.

In the last section of the paper, we introduce *supmetrizable* spaces. The concept of supmetrizability makes it possible to consider simultaneously the two closely related classes of spaces, Banach spaces equipped with weak topology and function spaces equipped with pointwise topology. We show that several results established earlier for B_w–spaces and for $C_p(K)$–spaces hold for all supmetrizable spaces.

Notation and terminology. Ordinals are sets of smaller ordinals, and cardinals are "initial ordinals". We also consider ordinals as topological spaces ("ordinal spaces"): where no other topology is implied for an ordinal α, we consider α to be equipped with its natural order topology. For example, the symbol ω also denotes a discrete topological space, while the ordinal space $\omega_1 + 1$ is compact.

For a completely regular topological space X, we denote by βX the maximal (Čech–Stone) compactification of X, and we denote by X^* the remainder $\beta X \setminus X$ of X in βX. Thus $\beta\omega$ and ω^* denote the Čech–Stone compactification and the remainder of the discrete space ω.

A family $\mathcal{N}$ of subsets of a topological space X is a *network* for the topology of X provided that every open subset of X can be written as the union of some subfamily of $\mathcal{N}$. Note that a network is a base for the topology if, and only if, it consists of open sets. The *weight* $w(X)$ (the *netweight* $nw(X)$) of a topological space X is the least cardinal of a base (of a network) for the topology of X and the *density* $d(X)$ of X is the least cardinal of a dense subset of X; if τ is the topology of X, then we sometimes denote by $d_\tau(A)$ the density of a subspace A of X. Note that $d(X) \le nw(X) \le w(X)$ and, if X is metric (in particular, if X is a Banach space with norm topology), then $d(X) = w(X)$. The space X is *monolithic* [Ar$_1$] provided that the equality $nw(A) = d(A)$ holds for every subspace A of X.

If B is a Banach space, we denote by B_w the set B equipped with the weak topology of B. For a compact Hausdorff space K, $C(K)$ denotes the Banach space that one obtains by giving the "supremum norm" for the linear space consisting of all real–valued continuous functions on K. We denote by $C_w(K)$ the set $C(K)$ equipped with the weak topology of $C(K)$ and by $C_p(K)$ the set $C(K)$ equipped with the topology of pointwise convergence (i.e., the topology inherited from the space $\mathbb{R}^K$). Note that the topology of pointwise convergence is weaker than the weak topology of $C(K)$.

For a family $\mathcal{L}$ of sets and for a set A, we let $(\mathcal{L})_A = \{L \in \mathcal{L} : L \cap A \ne \emptyset\}$; if $A = \{a\}$, then we write $(\mathcal{L})_x$ in room of $(\mathcal{L})_A$. A family $\mathcal{G}$ of sets is *point–finite* (*point–countable*) provided that, for every $x \in \bigcup \mathcal{G}$, the family $(\mathcal{G})_x$ if finite (countable). A topological space is *metacompact* (*metaLindelöf*) provided that every open cover of the space has a point–finite (point–countable) open refinement.

Let $\mathcal{L}$ be a family of subsets of a topological space X. We say that $\mathcal{L}$ is a *relatively open* family in X if every $L \in \mathcal{L}$ is open in the subspace $\bigcup \mathcal{L}$ of X. Further, we say that $\mathcal{L}$ is *σ–relatively open* if $\mathcal{L}$ can be written as the union of countably many relatively open families. A topological space X is *σ–relatively metacompact* if every open cover of X has a point–finite σ–relatively open refinement. This terminology was introduced in [DoJuPe]; earlier, σ–relatively metacompact spaces have been called either "weakly θ–refinable" or "weakly submetacompact". We shall also employ later the characterization of σ–relative metacompactness by the property that every open cover has a σ–relatively discrete refinement (i.e., a refinement which can be decomposed into countably many subfamilies each of which is discrete in its union).

We refer the reader to [En], [Ar$_3$], [vM], [Ku] and [DeGoZi] for general background and for definitions of terms and concepts used below without definition.

1. σ–relative metacompactness in Banach spaces. In [DoJuPe] we showed that for every compactified tree space $\bar{T}$, the space $C_p(\bar{T})$ is hereditarily σ–metacompact while the space $C_w(\bar{T})$ is hereditarily σ–relatively metacompact. S. Gul'ko has shown in [Gu] that a space $C_p(K)$, where K is Corson compact, is hereditarily σ–metacompact. Moreover, R.W. Hansell has showed in [Ha] that if a Banach space B admits a Kadeč norm, then the space B_w is hereditarily σ–relatively metacompact; in particular, if B admits a "locally uniformly convex" (LUR) norm, then B_w is hereditarily σ–relatively metacompact. We shall now extend the last–mentioned result by showing that already the existence of an equivalent "weakly locally uniformly convex" (WLUR) norm on B suffices for B_w to be hereditarily σ–relatively metacompact.

We recall some relevant definitions. A norm $\|\cdot\|$ on a Banach space B is a *Kadeč–norm* provided that the weak topology and the norm topology coincide on the set $S_{\|\cdot\|} = \{x \in B : \|x\| = 1\}$. The norm $\|\cdot\|$ is *(weakly) locally uniformly convex* ((W)LUR) provided that whenever $x \in S_{\|\cdot\|}$ and $x_n \in S_{\|\cdot\|}$, $n \in \omega$, are such that $\lim_{n\to\infty} \|x + x_n\| = 2$, then $x_n \to x$ in the norm topology (in the weak topology) of B. It is well known (see, e.g., [DeGoZi], Proposition II.1.4) that every LUR norm is a Kadeč norm, and it is obvious that every WLUR norm $\|\cdot\|$ is *strictly convex*, i.e., satisfies the condition that $x = y$ whenever $x, y \in S_{\|\cdot\|}$ are such that $\|x + y\| = 2$.

If $\|\cdot\|$ is a Kadeč–norm on a Banach space B, then the space $(S_{\|\cdot\|})_w$, i.e., the set $S_{\|\cdot\|}$ equipped with the weak topology, is metrizable; in particular, $(S_{\|\cdot\|})_w$ is metrizable whenever the norm $\|\cdot\|$ is LUR. We do not know whether $(S_{\|\cdot\|})_w$ is metrizable for a WLUR norm $\|\cdot\|$, but we can prove the weaker result that $(S_{\|\cdot\|})_w$ is semi–metrizable. Recall that a *semi–metric* on a set H is a non–negative real–valued function d on $X \times X$ which is symmetric (d(x,y)=d(x,y) for all x, y) and satisfies the condition that $d(x, y) = 0$ if, and only if, $x = y$. A topological space X is *semi–metrizable* provided that X has a semi–metric d which is *compatible* with the topology of X, in other words, for every $x \in X$, the sets $B_d(x, r) = \{y \in X : d(y, x) < r\}$, for $r > 0$, form a neighbourhood base at x.

Remark We use the term "semi–metric" here as it is traditionally used in general topology. In functional analysis, this term often refers to another type of generalized metric (known as "pseudo–metric" in topology), i.e., a function d with all the properties of a metric except that the condition "$d(x, y) = 0$ if, and only if, $x = y$" has been weakened to "$d(x, x) = 0$".

1.1 Lemma *Let $\|\cdot\|$ be a WLUR norm of a Banach space B. Then the space $(S_{\|\cdot\|})_w$ is semi–metrizable.*

Proof. Define the function d on the set $S_{\|\cdot\|} \times S_{\|\cdot\|}$ by the formula $d(x, y) = 2 - \|x + y\|$. It is easily seen that d is a semi–metric. To show that d is compatible with the weak topology of $S_{\|\cdot\|}$, let $x \in S_{\|\cdot\|}$. Note that it follows from the definition of d and from the defining condition of the WLUR–property that, for all $x_n \in S_{\|\cdot\|}$, if $d(x_n, x) \to 0$, then $x_n \to x$ in the weak topology. As a consequence, every weak neighbourhood of x contains some set $B_d(x, r)$. Compatibility of d with the relative weak topology of $S_{\|\cdot\|}$ now follows from the observation that any set $B_d(x, r)$ is open in that topology, being the complement in $S_{\|\cdot\|}$ of the closed convex subset $\{y : \|y - (-x)\| \leq 2 - r\}$ of B. $\square$

To be able to use the preceding lemma, we need to know that the weak topology of a Banach space is hereditary σ–relatively metacompact provided that the unit sphere has this property; to prove this result, we need an auxiliary lemma. This lemma is probably part of the "folklore" of covering theory, but we shall nevertheless include the straightforward proof of it.

1.2 Lemma *Let X be a hereditarily σ–relatively metacompact space and let Y be a topological space with a countable network. Then the space $X \times Y$ is hereditarily σ–relatively metacompact.*

Proof. It suffices to show that any family consisting of open rectangles of $X \times Y$ has a σ–relatively discrete refinement. Let $\mathcal{G} = \{U_s \times V_s : s \in S\}$ be a family of open rectangles. Let $\{H_n : n \in \omega\}$ be a network for the topology of Y. For every $n \in \omega$, let $\mathcal{U}_n = \{U_s : s \in S \text{ and } H_n \subset V_s\}$; since X is hereditarily σ–relatively metacompact, the open family $\mathcal{U}_n$ has a σ–relatively discrete refinement, say $\mathcal{L}_n$. It is easy to see that the family $\{L \times H_n : n \in \omega \text{ and } L \in \mathcal{L}_n\}$ is a σ–relatively discrete refinement of the family $\mathcal{G}$. □

Next we combine the result of the above lemma with some results from [Ha] to obtain the following.

1.3 Lemma *Let B be a Banach space with unit sphere S. Assume that S is hereditarily σ–relatively metacompact in the relative weak topology. Then B is hereditarily σ–relatively metacompact in the weak topology.*

Proof. In this proof, we consider B and its subspaces topologized by the weak topology. It suffices to show that the subspace $B \setminus \{0\}$ is hereditarily σ–relatively metacompact.

As observed by Hansell in the proof of Theorem 7.2 of [Ha], the mapping $(t, x) \mapsto (t, tx)$ is a homeomorphism from the product space $(0, \infty) \times S$ onto some subspace M of the product $(0, \infty) \times (B \setminus \{0\})$; moreover, M has the property that $p(M) = B \setminus \{0\}$, where p is the projection mapping $(0, \infty) \times (B \setminus \{0\}) \to (B \setminus \{0\})$. Since S is hereditarily σ–relatively metacompact, it follows from Lemma 1.2 that the product space $(0, \infty) \times S$ has the same property; as a consequence, the subspace M of $(0, \infty) \times (B \setminus \{0\})$ is hereditarily σ–relatively metacompact. According to Lemma 7.1 of [Ha], the projection map p is "index–σ–relatively discrete", in other words, for every relatively discrete family $\mathcal{L}$ of subsets of the space $(0, \infty) \times (B \setminus \{0\})$, the family $\{p(L) : L \in \mathcal{L}\}$ has a σ–relatively discrete refinement in the space $B \setminus \{0\}$. It follows from the foregoing that every open family of subsets of $B \setminus \{0\}$ has a σ–relatively discrete refinement, in other words, that $B \setminus \{0\}$ is hereditarily σ–relatively metacompact. □

An examination of Theorem 7.2 of [Ha] and the proofs of that theorem and of the above lemma shows that the result of the lemma remains valid in the case that we have $B = C(K)$, with K compact, and we change "weak topology" to "topology of pointwise convergence" in the statement of the lemma.

A simple application of the preceding lemmas and a result from [McA] now yields the result on WLUR–renormable Banach spaces mentioned above.

1.4 Proposition *Let B be a Banach space which has an equivalent WLUR norm. Then the weak topology of B is hereditarily σ–relatively metacompact.*

Proof. Also here we consider B and its subspaces topologized by the weak topology. We denote by S the unit sphere associated with a WLUR norm of B. By Lemma 1.1, the space S is semi–metrizable. It follows, by a result of L.F. McAuley [McA], that S is hereditarily σ–relatively metacompact. By Lemma 1.3, B is hereditarily σ–relatively metacompact. $\square$

In the remaining part of this section, we show that the weak topology of the Banach space ℓ^∞ fails to be σ–relatively metacompact, and we consider some consequences of this failure.

1.5 Example The Banach space ℓ^∞ is not σ–relatively metacompact in its weak topology.

Proof. Since ℓ^∞ is linearly isometric with the Banach space $C(\beta\omega)$, it suffices to show that the latter space fails to be σ–relatively metacompact in its weak topology; this in turn follows once we show that $C_w(\beta\omega)$ has a closed subspace X which fails to be σ–relatively metacompact.

For every $B \subset \omega$, denote by $\bar{B}$ the closure of B in $\beta\omega$ and let $B^* = \bar{B} \setminus B$. Fix any sequence $\{A_\alpha : \alpha < \omega_1\}$ of subsets of ω which is srictly increasing in the $\subseteq$ (mod finite) partial order. Let $Z = \bigcup_{\alpha<\omega_1} \bar{A}_\alpha$ (we may assume that each n is in some A_α so $\omega \subset Z$) and let $K = \beta\omega \setminus Z$.

Let

$$X = \{f \in C(\beta\omega, \{0,1\}) : f(p) = 0 \text{ for every } p \in K\}$$

and note that X is closed in $C_w(\beta\omega)$. To show that the subspace X fails to be σ–relatively metacompact, we have to describe its topology. It is well known that the weak topology of $C(\beta\omega)$, and hence also the relative weak topology of X, can be described in terms of positive Radon measures of the space $\beta\omega$ as follows. Denote by $\mathcal{M}$ the collection of all positive Radon measures on $\beta\omega$. A subbase for the neighbourhoods of $f \in X$ in the relative weak topology is formed by the sets

$$[f, \mu, \varepsilon] = \{g \in X : \int_{\beta\omega} |g - f| \, d\mu < \varepsilon\} ,$$

where $\mu \in \mathcal{M}$ and $\varepsilon > 0$. We shall use the shorter notation $[f, \mu]$ for the set $[f, \mu, 1]$. Note that, for all $\mu \in \mathcal{M}$ and $\varepsilon > 0$, we have $[f, \mu, \varepsilon] = [f, \frac{1}{\varepsilon} \cdot \mu]$; as a consequence, the sets $[f, \mu]$, for $\mu \in \mathcal{M}$, form a subbase for the neighbourhoods of f in the relative weak topology. Actually, these sets form a *base* of neighborhoods, because we have $[f, \mu + \nu] \subset [f, \mu] \cap [f, \nu]$ for all $\mu, \nu \in \mathcal{M}$.

To simplify our argument, we shall reduce the number of measures that are needed to define the relative weak topology of X. For every $\alpha \in \omega_1$, let $D_\alpha = A^*_\alpha \setminus \bigcup_{\beta<\alpha} A^*_\beta$. Note that we have $Z = \omega \cup \bigcup_{\alpha<\omega_1} D_\alpha$. We call a set $L \subset \beta\omega$ *finitary* provided that L is a Borel set, $L \subset Z$ and the sets $L \cap \omega$ and $\{\alpha \in \omega_1 : D_\alpha \cap L \neq \emptyset\}$ are finite. Recall that a measure $\mu \in \mathcal{M}$ is said to be *supported* by a Borel set $E \subset \beta\omega$ provided that $\mu(\beta\omega \setminus E) = 0$.

Claim. To define bases for the neighbourhoods of points of X in the relative weak topology, we only need to consider measures $\nu \in \mathcal{M}$ supported by finitary sets.

Proof of Claim. Let $f \in X$ and $\mu \in \mathcal{M}$. For every Borel set $E \subset \beta\omega$, we denote by μ_E the "restriction" of μ to the set E, which is defined by the condition $\mu_E(H) = \mu(H \cap E)$; it is easy to see that we have $\mu_E \in \mathcal{M}$. To establish the Claim, it suffices to show that there exists a finitary set L such that we have $[f, 2 \cdot \mu_L] \subset [f, \mu]$. First we note that, for every $h \in X$, since $h(p) = 0$ for every $p \in K$, we have that $\int_{\beta\omega} h\,d\mu = \int_Z h\,d\mu$. Next we observe that it follows from the regularity of μ that there exists $\alpha_0 < \omega_1$ such that we have $\mu(A^*_{\alpha_0}) = \mu(\bigcup_{\alpha<\omega_1} A^*_\alpha)$. Let $B = A^*_{\alpha_0} \cup \omega$ and note that we have, for every $h \in X$, that $\int_Z h\,d\mu = \int_B h\,d\mu$. Let $\mathcal{E} = \{D_\alpha : \alpha \leq \alpha_0\} \cup \{\{n\} : n \in \omega\}$. Then $\mathcal{E}$ is a countable collection of sets such that $\bigcup \mathcal{E} = B$, and therefore there exists a finite subcollection $\mathcal{L}$ of $\mathcal{E}$ such that we have, for the finitary set $L = \bigcup \mathcal{L}$, that $\mu(B) < \mu(L) + \frac{1}{2}$. Note that it follows from the foregoing that we have, for every $h \in X$, that

$$\int_{\beta\omega} h\,d\mu - \int_{\beta\omega} h\,d\mu_L = \int_B h\,d\mu - \int_L h\,d\mu = \int_{B\setminus L} h\,d\mu \leq \mu(B \setminus L) < \frac{1}{2}\,.$$

Let $\nu = 2 \cdot \mu_L$. To show that $[f, \nu] \subset [f, \mu]$, let $g \in [f, \nu]$. Then we have that

$$\int_{\beta\omega} |g - f| d\mu_L = \frac{1}{2} \int_{\beta\omega} |g - f| d\nu < \frac{1}{2}$$

and it follows from the above inequalities, setting $h = |g - f|$, that $\int_{\beta\omega} |g - f| d\mu < 1$. We have shown that $[f, \nu] \subset [f, \mu]$. Hence we have established the claim that measures supported by finitary sets suffice to define the weak topology on X. We denote by $\mathcal{N}$ the family of all such measures. For every $\nu \in \mathcal{N}$, we denote by $s_\omega(\nu)$ and $s_{\omega_1}(\nu)$ the finite sets $\{n \in \omega : \nu(\{n\}) > 0\}$ and $\{\alpha \in \omega_1 : \nu(D_\alpha) > 0\}$, respectively, and we let $S_{\omega_1}(\nu) = \bigcup_{\alpha \in s_{\omega_1}(\nu)} D_\alpha$; we note that ν is supported by the finitary set $s_\omega(\nu) \cup S_{\omega_1}(\nu)$.

For each $\alpha \in \omega_1$, let

$$O_\alpha = \{f \in X : f(p) = 0 \text{ for some } p \in A^*_\alpha\}\,.$$

Note that the family $\mathcal{O} = \{O_\alpha : \alpha < \omega_1\}$ covers X and consists of sets that are open in the relative pointwise topology of X and therefore also in the relative weak topology. We show that $\mathcal{O}$ has no point–finite σ–relatively open refinement. Assume that $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is a refinement of $\mathcal{O}$ such that each $\mathcal{U}_n$ consists of relatively open subsets of $\bigcup \mathcal{U}_n$. We shall show that some $\mathcal{U}_n$ is not point–finite.

For $n \in \omega$, we will say that a collection

$$\Delta = \{(f_\alpha, \nu_\alpha) : \alpha \in \omega_1\} \subset X \times \mathcal{N}$$

is a Δ*–system for* $\mathcal{U}_n$ if the following conditions hold. We require that, for every $\alpha \in \omega_1$, we have $f_\alpha \notin O_\alpha$ and there exists $U \in \mathcal{U}_n$ such that $[f_\alpha, \nu_\alpha] \subset U$. Moreover, we require that we have $s_\omega(\nu_\alpha) = s_\omega(\nu_\beta)$ and $f_\alpha | s_\omega(\nu_\alpha) = f_\beta | s_\omega(\nu_\beta)$ for all $\alpha, \beta \in \omega_1$; we shall denote the common restriction $f_\alpha | s_\omega(\nu_\alpha)$ by ψ_Δ. Finally, we require that the family $\{s_{\omega_1}(\nu_\alpha) : \alpha \in \omega_1\}$ is a Δ–system (in the ordinary sense; see, e.g., [Ku], p.49); we shall denote by $R(\Delta)$ the root of this Δ–system.

Let $F = \{\varphi \subset \omega \times \{0,1\} : \varphi \text{ is a finite function}\}$ and let H denote the set consisting of all pairs $(\varphi, n) \in F \times \omega$ for which there exists a Δ–system Δ for $\mathcal{U}_n$ such that $\psi_\Delta = \varphi$. For every $(\varphi, n) \in H$, let $\Delta_{\varphi,n}$ be a Δ–system for $\mathcal{U}_n$ such that $\psi_{\Delta_{\varphi,n}} = \varphi$. Denote by β_0 the supremum of the countable subset $\bigcup_{(\varphi,n)\in H} R(\Delta_{\varphi,n})$ of ω_1.

We observe that, for every $(\varphi, n) \in F \times \omega \setminus H$, there exists $\alpha_{\varphi,n} \in \omega_1$ such that, for all $g \in C(\beta\omega)$ and $\nu \in \mathcal{N}$, if we have that $g | s_\omega(\nu) = \varphi$ and if the set $[g, \nu] \cap \bigcup \mathcal{U}_n$ is contained in some $U \in \mathcal{U}_n$, then $g \in O_{\alpha_{\varphi,n}}$. To see this, assume on the contrary that $(\varphi, n) \notin H$ and that no such $\alpha_{\varphi,n}$ exists. Then we can find $g_\alpha \in C(\beta\omega)$ and $\nu_\alpha \in \mathcal{N}$, for $\alpha < \omega_1$, such that, for each α, we have $g_\alpha | s_\omega(\nu_\alpha) = \varphi$, $g_\alpha \notin O_\alpha$ and $[g_\alpha, \nu_\alpha] \cap \bigcup \mathcal{U}_n \subset U$ for some $U \in \mathcal{U}_n$. By the Δ–system Lemma (see, e.g., [Ku], p.49), we can find an uncountable set $A \subset \omega_1$ such that the family $\{s_{\omega_1}(\nu_\alpha) : \alpha \in A\}$ is a Δ–system (in the ordinary sense). If we now let $\{\gamma_\alpha : \alpha \in \omega_1\}$ be an increasing enumeration of A, then the collection $\Delta = \{(g_{\gamma_\alpha}, \nu_{\gamma_\alpha}) : \alpha \in \omega_1\}$ is a Δ–system for $\mathcal{U}_n$ with $\psi_\Delta = \varphi$. This, however, is impossible since we have that $(\varphi, n) \notin H$. By the foregoing, the ordinal $\alpha_{\varphi,n}$ exists. We let $\delta_0 = \sup\{\alpha_{\varphi,n} : (\varphi, n) \in F \times \omega \setminus H\}$.

Enumerate H as $\{(\varphi_k, n_k) : k \in \omega\}$ so that every element is repeated infinitely many times. Let $\alpha_0 = \beta_0 + \delta_0 + 1$. For $k = 1, 2, 3, \ldots$, define $(f_k, \nu_k) \in \Delta_{\varphi_k, n_k}$, $U_k \in \mathcal{U}_{n_k}$ and $\alpha_k \in \omega_1$ by induction so that the following conditions hold for every $k > 0$:

1^o $[f_k, \nu_k] \cap \bigcup \mathcal{U}_{n_k} \subset U_k \subset O_{\alpha_k}$.

2^o $f_k \notin O_{\alpha_{k-1}}$.

3^o $s_{\omega_1}(\nu_k) \subset \alpha_0 \cup (\alpha_k \setminus \alpha_{k-1})$.

(To get condition 3^o to hold, use the observation that whenever $\Delta = \{(f_\alpha, \nu_\alpha) : \alpha \in \omega_1\}$ is a Δ–system for some $\mathcal{U}_n$, then we have, for every $\gamma < \omega$, that $s_{\omega_1}(\nu_\alpha) \cap \gamma \subset R(\Delta)$ for all but at most countably many α's).

Set $\rho_0 = \sup\{\alpha_k : k \in \omega\}$. The subspace

$$E = A^*_{\alpha_0} \cup \left(\beta\omega \setminus \bar{A}_{\rho_0}\right) \cup \bigcup_{k=1}^{\infty} S_{\omega_1}(\nu_k)$$

of $\beta\omega$ is σ–compact and hence C^*–embedded ([CoHiNe], or see [vM], Theorem 3.5.2). Define a function $h : E \to \{0,1\}$ by setting $h(p) = 1$ for $p \in A^*_{\alpha_0}$, $h(p) = 0$ for $p \in \beta\omega \setminus \bar{A}_{\rho_0}$ and $h(p) = f_k(p)$ for $k > 0$ and $p \in S_{\omega_1}(\nu_k) \setminus A^*_{\alpha_0}$. The function h is continuous on E and hence there exists $f \in C(\beta\omega, \{0,1\})$ such that $f|E = h$. Note that we have $f \in X$ and $f \notin O_{\alpha_0}$. Let $n \in \omega$ and $U \in \mathcal{U}_n$ be such that $f \in U$. Since the family $\mathcal{U}_n$ is relatively open, there exists $\nu \in \mathcal{N}$ such that $[f, \nu] \cap \bigcup \mathcal{U}_n \subset U$. Set $\varphi = f | s_\omega(\nu)$, and note that $(\varphi, n) \in H$: otherwise we would have that $f \in [f, \nu] \cap \bigcup \mathcal{U}_n \subset O_{\alpha_{\varphi,n}}$ and this would lead to a contradiction since we have $f \notin O_{\alpha_0}$ and $\alpha_0 \geq \alpha_{\varphi,n}$. Consequently, there are infinitely many k's such that

$(\varphi_k, n_k) = (\varphi, n)$. It is easily checked that we have $f|S(\nu_k) = f_k|S(\nu_k)$ and therefore $f \in [f_k, \nu_k] \subset U_k$ for each such k. The selection of the corresponding U_k's guarantee that they are distinct members of $\mathcal{U}_n$. Hence the family $\mathcal{U}_n$ is not point–finite at f. $\square$

Remarks (i) Proposition 1.4 and Example 1.5 provide a new proof for the result of J. Lindenstrauss ([Li]) that the Banach space ℓ^∞ does not admit a WLUR norm.

(ii) Lemma 1.3 and Example 1.5 show that the Banach space ℓ^∞ does not admit any equivalent norm such that the unit sphere is hereditarily σ–relatively metacompact in its relative weak topology. As a corollary, we obtain the result (see [DeGoZi], Theorem 7.10) that ℓ^∞ does not admit a Kadeč norm.

Hansell has asked whether $C_p(K)$ has to be σ–relatively metacompact (a) for every compact space? (b) for every scattered compact space? (c) for every separable compact space? In our previous paper we gave negative answers to question (a) and (under the set–theoretic axiom $\diamondsuit$) to question (b). An examination of the proof of Example 1.5 shows that also question (c) has a negative answer:

1.5′ Example The space $C_p(\beta\omega)$ is not σ–relatively metacompact.

Next we observe that Examples 1.5 and 1.5′ provide us with many more examples of non–σ–relatively metacompact function spaces. To see this, we need the following auxiliary result.

1.6 Lemma *Let X be a completely regular space which is not pseudocompact. Then there is a isometric embedding $\psi : C(\beta\omega) \to C(\beta X)$ such that the subspace $\psi(C(\beta\omega))$ is closed in $C_p(\beta X)$ and ψ is a homeomorphism $C(\beta\omega) \to \psi(C(\beta\omega))$ also with respect to the point–wise topologies.*

Proof. Let $\{G_n : n \in \omega\}$ be a discrete family of open subsets of X and, for each $n \in \omega$, let $z_n \in G_n$. Further, for every n, let $g_n : X \to [0,1]$ be continuous and such that $s(g_n) \subset G_n$ and $g_n(z_n) = 1$. For every $f \in C(\beta\omega)$, let $\psi(f) \in C(\beta X)$ be defined by the formula:

$$\psi(f) = \beta\left(\Sigma_{n\in\omega} f(n) \cdot g_n\right) .$$

It is easy to see that the mapping ψ has the required properties. $\square$

1.7 Example Let X be a completely regular non–pseudocompact space. Then the spaces $C_p(\beta X)$ and $C_w(\beta X)$ are not σ–relatively metacompact.

If we let $X = \mathbb{R}$ in the preceding example, we see that there exists a connected separable compact space K such that $C_p(K)$ and $C_w(K)$ fail to be σ–relatively metacompact.

Other compact spaces K whose function spaces $C_p(K)$ and $C_w(K)$ fail to be σ–relatively metacompact can be obtained from Examples 1.5 and 1.5′ by considering compact spaces K which map continuously onto $\beta\omega$; also for such a space K we have that $C(\beta\omega)$ embeds isometrically and linearly in $C(K)$ in such a way that we also have an embedding with respect to the point–wise topologies. In this way we obtain the following example.

1.8 Example The spaces $C_p(\omega^*)$ and $C_w(\omega^*)$ are not σ–relatively metacompact.

Proof. It is well known and easy to see that the space ω^* maps continuously onto $\beta\omega$ (if $f:\omega\to\omega$ is such that $|f^{-1}(n)|=\omega$ for every n, then the restriction to ω^* of the extension mapping $\beta f:\beta\omega\to\beta\omega$ is a continuous onto map $\omega^*\to\beta\omega$). $\square$

To finish our study of the failure of weak covering properties in $C_p(K)$–spaces, we observe that every non–metacompact $C_p(X)$–space is non–orthocompact. This follows from results of Peregudov ([Pe]) which show that, for every completely regular space X, the topology of $C_p(X)$ has a "strictly open–finite" base and therefore orthocompactness implies metacompactness in $C_p(X)$. An extension of Peregudov's results given in [FlLi] shows that a non– metacompact $C_p(X)$–space even fails to be "preorthocompact".

2. Compact spaces with strongly ccc function spaces. The *point–finite cellularity* of a topological space X is the cardinal number $p(X)$ defined as the supremum of all numbers $|\mathcal{U}|$, where $\mathcal{U}$ is a point–finite family consisting of open subsets of X ([Tk]). If $p(X)\le\omega$, then X satisfies the countable chain condition (ccc); therefore we use the term *strong ccc space* for a space X with $p(X)\le\omega$.

To construct the space K, we will use the set–theoretic principle $\diamondsuit$ which is valid in the constructible universe. Recall that $\diamondsuit$ is the following statement:
There are sets $A_\alpha\subset\alpha$, for $\alpha\in\omega_1$, such that

$$\text{for every } A\subset\omega_1,\ \{\alpha<\omega_1 : A\cap\alpha=A_\alpha\} \text{ is stationary .}$$

2.1 Example ($\diamondsuit$) There exists a non–metrizable compact scattered space K such that $p(C_p(K))=\omega$.

Proof. We assume that axiom $\diamondsuit$ holds, and we use it to construct a compact topology on ω_1+1. Intuitively, we construct the space K from the linearly ordered space ω_1+1 by "splitting" some limit ordinals $\alpha<\omega_1$ and their neighbourhoods into finitely many parts. For technical reasons, we do not want to introduce any new points to ω_1+1, and therefore we use $\alpha,\ \alpha+1,\ \dots\ ,\alpha+n$ as the "new" points. To construct the new topology, we first define by induction a set $W_\alpha\subset\alpha+1$ for each $\alpha\le\omega_1$; then we shall use these sets to define the topology.

For every $n\in\omega$, let F_n denote the set consisting of all finite surjections $A\to\{0,...,n-1\}$, where $A\subset\omega_1$. Let $F=\bigcup_{n\in\omega}F_n$ and let $\langle t_\gamma:\gamma<\omega_1\rangle$ be an enumeration of F. Let $\delta\le\omega_1$ and let $\gamma_\iota\in\omega_1$ for each $\iota<\delta$. We say that the transfinite sequence $\langle\gamma_\iota:\iota<\delta\rangle$ is an "orderly sequence" provided that for all $\iota<\nu<\delta$, we have that

$$\max\left(\{\gamma_\iota\}\cup\operatorname{dom}t_{\gamma_\iota}\right)<\min\left(\{\gamma_\nu\}\cup\operatorname{dom}t_{\gamma_\nu}\right)\quad .$$

The limit ordinals which will be "split" are elements of a set $S=\bigcup_{k\in\omega}S_k$, defined as follows. For $k\in\omega$, the set S_k consists of those elements α of ω_1 for which we have $\{t_\sigma:\sigma\in A_\alpha\}\subset F_k$ and there exists an orderly sequence $\langle\sigma_n:n<\omega\rangle$

such that $\sup_{n\in\omega} \sigma_n = \alpha$ and $\sigma_i \in A_\alpha$ for every $i \in \omega$. Note that the sets F_k, for $k \in \omega$, are mutually disjoint.

For $\alpha \in S = \bigcup_{k\in\omega} S_k$, we define $k(\alpha) \in \omega$ by the condition $\alpha \in S_{k(\alpha)}$. We set $\hat{S} = \{\alpha + i : \alpha \in S \text{ and } i \le k(\alpha)\}$, and for $\beta = \alpha + i \in \hat{S}$, we set $\bar{\beta} = \alpha + k(\alpha)$. For $\beta \in \omega_1 \setminus \hat{S}$, we set $\bar{\beta} = \beta$.

We now define the sets W_α. For every $\alpha \in (\omega_1+1)\setminus\hat{S}$, we set $W_\alpha = \alpha+1$. For $\alpha \in S_k$, we define the sets $W_{\alpha+i}$, for $i \le k$ inductively as follows. Let $\langle \sigma_n : n < \omega\rangle$ be an orderly sequence with the properties mentioned above. Denote by $t^{(\alpha)}$ the function $\bigcup_{n\in\omega} t_{\sigma_n}$. Let $\{\rho_n : n \in \omega\}$ be an increasing enumeration of $\operatorname{dom} t^{(\alpha)}$. For every $n \in \omega$, we let $\nu_n = \max A_n$, where $A_n = \{0\} \cup \{\bar{\rho}_m : m < n \text{ and } \bar{\rho}_m < \rho_n\}$. We set $V_{\alpha+k} = \alpha$ and, for each $i < k$, we set

$$V_{\alpha+i} = \bigcup \{W_{\rho_n} \cap (\nu_n, \rho_n] : n \in \omega \text{ and } t^{(\alpha)}(\rho_n) = i\} \ .$$

Finally, we set

$$W_{\alpha+i} = \{\alpha + i\} \cup \left(V_{\alpha+i} \setminus \bigcup_{j<i} V_{\alpha+j} \right) \ .$$

Note that it follows from the above construction that the sets $W_{\alpha+i}$, for $i \le k$, form a partition of the set $\alpha + k + 1$ and that $t^{(\alpha)} \subset \Sigma_{i=0}^{k} i\chi_{W_{\alpha+i}}$.

By induction, it is easy to verify the following property of the sets W_α: for all $\alpha, \beta \in \omega_1 + 1$, if $\beta \in W_\alpha$, then there exists $\gamma \in \beta$ such that $\gamma + 1 \notin \hat{S}$ and $W_\beta \cap (\gamma, \beta] \subset W_\alpha$. Using this property, we see that there exists a topology τ on the set ω_1+1 such that, for every $\alpha \le \omega_1$, the family $\{W_\alpha \cap (\gamma, \alpha] : \gamma \in \alpha \text{ and } \gamma+1 \notin \hat{S}\}$ is a clopen neighbourhood base of α in (ω_1+1, τ). Let us denote the space (ω_1+1, τ) by K. Note that, since $W_\alpha \subset \alpha+1$ for every α, the space K is scattered. Clearly, K is a Hausdorff space. From the definition of the sets W_α it follows that the finite–to–one mapping f from K onto the ordinal space $\omega_1 + 1$, defined by setting $f(\alpha) = \alpha$ if $\alpha \notin \hat{S}$ and $f(\alpha + i) = \alpha$ if $\alpha \in S$ and $i \le k(\alpha)$, is closed and continuous. As a consequence, the space K is compact and non–metrizable.

We show that $p(C_p(K)) = \omega$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1\}$ be a family of non–empty open subsets of $C_p(K)$. We must show that some $f \in C(K)$ belongs to U_α for infinitely many ordinals α.

For every $\alpha \in \omega_1$, let $f_\alpha \in U_\alpha$. Since the set U_α is open in $C_p(K)$, there exists $E_\alpha \in [\omega_1 + 1]^{<\omega}$ and $\varepsilon_\alpha > 0$ such that for every $g \in C(K)$, if $|f_\alpha(\delta) - g(\delta)| < \varepsilon_\alpha$ for every $\delta \in E_\alpha$, then $g \in U_\alpha$; it follows that there exists a finite function $h_\alpha \subset (\omega_1 + 1) \times \mathbb{Q}$ such that $\{g \in C(K) : h_\alpha \subset g\} \subset U_\alpha$. Let $L \in [\omega_1]^{\omega_1}$ be such that the family $\{\operatorname{dom} h_\alpha : \alpha \in L\}$ is a Δ–system, with root R. For every $\alpha \in L$, let $\varphi_\alpha = h_\alpha|(\operatorname{dom} h_\alpha \setminus R)$. Without loss of generality, we may assume that $h_\alpha|R = h_\beta|R$ and $\operatorname{range}(\varphi_\alpha) = \operatorname{range}(\varphi_\beta)$ for all $\alpha, \beta \in L$. Denote the common range of the functions φ_α, $\alpha \in L$, by A, let $k = |A|$ and let φ be a bijection $A \to \{0, ..., k-1\}$; note that we have $\varphi \circ \varphi_\alpha \in F_k$ for every $\alpha \in L$. Let $\alpha_0 \in \omega_1 \setminus \hat{S}$ be such that $\alpha_0 > \beta$ for every $\beta \in R \setminus \{\omega_1\}$. Without loss of generality, we may assume that $\min \operatorname{dom} \varphi_\alpha > \alpha_0$ for every $\alpha \in L$.

By induction, one can construct an orderly sequence $\langle \eta_\iota : \iota < \omega_1 \rangle$ such that $t_{\eta_\iota} \in \{\varphi \circ \varphi_\alpha : \alpha \in L\}$ whenever $\iota \in \omega_1$. Let $E = \{\eta_\iota : \iota < \omega_1\}$ and denote by E' the set of all accumulation points of E in the order topology of ω_1; note that E' is uncountable and closed in the order topology. By the principle $\diamondsuit$, we know that there will be a $\theta \in E'$ such that $A_\theta = E \cap \theta$. Note that we have $\theta \in S_k$ since $\theta \in E'$, $A_\theta \subset E$ and $\{t_\sigma : \sigma \in E\} \subset F_k$.

Let $\langle \sigma_n : n < \omega \rangle$ be that orderly sequence which was used in the definition of W_θ. Note that since $A_\theta = E \cap \theta$, we have, for each $j \in \omega$, that $\sigma_j \in E$ and it follows from this that there exists $\delta_j \in L$ such that $t_{\sigma_j} = \varphi \circ \varphi_{\delta_j}$. Note that the ordinals δ_j, $j \in \omega$, are distinct. Also note that, for every $j \in \omega$, we have

$$\varphi \circ \varphi_{\delta_j} \subset \bigcup_{i \in \omega} t_{\sigma_i} = t^{(\theta)} \subset \Sigma_{i=0}^{k-1} i \chi_{W_{\theta+i}} \quad \text{and therefore} \quad \varphi_{\delta_j} \subset \varphi^{-1} \circ \Sigma_{i=0}^{k-1} i \chi_{W_{\theta+i}} .$$

Let $d \in C(K)$ be such that $d|R = h_\alpha|R$ for $\alpha \in L$. The mapping $\Sigma_{i=0}^{k-1} i \chi_{W_{\theta+i}}$ is continuous $K \to \mathbb{R}$ and it follows that also the mapping $l = \varphi^{-1} \circ \Sigma_{i=0}^{k-1} i \chi_{W_{\theta+i}}$ is continuous. We define $f \in C(K)$ by setting $f(\beta) = d(\beta)$ if $\beta \le \alpha_0$ and $f(\beta) = l(\beta)$ if $\alpha_0 < \beta \le \bar{\theta}$; for $\beta > \bar{\theta}$ we set $f(\beta) = c$, where c is the common value of the mappings h_α, $\alpha \in L$, at the point $\omega_1 + 1$ in case $\omega_1 + 1 \in R$ and otherwise $c = 0$. Note that we have $h_{\delta_j} \subset f$ for every $j \in \omega$. As a consequence, we have $f \in \bigcap_{j \in \omega} U_{\delta_j}$. We have shown that the family $\mathcal{U}$ is not point–finite. □

Since the space K admits a finite–to–one continuous mapping into the ordinal space $\omega_1 + 1$, we see that every separable subspace of K is countable. It follows that we have $|\bar{A}| = |A|$ for every $A \subset K$. As a consequence, the space K is monolithic. Since K is non–metrizable, we have a counterexample, under $\diamondsuit$, to Problem 72 of [Ar$_4$]; moreover, by Proposition 13.19 of the same paper, there cannot exist any continuous one–to–one mapping from $C_p(K)$ into a Σ–product of real lines; hence we also have a counterexample, under $\diamondsuit$, to Problem 74 of [Ar$_4$].

(Our example is also a counterexample, under $\diamondsuit$, to Problem 71 of [Ar$_4$]; however, that problem had already been solved before it was posed, by the result of [ArTk] that $C_p(K)$ is strongly ccc for every extremally disconnected compact space K.)

3. Supmetrizable spaces. Our examples in this and in our previous paper show that pointwise function spaces over compact spaces and weak Banach spaces may fail to have certain relatively weak covering properties. Nevertheless, it is conceivable that there would exist some even weaker covering properties satisfied by all B_w–spaces and $C_p(K)$–spaces; for example, Arhangel'skij asks in [Ar$_5$] whether every $C_p(K)$–space is "astral". To facilitate the further study of weak covering properties in linear spaces, we shall now introduce a class of spaces, which we call "supmetrizable", containing all B_w–spaces and all $C_p(K)$–spaces. These spaces are technically simpler to work with than the weak linear spaces, and we hope that they are not too general to be useful in connection with problems like the one of Arhangel'skij's just mentioned. We shall show below that supmetrizable spaces have, at least, so much structure that they share certain properties that have been earlier established for weak Banach spaces and $C_p(K)$–spaces spaces.

The essential feature of supmetrizable spaces is that they have a finer metrizable topology which is linked to the original topology. Recall that a topological space X is called *submetrizable* if there exists a coarser metrizable topology on X. Analoguously, we could define "supmetrizable" spaces by requiring that there exists a finer metrizable topology, but this definition does not make sense since the discrete topology is always such a topology. To arrive at a meaningful property we must require that the finer metrizable topology is more closely related to the original topology. The property introduced in the following definition appears useful and interesting.

3.1 Definition A topological space (X,τ) is *supmetrizable* if there exists a metrizable topology $\pi \supset \tau$ on X and a family $\mathcal{L}$ of subsets of X with the following properties:

(i) The family $\mathcal{L}$ is closed under monotone unions.
(ii) We have $\mathrm{Cl}_\pi L = \mathrm{Cl}_\tau L$ for every $L \in \mathcal{L}$.
(iii) Every infinite set $A \subset X$ is contained in a set $L \in \mathcal{L}$ with $d_\pi(L) = d_\tau(A)$.

We shall say that the family $\mathcal{L}$ is a *linking family* between the topologies τ and π.

Before considering properties of supmetrizable spaces, we shall indicate some classes of these spaces. Obviously, all metrizable spaces are supmetrizable. However, the main motivation for introducing this new class of spaces is the following result; the proof of the result explains the conditions that we have put on the linking family in the definition of supmetrizability.

3.2 Theorem *If X is either a $C_p(K)$–space or a Banach space with its weak topology, then X is supmetrizable.*

Proof. Assume that $(X,\tau) = C_p(K)$. For π we take the sup–norm topology of $C(K)$, and for $\mathcal{L}$, we take the family consisting of all subalgebras of $C(K)$. It is obvious that $\mathcal{L}$ is closed under monotone unions, and it is a consequence of the Stone–Weierstrass Theorem that we have $\mathrm{Cl}_\pi L = \mathrm{Cl}_\tau L$ for every $L \in \mathcal{L}$. To see that Condition (iii) of the definition holds, let A be an infinite subset of $C(K)$. Let B be a τ–dense subset of A with $|B| = d_\tau(A)$. Further, let H be the subalgebra of $C(K)$ generated by B, and let $L = \mathrm{Cl}_\pi H$. Note that we have $d_\pi(L) \le |H| = |B| = d_\tau(A)$. By the above–mentioned result, the set L is τ–closed; it follows that $A \subset \mathrm{Cl}_\tau B \subset L$. Moreover, the π–closure of a subalgebra is a subalgebra, and therefore we have that $L \in \mathcal{L}$. We have shown that also Condition (iii) holds.

If we replace above subalgebras by linear subspaces (or by convex sets), and if we use the Hahn–Banach Theorem instead of the Stone–Weierstrass Theorem, then we obtain a proof for the statement concerning weak Banach spaces. □

Besides metrizable spaces, another obvious class of supmetrizable spaces is formed by the spaces which admit a finer separable metrizable topology, i.e., by the spaces which have a countable network (in this case we can take $\mathcal{L} = \{X\}$); in the following result, we indicate an even larger class of supmetrizable spaces.

3.3 Lemma *If a space X has a closed subspace S with a countable network such that the subspace $X \setminus S$ is supmetrizable, then X is supmetrizable.*

Proof. Let τ be the topology of X. Denote by τ' the relative topology on $X \setminus S$.

Let $\pi' \supset \tau'$ be a metric topology on $X \setminus S$ and let $\mathcal{L}'$ be a linking family for τ' and π'.

Let $\mathcal{N}$ be a countable network for the subspace S of X. Let $\pi^{\#}$ be the (separable and metrizable) topology on S generated by the family $\mathcal{N} \cup \{S \setminus N : N \in \mathcal{N}\}$, and let π be the topology on X generated by the family $\pi' \cup \pi^{\#}$. Note that the topology π is metrizable since the space (X, π) is the union of two closed metrizable subspaces S and $X \setminus S$. Also note that $\tau \subset \pi$. If we let $\mathcal{L} = \{L' \cup S : L' \in \mathcal{L}'\}$, then we can easily see that $\mathcal{L}$ is a linking family for τ and π. □

Several familiar examples of non–metrizable spaces, for instance many "bow–tie spaces", can be seen to be supmetrizable with the help of the above lemma.

Next we shall observe that a non–discrete topology can be linked with the discrete topology in the sense of the above definition:

3.4 Lemma *If the space (X, τ) has a closure–preserving cover by countable closed sets, then Definition 3.1 is satisfied with π the discrete topology of X.*

Proof. Let $\mathcal{F}$ be a closure–preserving cover by countable closed subsets of (X, τ), and let $\mathcal{L} = \{\bigcup \mathcal{L}' : \mathcal{L}' \subset \mathcal{L}\}$. Then $\mathcal{L}$ is a family of τ–closed sets closed under arbitary unions; hence Conditions (i) and (ii) of Definition 3.1 hold with π the discrete topology of X. To see that also Condition (iii) holds, let $A \subset X$ be infinite, and let B be a τ–dense subset of A such that $|B| = d_\tau(A)$. For every $x \in B$, let $F_x \in (\mathcal{F})_x$. Let $L = \bigcup_{x \in B} F_x$, and note that $L \in \mathcal{L}$ and $|L| = |B| = d_\tau(A)$. Moreover, we have that $A \subset \mathrm{Cl}_\tau B \subset L$. We have shown that also Condition (iii) holds with π the discrete topology of X. □

In particular, we see that if a space has only one non–isolated point, then the space is supmetrizable.

Further examples of supmetrizable spaces can now be obtained with the help of the following result.

3.5 Lemma A. *Every open subspace of a supmetrizable space is supmetrizable.*
B. *The product of countably many supmetrizable spaces is supmetrizable.*

Proof. A. Let (X, τ) be supmetrizable and let π and $\mathcal{L}$ be as in the definition. Let $G \in \tau$. If we let τ_G and π_G be the respective relative topologies on G and $\mathcal{L}_G = \{L \cap G : L \in \mathcal{L}\}$, then we can easily see that the conditions defining supmetrizability are satisfied for (G, τ_G), π_G and $\mathcal{L}_G$.
B. For every $n \in \omega$, let (X_n, τ_n) be supmetrizable and let π_n and $\mathcal{L}_n$ be as in the definition. To show supmetrizability of the product space $(X, \tau) = \prod_{n \in \omega} (X_n, \tau_n)$, let $\pi = \prod_{n \in \omega} \pi_n$ and $\mathcal{L} = \{\prod_{n \in \omega} L_n : L_n \in \mathcal{L}_n \text{ for every } n \in \omega\}$. It is easy to see that the conditions of Definition 3.1 are then satisfied. □

3.6 Problems A. *Is every closed subspace of a supmetrizable space supmetrizable?*
B. *Is every Eberlein compact space supmetrizable?*
C. *Is every compact supmetrizable space Eberlein compact?*

Much research has been conducted recently on the properties of spaces which can be embedded in $C_p(K)$–spaces; Arhangel'skij calls this kind of spaces *Eberlein–Grothendieck spaces* in [Ar 3]. By Theorem 3.2 above, every Eberlein–Grothendieck space can be embedded in a supmetrizable space; in the following, we will show that some of the established properties of Eberlein–Grothendieck spaces follow from properties of supmetrizable spaces.

To start our study on the properties of supmetrizable spaces, we note that it follows from Condition (iii) of Definition 3.1 that the linked topologies τ and π of X are "d–equivalent": for every $A \subset X$ we have that $d_\pi(A) = d_\tau(A)$; in the following, we shall denote by $d(A)$ the common value of the densities, with respect to the two topologies, of a subspace A of a supmetrizable space.

Arhangels'kij [Ar 2] has shown that all B_w–spaces and all $C_p(K)$–spaces are monolithic. Since the topology of an supmetrizable space is "d–equivalent" with a finer metrizable topology, the following result obtains.

3.7 Proposition *Every supmetrizable space is monolithic.*

In the following technical lemma, we combine two induction arguments that we need to prove our remaining results. We say that a sequence $\langle L_\alpha \rangle_{\alpha<\delta}$ of sets is *continuously increasing* provided that we have $L_\alpha = \bigcup\{L_{\beta+1} : \beta < \alpha\}$ for every $0 < \alpha < \delta$.

3.8 Lemma *Let (X,τ) be a supmetrizable space with linking family $\mathcal{L}$ and let κ be an uncountable cardinal. Set $\mathcal{L}' = \{L \in \mathcal{L} : d(L) < \kappa\}$, and let $\{A(L) : L \in \mathcal{L}'\}$ be a family of sets such that $A(L) \subset A(L')$ whenever $L \subset L'$. Let E be a subset of X that is not contained in any $A(L)$, and let $\{z_\alpha : \alpha < \kappa\} \subset X$. Then there exists a continuously increasing sequence $\langle L_\alpha \rangle_{\alpha<\kappa}$ of members of $\mathcal{L}'$ such that we have, for every $\alpha < \kappa$, that $d(L_\alpha) \leq \max(\alpha, \omega)$, $z_\alpha \in L_{\alpha+1}$ and $L_{\alpha+1} \cap E \not\subset A(L_\alpha)$.*

Proof. We let L_0 be any separable member of $\mathcal{L}'$, and then we construct the sequence by induction as follows: we assume that $0 < \alpha < \kappa$ and that the sets L_β, for $\beta < \alpha$, have already been constructed so that $d(L_\beta) \leq \max(\beta, \omega)$ for every $\beta < \alpha$. If α is a limit ordinal, then we set $L_\alpha = \bigcup\{L_\beta : \beta < \alpha\}$, and we note that $d(L_\alpha) \leq \alpha < \kappa$ and hence $L_\alpha \in \mathcal{L}'$. If α is a successor, say $\alpha = \beta + 1$, then we have that $E \not\subset A(L_\beta)$ and hence there exists a point a in $E \setminus A(L_\beta)$. Note that we have $d\left(L_\beta \cup \{a, z_\beta\}\right) \leq |\max(\beta, \omega) + 2| \leq \max(\beta, \omega)$; it follows, by Condition (iii) of Definition 3.1, that there exists $L_\alpha \in \mathcal{L}$ such that $L_\beta \cup \{a, z_\beta\} \subset L_\alpha$ and $d(L_\alpha) \leq \max(\beta, \omega) \leq \max(\alpha, \omega) < \kappa$. □

Let κ be a cardinal. We say that a family $\mathcal{E}$ of sets is *pointwise* $< \kappa$ provided that $|(\mathcal{E})_x| < \kappa$ for every $x \in \bigcup\mathcal{E}$. Further, we say that $\mathcal{E}$ is σ*–pointwise* $< \kappa$ provided that $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$, where each $\mathcal{E}_n$ is pointwise $< \kappa$. Note that, if the cofinality of κ is uncountable, then every σ–pointwise $< \kappa$ family is actually pointwise $< \kappa$. Our next result shows that, in a subspace of a supmetrizable space, every open cover has an open refinement which is σ–pointwise less than the density of the subspace. The proof of the result is a modification of that of Lemma 1.1 of [DoJuPe].

3.9 Proposition *Let Z be an infinite subspace of a supmetrizable space. Then every open cover of Z has an open refinement which is σ–pointwise $< d(Z)$.*

Proof. Denote the cardinal $d(Z) = d_\tau(Z) = d_\pi(Z)$ by κ. Assume first that the cofinality of κ is ω. Note that it follows from Proposition 3.7 that every open cover of Z has a subcover of cardinality at most κ. Moreover, since $\mathrm{cof}(\kappa) = \omega$, every family of cardinality at most κ is σ–pointwise $< \kappa$. Hence the result holds in the case that $\mathrm{cof}(\kappa) = \omega$. For the remainder of the proof, we assume that $\mathrm{cof}(\kappa) > \omega$.

Let (X, τ) be a supmetrizable space, with a finer metrizable topology π and a linking family $\mathcal{L}$, such that Z is a subspace of (X, τ). Let d be a metric on X which induces the topology π; we may assume that d is bounded. For all $x \in X$ and $r > 0$, denote by $B(x, r)$ the d–ball of radius r with center x.

Let $\mathcal{O}$ be an open cover of Z, and let $\mathcal{V}$ be an open family in (X, τ) such that $\mathcal{O} = \{V \cap Z : V \in \mathcal{V}\}$. Let $Y = \bigcup \mathcal{V}$, and for every $y \in Y$, let

$$r_y = \sup\{r : \text{ there exists } V \in \mathcal{V} \text{ such that } B(y, r) \subset V\};$$

note that $r_y > 0$, because $\tau \subset \pi$; further, let $V_y \in \mathcal{V}$ be such that $B(y, \frac{3}{4}r_y) \subset V_y$, and set $O_y = V_y \cap Z$.

Let $E = \{z_\alpha : \alpha < \kappa\}$ be a π–dense subset of Z, and let $\mathcal{L}' = \{L \in \mathcal{L} : d(L) < \kappa\}$. By Lemma 3.8, there exists a continuously increasing sequence $\langle L_\alpha \rangle_{\alpha<\kappa}$ from $\mathcal{L}'$ such that we have $d(L_\alpha) \leq \max(\alpha, \omega)$ and $z_\alpha \in L_{\alpha+1}$ for every $\alpha < \kappa$. We let $F_\alpha = \mathrm{Cl}_\tau L_\alpha$ for every $\alpha < \kappa$. Note that since we have that $E \subset \bigcup_{\alpha=\omega}^{\kappa} L_\alpha$ and $\mathrm{cof}(\kappa) > \omega$ and since every point of Z is in the π–closure of some countable subset of E, we have that $Z \subset \bigcup_{\alpha<\kappa} F_\alpha$. For every $\alpha < \kappa$, let C_α be a π–dense subset of $F_{\alpha+1} \setminus F_\alpha$ with $|C_\alpha| \leq \max(\alpha, \omega)$. Further, let

$$\mathcal{U} = \{O_y \setminus F_\alpha : \alpha < \kappa \text{ and } y \in C_\alpha \cap Y\} .$$

It is easy to see that the family $\mathcal{U}$ is point–wise $< \kappa$. Moreover, for every $U \in \mathcal{U}$, the set U is open in Z and it is contained in some set of the family $\mathcal{O}$. To complete the proof, we show that $\mathcal{U}$ covers Z. Let $z \in Z$. Denote by η_z the least $\eta \leq \kappa$ such that $z \in F_\eta$. We show that there exists $\beta < \eta_z$ such that $(F_{\beta+1} \setminus F_\beta) \cap B(z, \frac{1}{4}r_z) \neq \emptyset$. Let δ be the least $\beta < \kappa$ such that $F_\beta \cap B(z, \frac{1}{4}r_z) \neq \emptyset$. We then have that $\delta \leq \eta_z$, and it follows from Condition (c) of Lemma 3.8 and Condition (ii) of Definition 3.1 that δ is not a limit ordinal. Let β be the predecessor of δ, and note that $\beta < \eta_z$. By the minimality of δ, we have that $F_\beta \cap B(z, \frac{1}{4}r_z) = \emptyset$. It follows that β satisfies the required condition $(F_{\beta+1} \setminus F_\beta) \cap B(z, \frac{1}{4}r_z) \neq \emptyset$. Since C_β is dense in $F_{\beta+1} \setminus F_\beta$, there exists a point $y \in C_\beta \cap B(z, \frac{1}{4}r_z)$. Note that $y \in B(z, \frac{1}{4}r_z) \subset Y$. We show that $z \in O_y$. Since $y \in B(z, \frac{1}{4}r_z)$ and $B(z, \frac{3}{4}r_z) \subset O_z$, we have that $B(y, \frac{1}{2}r_z) \subset O_z$ and hence that $r_y \geq \frac{1}{2}r_z$; further, since $B(y, \frac{3}{4}r_y) \subset O_y$, we have that $B(y, \frac{3}{8}r_z) \subset O_y$ and hence that $z \in O_y$. Since $\beta < \eta_z$, we have that $z \notin F_\beta$. It follows that $z \in O_y \setminus F_\beta \in \mathcal{U}$. □

3.10 Corollary *In a supmetrizable space, every subspace of density at most ω_1 is meta–Lindelf.*

In particular, no supmetrizable space contains a copy of the ordinal space ω_1.

3.11 Proposition *Let Z be a subset of the supmetrizable space X, let κ be an infinite cardinal number and let $\mathcal{U}$ be an open cover of X such that no subfamily of $\mathcal{U}$ of cardinality $\leq \kappa$ covers Z. Then Z contains a closed and discrete subset of X of cardinality κ^+.*

Proof. We show first that there exists a subset $H = \{z_\alpha : \alpha < \kappa^+\}$ of Z and a subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ covers the set $\mathrm{Cl}_X H$ and, for every $V \in \mathcal{V}$, we have $|V \cap H| \leq \kappa$.

Let τ be the topology of X, $\pi \supset \tau$ a finer metrizable topology and $\mathcal{L}$ a linking family between τ and π. Let $\mathcal{B}$ be a point–countable base for the topology π. Without loss of generality, we may assume that $\mathcal{B}$ refines $\mathcal{U}$. For every $B \in \mathcal{B}$, let $U_B \in \mathcal{U}$ be such that $B \subset U_B$. For every $\mathcal{C} \subset \mathcal{B}$, let $\mathcal{U}(\mathcal{C}) = \{U_B : B \in \mathcal{C}\}$.

Let $\mathcal{L}' = \{L \in \mathcal{L} : d(L) < \kappa^+\}$. For every $L \in \mathcal{L}'$, since $\mathcal{B}$ is a point–countable family of π–open sets and $d_\pi(L) \leq \kappa$, we have that $|(\mathcal{B})_L| \leq \kappa$; as a consequence, we have that $|\mathcal{U}(\mathcal{B}_L)| \leq \kappa$ and hence that $Z \not\subset \bigcup \mathcal{U}(\mathcal{B}_L)$. Set $A(L) = \bigcup \mathcal{U}(\mathcal{B}_L)$ for every $L \in \mathcal{L}'$, and note that $A(L) \subset A(L')$ whenever $L \subset L'$. By Lemma 3.8, there exists a continuously increasing sequence $\langle L_\alpha \rangle_{\alpha<\kappa^+}$ of members of $\mathcal{L}'$ such that we have, for every $\alpha < \kappa^+$, that $L_{\alpha+1} \cap Z \not\subset A(L_\alpha)$.

Let $L = \bigcup_{\alpha<\kappa^+} L_\alpha$ and $S = \mathrm{Cl}_\tau L$. By Conditions (i) and (ii) of Definition 3.1, we have that $S = \mathrm{Cl}_\pi L$. Since $\mathcal{B}$ is an open cover of (X, π), we have that $\mathrm{Cl}_\pi L \subset \bigcup (\mathcal{B})_L$. Set $\mathcal{V} = \mathcal{U}((\mathcal{B})_L)$. By the foregoing, $\mathcal{V}$ covers the set S. Moreover, since $L = \bigcup_{\alpha<\kappa^+} L_\alpha$, we have that $(\mathcal{B})_L = \bigcup_{\alpha<\kappa^+} (\mathcal{B})_{L_\alpha}$. It follows that $\mathcal{V} = \bigcup_{\alpha<\kappa^+} \mathcal{U}((\mathcal{B})_{L_\alpha})$. For every $\alpha < \kappa^+$, let $z_\alpha \in Z \cap [L_{\alpha+1} \setminus A(L_\alpha)]$. Let $H = \{z_\alpha : \alpha < \kappa^+\}$. Note that we have $H \subset L$ and hence $\mathrm{Cl}_\tau H \subset S$; as a consequence, the family $\mathcal{V}$ covers the set $\mathrm{Cl}_\tau H$. Note that, for every $\alpha < \kappa^+$, we have $z_\beta \notin A(L_\alpha)$ for every $\beta > \alpha$, and it follows that we have, for every $U \in \mathcal{U}(\mathcal{B}_\alpha)$, that $U \cap \{z_\beta : \beta > \alpha\} = \emptyset$. By the foregoing, we have that $|U \cap H| \leq \kappa$ for every $U \in \mathcal{V}$.

To complete the proof, we shall use the result of Proposition 3.9 on the subspace $F = \mathrm{Cl}_\tau H$ of X. Since $F \subset \bigcup \mathcal{V}$ and $d(F) \leq \kappa^+$, it follows from the lemma that the subspace F has an open cover $\mathcal{O}$ refining $\mathcal{V}$ such that $\mathcal{O}$ is σ–pointwise $< \kappa^+$; since $\mathrm{cof}(\kappa^+) = \kappa^+ > \omega$, the family $\mathcal{O}$ is pointwise $< \kappa^+$. Since $\mathcal{O}$ refines $\mathcal{V}$, we have that $|O \cap H| \leq \kappa$ for every $O \in \mathcal{O}$. It follows, since H is dense in F and $|(\mathcal{O})_x| \leq \kappa$ for every $x \in H$, that we have $|(\mathcal{O})_O| \leq \kappa$ for every $O \in \mathcal{O}$. As a consequence, we can write $\mathcal{O}$ as $\bigcup_{i\in I} \mathcal{O}_i$ so that $|\mathcal{O}_i| \leq \kappa$ for every $i \in I$ and $\bigcup \mathcal{O}_i \cap \bigcup \mathcal{O}_j = \emptyset$ whenever $i \neq j$. Since $\mathcal{O}$ covers the set H of cardinality κ^+ and since we have $|O \cap H| \leq \kappa$ for every $O \in \mathcal{O}$, we must have $|I| > \kappa$. For every $i \in I$, let $p_i \in H \cap \bigcup \mathcal{O}_i$. Then the set $P = \{p_i : i \in I\}$ is closed and discrete in F and hence in X. Moreover, we have that $|P| > \kappa$. □

Let X be a topological space. Recall that $e(X)$, the "extent" of X is the supremum of the cardinalities of closed discrete subsets of X while $\ell(X)$, the "Lindelöf number" of X is the smallest cardinal λ such that every open cover of X has a subcover of size $\leq \lambda$. For every space X, the inequality $e(X) \leq \ell(X)$ holds. Extending earlier results of Eberlein and Grothendieck, D.P. Baturov [Ba] has proved that for subspaces Z of B_w– or $C_p(K)$–spaces the equality $e(Z) = \ell(Z)$ holds. The previous proposition implies that this equality holds also for subspaces of supmetrizable spaces.

3.12 Corollary *The equality $e(Z) = \ell(Z)$ holds for a subspace Z of a supmetrizable space.*

Proof. It suffices to show that, whenever $\kappa < \ell(Z)$, then we have that $\kappa < e(Z)$. This is obvious for κ finite, so we can assume that κ is infinite. Since $\kappa < \ell(Z)$, there exists an open cover $\mathcal{G}$ of Z such that no subfamily of $\mathcal{G}$ of size $\leq \kappa$ covers Z.

Let Z be a subspace of the supmetrizable space X. There exists an open family $\mathcal{U}$ in X such that $\mathcal{G} = \{U \cap Z : U \in \mathcal{U}\}$. Let $Y = \bigcup \mathcal{U}$. By Lemma 3.5, the subspace Y of X is supmetrizable. Moreover, $Z \subset Y$ and $\mathcal{U}$ is an open cover of Y. By the previous proposition, the subspace Z of Y has a closed discrete subset of size $> \kappa$. Hence $e(Z) > \kappa$. □

3.13 Corollary *Every countably compact subspace of a supmetrizable space is compact.*

With the help of the last result, we can show that regular supmetrizable spaces are "angelic", in the sense of D.H. Fremlin (see [Flo]). Recall that a subset A of a topological space X is said to be *relatively countably compact* in X provided that every sequence from A has a cluster point in X while A is said to be *relatively compact* in X provided that $\overline{A}$ is compact. Note that A is relatively countably compact in X if, and only if, every countable open cover of X has a finite subfamily covering A. A topological space X is *angelic* provided that the following two conditions hold for every relatively countably compact subset A of X:
(i) A is relatively compact;
(ii) for every $x \in \overline{A}$, there exists a sequence from A converging to x.

The importance of angelic spaces is that for their subspaces the notions of (relative) sequential and countable compactness coincide with (relative) compactness. It has been shown by J.D. Pryce in [Pr] that all $C_p(K)$–spaces and B_w–spaces are angelic; here we show that all regular supmetrizable spaces have this property.

3.14 Lemma *Let X be a regular space such that every separable closed subspace of X is Lindelöf, and let $A \subset X$ be relatively countably compact. Then the set $\bigcup\{\overline{C} : C \in [A]^{\leq\omega}\}$ is countably compact.*

Proof. Denote the set in question by B. Assume that B is not countably compact. Then there exists a sequence $(x_n)_{n\in\omega}$ from B without a cluster point in B. For every $n \in \omega$, let $C_n \in [A]^{\leq\omega}$ be such that $x_n \in \overline{C_n}$. Let $C = \bigcup_{n\in\omega} C_n$. Then we have that $\{x_n : n \in \omega\} \subset \overline{C} \subset B$ and it follows that the subspace $\overline{C}$ of X is not countably compact. However, since $\overline{C}$ is separable, it is Lindelöf.

We shall derive a contradiction by showing that $\overline{C}$ is compact. Let $\mathcal{G}$ be an open cover of $\overline{C}$. By regularity, there exists an open cover $\mathcal{V}$ of $\overline{C}$ such that the family $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{G}$. Let $\hat{\mathcal{V}}$ be a countable subfamily of $\mathcal{V}$ covering $\overline{C}$. Further, let $\mathcal{W}$ be a countable family of open subsets of X such that $\{W \cap \overline{C} : W \in \mathcal{W}\} = \hat{\mathcal{V}}$, and let $\mathcal{W}' = \mathcal{W} \cup \{X \setminus \overline{C}\}$. Then $\mathcal{W}'$ is a countable open cover of X, and hence there exists a finite subfamily $\mathcal{W}''$ of $\mathcal{W}'$ covering the relatively countably compact set A. Note that the finite subfamily $\mathcal{W}^* = \mathcal{W}'' \setminus \{X \setminus \overline{C}\}$ of $\mathcal{W}$ covers the set C. It follows that $\overline{C} \subset \bigcup\{\overline{W \cap C} : W \in \mathcal{W}^*\}$. For every $W \in \mathcal{W}^*$, there exists $G_W \in \mathcal{G}$ such that $\overline{W \cap \overline{C}} \subset G_W$. Now $\{G_W : W \in \mathcal{W}^*\}$ is a finite subfamily of $\mathcal{G}$ covering $\overline{C}$. We have shown that the set $\overline{C}$ is compact – a contradiction, since the set is not countably compact. □

3.15 Proposition *Let X be a regular monolithic space such that every countably compact subspace of X is compact. Then X is angelic.*

Proof. Let $A \subset X$ be relatively countably compact. Let $B = \bigcup\{\overline{C} : C \in [A]^{\leq\omega}\}$. By the above lemma, B is countably compact; it follows that B is compact in X and hence closed. As a consequence, $\overline{A} = B$, and A is relatively compact. Now let $x \in \overline{A}$. Then $x \in B$, and there exists $C \in [A]^{\leq\omega}$ such that $x \in \overline{C}$. The set $\overline{C}$ is compact and separable and it follows, since X is monolithic, that $\overline{C}$ is metrizable. Hence there exists a sequence from C converging to x. □

By Propositions 3.7 and 3.15 and Corollary 3.13, we have the following result.

3.16 Corollary *Every regular supmetrizable space is angelic.*

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